The theory of modal and intermediate logics

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Modal logic and intuitionistic logic

Modal logic is an **expansion** of classical logic.

Additional modal operators have different meanings:

- alethic modalities (possibility, necessity),
- temporal modalities (since, until),
- deontic modalities (obligation, permission),
- epistemic modalities (knowledge),
- doxastic modalities (belief), etc.

Intuitionistic logic is a **subsystem** of classical logic.

Constructive viewpoint: Truth = Proof.
The law of excluded middle $p \vee \neg p$ is rejected.

Surprisingly: intuitionistic and modal logic are

closely connected!

Background

We will assume the following basic knowledge.

- Syntax and Kripke semantics of modal logic,
- The modal systems **S4**, **S5** and their basic properties.

Familiarity with the following notions is advantageous, but not required.

- Basics of first-order logic, set-theory and topology.

References

1. Blackburn, de Rijke, Venema, Modal Logic, Cambridge University Press, 2001.
2. Chagrov and Zakharyaschev, Modal Logic, Clarendon Press, 1997.
3. Rasiowa and Sikorski, The Mathematics of Metamathematics, Pantswowe Wydaw, 1963.
4. N. Bezhanishvili and D. de Jongh, Intuitionistic logic, ILLC, University of Amsterdam, 2006.

Main topics

- A translation of intuitionistic logic into modal logic.
- Completeness: Kripke completeness, the finite model property (FMP), algebraic completeness, topological completeness.
- Axiomatization and decidability of modal and intermediate logics.
- Classes of modal and intermediate logics.

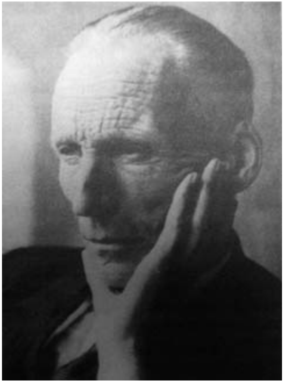
Overview of today's lecture

1. Intuitionistic logic and its Kripke semantics,
2. Intermediate logics,
3. Gödel translation,
4. Modal companions of intermediate logics,
5. Least and greatest modal companions,
6. Blok-Esakia theorem: an overview.

Intuitionistic logic

One of the cornerstones of classical reasoning is the law of excluded middle $p \vee \neg p$.

On the grounds that the only accepted reasoning should be constructive, the Dutch mathematician L. E. J. Brouwer rejected this law, and hence classical reasoning.



Luitzen Egbertus Jan Brouwer (1881 - 1966)

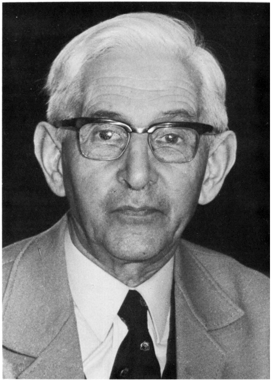
Intuitionistic logic

This resulted in serious debates between Hilbert and Brouwer. Other leading mathematicians of the time were also involved in this debate.

David Hilbert (1862 - 1943)

Intuitionistic logic

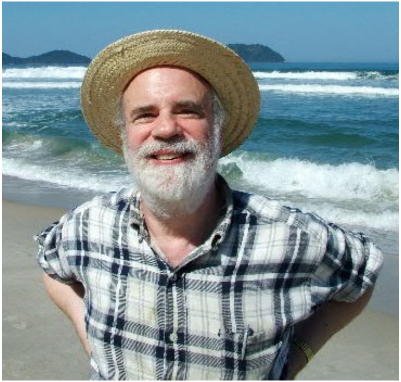
In 30's Brouwer's ideas led his student Heyting to axiomatize intuitionistic logic.



Arend Heyting (1898 - 1980)

Kripke semantics

In 50's and 60's Kripke discovered a relational (Kripke) semantics for intuitionistic and modal logic and proved completeness of intuitionistic logic wrt this semantics.



Saul Kripke

Kripke semantics

An intuitionistic Kripke frame is a pair $\mathfrak{F} = (W, R)$, where W is a set and R is a partial order; that is, a reflexive, transitive and anti-symmetric relation on W.

An intuitionistic Kripke model is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ such that $\mathfrak{F}$ is an intuitionistic Kripke frame and V is an intuitionistic valuation; that is, a map $V : \text{PROP} \to \mathcal{P}(W)$ such that:

$$w \in V(p) \text{ and } wRv \text{ implies } v \in V(p).$$

Persistence: Information is never lost.

Sets satisfying the above property are called upward closed.

A frame is rooted if there is a point x that sees every point in the frame.

We will consider only rooted frames.

Kripke semantics

$\mathfrak{M} = (W, R, V)$ intuitionistic model, $w \in W$, and $\varphi \in \textsc{Form}$.

Satisfaction $\mathfrak{M}, w \models \varphi$ defined inductively:

$\mathfrak{M}, w \models p$	if	$w \in V(p)$;
$\mathfrak{M}, w \models \bot$		never;
$\mathfrak{M}, w \models \varphi \wedge \psi$	if	$\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$;
$\mathfrak{M}, w \models \varphi \vee \psi$	if	$\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$;
$\mathfrak{M}, w \models \varphi \to \psi$	if	$\forall v$, if (wRv and $\mathfrak{M}, v \models \varphi$) then $\mathfrak{M}, v \models \psi$;
$\mathfrak{M}, w \models \neg\varphi$	if	$\mathfrak{M}, v \not\models \varphi$ for all v with wRv.

Validity $\mathfrak{F} \models \varphi$ is satisfaction at every w and for each V.

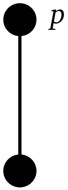
The satisfaction clause for intuitionistic $\varphi \to \psi$ resembles the satisfaction clause for modal $\Box(\varphi \to \psi)$.

$\mathbf{IPC} \subsetneq \mathbf{CPC}$

CPC = classical propositional calculus
IPC = intuitionistic propositional calculus.

$$\mathbf{CPC} = \mathbf{IPC} + (p \vee \neg p).$$

The law of excluded middle $p \vee \neg p$ is not derivable in intuitionistic logic.



Assuming completeness, this shows that $\mathbf{IPC} \subsetneq \mathbf{CPC}$.

Intermediate logics

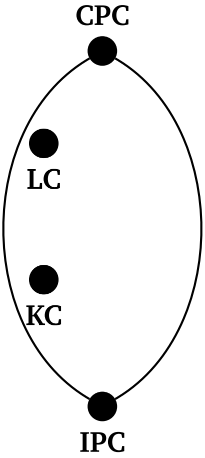
Logics in between **IPC** and **CPC** are called intermediate logics.

$\mathbf{LC} = \mathbf{IPC} + (p \to q) \vee (q \to p)$

Gödel-Dummett calculus

$\mathbf{KC} = \mathbf{IPC} + (\neg p \vee \neg\neg p)$

weak law of excluded middle



Intermediate logics

Theorem.

1. **IPC** is the logic of all intuitionistic frames.
2. **CPC** is the logic of a one-point frame.
3. **KC** is the logic of directed intuitionistic frames.
4. **LC** is the logic of linear intuitionistic frames.

Gödel translation

In the 30's Gödel defined a translation of intuitionistic logic into the modal logic **S4**.

Kurt Gödel (1906 - 1978)

Gödel translation

$$
\begin{aligned}
(\bot)^* &= \bot, \\
(p)^* &= \Box p, \text{ where } p \in \mathsf{Prop}, \\
(\varphi \wedge \psi)^* &= \varphi^* \wedge \psi^*, \\
(\varphi \vee \psi)^* &= \varphi^* \vee \psi^*, \\
(\varphi \to \psi)^* &= \Box(\varphi^* \to \psi^*).
\end{aligned}
$$

S4 is the modal logic of reflexive and transitive frames.

Gödel translation

McKinsey and Tarski proved in the 40's that Gödel's translation is full and faithful.

Theorem (Gödel-McKinsey-Tarski) For each formula φ in the propositional language we have

$$\mathbf{IPC} \vdash \varphi \text{ iff } \mathbf{S4} \vdash \varphi^*.$$

Topological semantics

They also defined topological semantics for modal and intuitonistic logic and proved that **S4** and **IPC** are complete wrt the real line $\mathbb{R}$.

Alfred Tarski (1901 - 1983)

Generalized Gödel embedding

Dummett and Lemmon in the 50's lifted the Gödel translation to intermediate logics and extensions of **S4**.

Michael Dummett (1925 - 2011)

Modal companions

A modal logic $M \supseteq \mathbf{S4}$ is a modal companion of an intermediate logic $L \supseteq \mathbf{IPC}$ if for any propositional formula φ we have

$$L \vdash \varphi \text{ iff } M \vdash \varphi^*.$$

Examples.

1. **S4** is a modal companion of **IPC**.
2. **S5** is a modal companion of **CPC**.
3. **S4.2** is a modal companion of **KC**.
4. **S4.3** is a modal companion of **LC**.

Recall that

- $\mathbf{S4.2} = \mathbf{S4} + \Diamond\Box p \to \Box\Diamond p$ is the logic of directed **S4**-frames.
- $\mathbf{S4.3} = \mathbf{S4} + \Box(\Box p \to \Box q) \vee \Box(\Box q \to \Box p)$ is the logic of linear **S4**-frames.

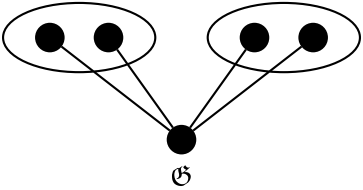
S4-frames and their skeletons

Let us look at an **S4**-frame $\mathfrak{G}$.

We say that an intuitionistic frame $\mathfrak{F}$ is the skeleton of $\mathfrak{G}$ if by identifying all the clusters in $\mathfrak{G}$ we obtain $\mathfrak{F}$.

A cluster is an equivalence class of the relation:

$$x \sim y \text{ if } (xRy \text{ and } yRx).$$



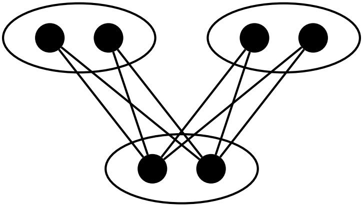
S4-frames and their skeletons

Let us look at preordered (relfexive and transitive) frame $\mathfrak{G}$.

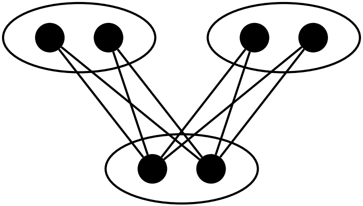
We say that an intuitionistic frame (reflexive, transitive, anti-symmetric) $\mathfrak{F}$ is the skeleton of $\mathfrak{G}$ if by identifying all the clusters in $\mathfrak{G}$ we obtain $\mathfrak{F}$.

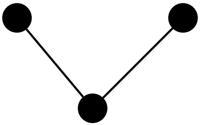
A cluster is an equivalence class of the relation:

$$x \sim y \text{ if } (xRy \text{ and } yRx).$$



S4-frames and their skeletons





Thus we can think of an **S4**-frame as a poset of clusters.

S4-frames and their skeletons

Lemma. Let $\mathfrak{G}$ be such that $\mathfrak{F}$ is its skeleton, then for any intuitionistic formula φ:

$$\mathfrak{F} \models \varphi \quad \text{iff} \quad \mathfrak{G} \Vdash \varphi^*.$$

Key idea: $\mathfrak{G}$ and $\mathfrak{F}$ have **matching** upward closed subsets.

Let $Log(\mathfrak{F}) = \{\varphi : \mathfrak{F} \models \varphi\}$. We call it the intermediate logic of $\mathfrak{F}$.

Let $\mathfrak{F}$ be a finite intuitionistic frame. We let K denote a class of **S4**-frames that have $\mathfrak{F}$ as their skeleton.

Theorem. An extension M of **S4** is a modal companion of $Log(\mathfrak{F})$ iff $M = Log(\mathsf{K})$ for some K.

To prove an analogue of this result for all intermediate logics we need algebras and duality.

Examples

Recall that **CPC** $= Log(\mathfrak{F}_1)$, where

$\mathfrak{F}_1$

Which modal logics are modal companions of **CPC**?

$\mathfrak{G}_1$ $\mathfrak{G}_2$ $\mathfrak{G}_3$

$$Log(\mathfrak{G}_1) \supsetneq Log(\mathfrak{G}_2) \supsetneq Log(\mathfrak{G}_3) \supsetneq \ldots \supsetneq \mathbf{S5}$$

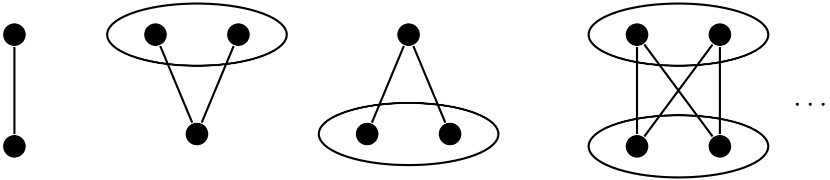
Exercise: Verify these inclusions. Find formulas showing that the inclusions are strict. (This will be easily done once we discuss the Jankov-de Jongh formulas.)

Examples

$$Log(\mathfrak{G}_1) \supsetneq Log(\mathfrak{G}_2) \supsetneq Log(\mathfrak{G}_3) \supsetneq \cdots \supsetneq \mathbf{S5}$$

We see that $Log(\mathfrak{G}_1)$ is the greatest modal companion of **CPC** and **S5** is the least one.

For the intermediate logic of the two-chain we have modal companions given by the following frames.



Exercise: Do these modal companions form a chain?

Greatest and least modal companions

Question: Do the least and greatest modal companions of any intermediate logic always exist?

Our examples were such that $Log(\mathfrak{F})$ is complete wrt one finite frame.

In general there exist logics that are not complete wrt one finite frame (non-tabular logics), a class of finite frames (logics without the FMP), or any class of Kripke frames (Kripke incomplete logics).

We overcome this problem by algebraic completeness.

In order to regain the intuition of the relational semantics we will develop duality between algebras and general frames.

Greatest and least modal companions

Esakia and independently Maksimova in the 70's developed the theory of Heyting and closure algebras. Esakia also developed an order-topological duality for closure and Heyting algebras.

Leo Esakia (1934 - 2010)

Larisa Maksimova

Grzegorczyk's logic

The logic of finite **S4**-frames without clusters is Grzegorczyk's modal system

$$\mathbf{Grz} = \mathbf{S4} + (\Box(\Box(p \to \Box p) \to p) \to p))$$

Theorem.

1. **Grz** is complete wrt partially ordered finite **S4**-frames.
2. **Grz** and **S4** are the greatest and least modal companions of **IPC**, respectively.
3. For an intermediate logic L its least and greatest modal companions exist. Moreover, the least modal companion is $\mathbf{S4} + \{\varphi^* : \varphi \in L\}$ and the greatest is $\mathbf{Grz} + \{\varphi^* : \varphi \in L\}$.

This gives a purely syntactic characterization of the least and greatest modal companions of an intermediate logic.

Grzegorczyk's logic

Andrzej Grzegorczyk

Mappings τ and σ

The least modal companion of L is denoted by $\tau(L)$ and the greatest by $\sigma(L)$.

That is, $\tau(L) = \mathbf{S4} + \{\varphi^* : \varphi \in L\}$ and $\sigma(L) = \mathbf{Grz} + \{\varphi^* : \varphi \in L\}$.

M is a modal companion of L iff $\tau(L) \subseteq M \subseteq \sigma(L)$.

Theorem.

1. $\tau(\mathbf{IPC}) = \mathbf{S4}$ and $\sigma(\mathbf{IPC}) = \mathbf{Grz}$.
2. $\tau(\mathbf{CPC}) = \mathbf{S5}$ and $\sigma(\mathbf{CPC}) = Log(\mathfrak{G}_1) = \mathbf{S5} \cap \mathbf{Grz}$.
3. $\tau(\mathbf{KC}) = \mathbf{S4.2}$ and $\sigma(\mathbf{KC}) = \mathbf{Grz.2}$
4. $\tau(\mathbf{LC}) = \mathbf{S4.3}$ and $\sigma(\mathbf{LC}) = \mathbf{Grz.3}$

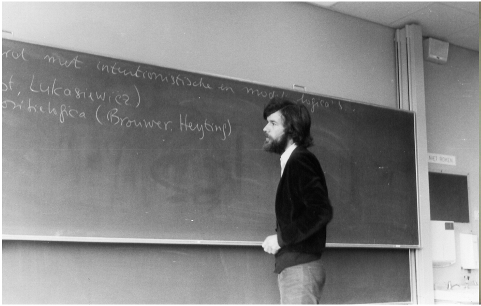
Blok-Esakia theorem

Let $\Lambda(\mathbf{IPC})$ denote the lattice of intermediate logics, let $\Lambda(\mathbf{S4})$ denote the lattice of extensions of **S4**, and let $\Lambda(\mathbf{Grz})$ denote the lattice of extensions of **Grz**.

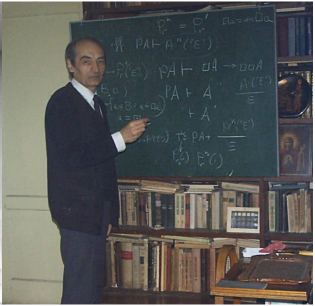
Theorem.

1. $\tau, \sigma : \Lambda(\mathbf{IPC}) \to \Lambda(\mathbf{S4})$ are lattice homomorphisms.
2. $\tau : \Lambda(\mathbf{IPC}) \to \Lambda(\mathbf{S4})$ is an embedding of the lattice of intermediate logics into the lattice of extensions of **S4**.
3. (Blok-Esakia) $\sigma : \Lambda(\mathbf{IPC}) \to \Lambda(\mathbf{Grz})$ is an isomorphism from the lattice of intermediate logics onto the lattice of extensions of **Grz**.

Blok-Esakia theorem



Wim Blok (1947 - 2003)



Leo Esakia (1934 - 2010)

Blok-Esakia theorem

Modern proof of the Blok-Esakia theorem uses Heyting and modal algebras, duality and canonical formulas.

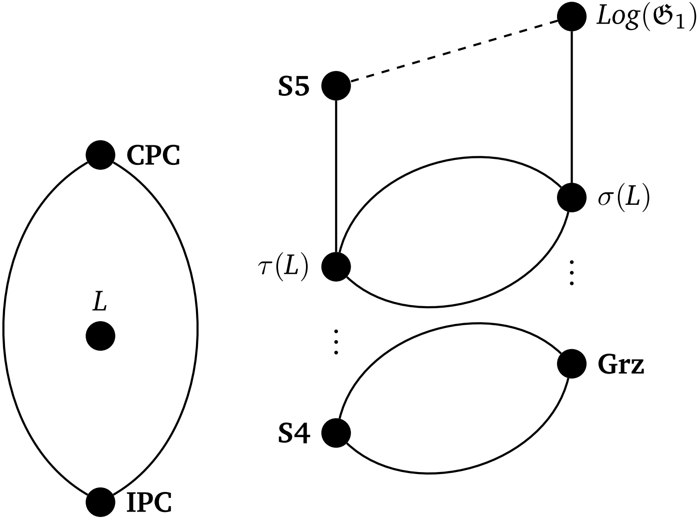
The method of canonical formulas is a powerful tool allowing to axiomatize all intermediate logics and all extensions of **S4**.

This method, developed by Zakharyaschev, builds on Jankov-de Jongh formulas and Fine's subframe formulas.

This method is very complex.

Nowadays we can provide a simplified algebraic approach to this method.

Picture of $\Lambda(\mathbf{IPC})$ and $\Lambda(\mathbf{S4})$



Exercises

1. Describe the intermediate logic whose modal companion is $\mathbf{S4.1} = \mathbf{S4} + (\Box\Diamond p \to \Diamond\Box p)$?

2. Is there a modal logic M with $\mathbf{S4} \subseteq M \subseteq \mathbf{S5}$ such that for no intermediate logic L we have $\tau(L) = M$? Justify your answer.

3. How many modal companions does the intermediate logic of the two element chain have? Justify your answer.

4. Is there an intermediate logic that has a finite number of modal companions? Justify your answer.

Thank you!