INTRODUCTION TO MODAL LOGIC
FINAL EXAM

21 DECEMBER 2017
UNIVERSITY OF AMSTERDAM

Name:

UvA Student ID:

General comments.

1. The time for this exam is 3 hours (180 minutes).
2. There are 100 points in the exam.
3. Please use a blue-ink or black-ink pen only, no pencils.
4. Make sure that you have your name and student ID on each of the sheets you are handing in.
5. If you have any questions, please indicate this silently and someone will come to you. Answers to questions that are relevant for anyone will be announced publicly.
6. No talking during the exam.
7. Cell phones must be switched off and stowed.

	Maximal Points	Your points
Exercise 1	10	
Exercise 2	30	
Exercise 3	20	
Exercise 4	20	
Exercise 5	20	
Total	100	

(1) (10pt) Let $\mathfrak{F} = (W, R)$ be a Kripke frame. The frame $\mathfrak{F}^r = (W, \overline{R})$ is called the *reflexivization of* $\mathfrak{F}$ if $\overline{R} = R \cup \{(x, x) \mid x \in W\}$. For a model $\mathfrak{M} = (W, R, V)$ let $\mathfrak{M}^r = (W, \overline{R}, V)$.

For each modal formula φ let φ^+ be the formula obtained from φ by replacing each subformula of the form $\Diamond\psi$ with $\Diamond\psi \vee \psi$ (we assume that $\bot, \vee, \neg$ and $\Diamond$ are the primitive symbols of the language).

Show that for each model $\mathfrak{M} = (W, R, V)$, each $w \in W$ and each modal formula φ:

$$\mathfrak{M}, w \Vdash \varphi^+ \text{ iff } \mathfrak{M}^r, w \Vdash \varphi.$$

(2) (30pt)

(a) Show, using the Sahlqvist algorithm, that the first-order correspondent of the formula

$$\Diamond\Box p \to \Box p$$

is the formula

$$\forall x \forall y \forall z((Rxy \wedge Rxz) \to Rzy)).$$

(b) Show that the modal logic

$$\mathbf{K5} = \mathbf{K} + (\Diamond\Box p \to \Box p)$$

is canonical and hence Kripke complete.

You are not allowed to use the Sahlqvist completeness theorem.

(c) Give an example of a model $\mathfrak{M} = (W, R, V)$ such that (W, R) is a **K5**-frame and of a filtration $\mathfrak{M}^f = (W^f, R^f, V^f)$ of $\mathfrak{M}$ such that (W^f, R^f) is not a **K5**-frame. Justify your solution.

(3) (20pt) Let $\mathfrak{g} := (W, R, \mathcal{A})$, where

$$W = \mathbb{N} \cup \{\omega_1, \omega_2\},$$
$$R = \{(\omega_i, \omega_j), (\omega_i, n) \mid i, j \in \{1, 2\}, n \in \mathbb{N}\} \cup \{(n, m) \mid n \geq m\},$$
$$\mathcal{A} = \{U \subseteq \mathbb{N} \mid U \text{ is finite}\} \cup \{U' \cup \{\omega_1, \omega_2\} \mid U' \subseteq \mathbb{N} \text{ is cofinite}\}.$$

See Figure 1 below.

(a) Show that $\mathfrak{g}$ is a general frame.

(b) Show that for every admissible valuation V (i.e., $V(p) \in \mathcal{A}$) we have

$$(\mathfrak{g}, V), \omega_1 \Vdash \Box(\Box(p \to \Box p) \to p) \to p.$$

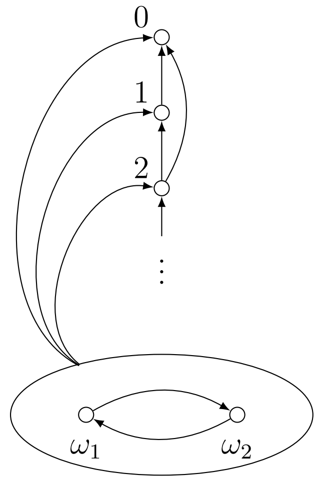

Figure 1: The frame $\mathfrak{g}$

(4) (20pt) For a program π and a natural number k we let

$$\langle\pi\rangle^0 p = p, \quad \langle\pi\rangle^1 p = \langle\pi\rangle p, \quad \langle\pi\rangle^{k+1} p = \langle\pi\rangle\langle\pi\rangle^k p.$$

For each $n \geq 0$ we define a normal extension $\mathbf{PDL}^n$ of $\mathbf{PDL}$ as follows. $\mathbf{PDL}^n$ is the least normal extension of $\mathbf{PDL}$ that contains all the instances of the formula:

$$\langle\pi^*\rangle p \leftrightarrow \bigvee_{i=0}^{n} \langle\pi\rangle^i p.$$

(a) Show that the logic $\mathbf{PDL}^n$ is sound with respect to regular frames $(W, \{R_\pi\}_{\pi\in\Pi})$ such that $R_{\pi^*} = \bigcup_{k=0}^{n} R_\pi^k$, where

$$R_\pi^0 := \{(w, w) \mid w \in W\}, \quad R_\pi^1 := R_\pi, \quad \text{and} \quad R_\pi^{k+1} := R_\pi \circ R_\pi^k.$$

(b) Show that $\mathbf{PDL} = \bigcap_{n\geq 0} \mathbf{PDL}^n$.

(Hint: For the right to left inclusion use the fact that $\mathbf{PDL}$ has the finite model property.)

(5) (20pt) Show that

(a) the formula $(\Box p \to \Diamond p) \to \Diamond\top$ is a theorem of the monotone modal logic $\mathbf{EM}$;

(b) the formula $\Diamond\top \to (\Box p \to \Diamond p)$ is not a theorem of the monotone modal logic $\mathbf{EM}$.

(Hint: use completeness of $\mathbf{EM}$ with respect to monotone neighborhood frames.)