INTRODUCTION TO MODAL LOGIC 2017

HOMEWORK 2

- Deadline: 5 October — at the **beginning** of class.
- Grading is from 0 to 100 points.
- Success!

Exercise 1. (30 points) Show that the following frame properties are not modally definable by a single formula in the basic modal language (i.e., that there is no basic modal formula ϕ such that a frame $\mathbb{F}$ has the given property iff $\mathbb{F} \Vdash \phi$):

(a) every point has at least two successors;
(b) every point lies on a cycle (i.e., from every node s there is a path of non-zero length from s to itself).

Exercise 2. (30 points) Let $\mathbb{M}$ and $\mathbb{M}'$ be image-finite Kripke models, and consider states s and s' in $\mathbb{M}$ and $\mathbb{M}'$, respectively. Assume that $\mathbb{M}, s \leftrightarroweq_n \mathbb{M}', s'$, for all natural numbers n.

(a) Prove that $\mathbb{M}, s \leftrightarroweq \mathbb{M}', s'$.
(b) Prove that $\mathbb{M}, s \leftrightarroweq \mathbb{M}', s'$ without using results from [BdRV].

Exercise 3. (40 points) In this exercise we consider a bimodal language with two diamonds, $\Diamond$ and $\langle * \rangle$. We call a frame $\mathbb{F} = (W, R_\Diamond, R_{\langle * \rangle})$ for this language *regular* if $R_{\langle * \rangle}$ is the reflexive-transitive closure of $R_\Diamond$: $R_{\langle * \rangle} = (R_\Diamond)^*$.

Verify that the formula $\langle * \rangle \phi \leftrightarrow (\phi \lor \Diamond \langle * \rangle \phi)$ is valid in every regular frame (but you do not need to hand in your proof).

Now let Σ be a set of formulas which is closed under taking subformulas, and in addition satisfies $\langle * \rangle \phi \in \Sigma \Rightarrow \Diamond \langle * \rangle \phi \in \Sigma$.

(a) Fix a regular frame $\mathbb{F} = (W, R_\Diamond, R_{\langle * \rangle})$ and assume that the relation $R^s \subseteq W_\Sigma \times W_\Sigma$ is the *smallest* filtration of $R_\Diamond$, that is:

$$R^s \overline{u}\,\overline{v} \text{ iff there are } u' \in \overline{u}, v' \in \overline{v} \text{ with } R_\Diamond u'v'.$$

Prove that the relation $(R^s)^*$ satisfies the filtration conditions for R^* and $\langle * \rangle$:

(S^*) if $R^* uv$ then $(R^s)^* \overline{u}\,\overline{v}$;
(L^*) if $(R^s)^* \overline{u}\,\overline{v}$ then, for all $\langle * \rangle \phi \in \Sigma$: $\mathbb{M}, v \Vdash \phi \Rightarrow \mathbb{M}, u \Vdash \langle * \rangle \phi$.

(b) Show that a formula in this language is satisfiable in a regular model if and only if is satisfiable in a finite regular model.