INTRODUCTION TO MODAL LOGIC.
FINAL EXAM

22 DECEMBER 2016
UNIVERSITY OF AMSTERDAM

Name:

UvA Student ID:

General comments.

1. The time for this exam is 3 hours (180 minutes).
2. There are 100 points in the exam.
3. Make sure that you have your name and student ID on each of the sheets you are handing in.
4. If you have any questions, please indicate this silently and someone will come to you. Answers to questions that are relevant for anyone will be announced publicly.
5. No talking during the exam.
6. Cell phones must be switched off and stowed.

	Maximal Points	Your points
Exercise 1	20	
Exercise 2	20	
Exercise 3	20	
Exercise 4	20	
Exercise 5	20	
Total	100	

(1) (20pt) Let (W, R) and (W', R') be Kripke frames.

(a) Define when a map $f : W \to W'$ is a bounded morphism.

(b) A map $f : W \to W'$ is called a *homomorphism* if for each $w, v \in W$ we have Rwv implies $R'f(w)f(v)$. Give an example of two frames (W, R) and (W', R') and a surjective map $f : W \to W'$ such that f is a homomorphism, but not a bounded morphism.

(c) Is validity of modal formulas preserved under surjective homomorphisms? In other words, if a modal formula φ is valid in (W, R) and if $f : W \to W'$ is a surjective homomorphism, is φ valid in (W', R')? If yes, provide a proof, if not give a counter-example.

(2) (20pt)

(a) Show, using the Sahlqvist algorithm, that the first-order correspondent of the formula

$$\Diamond\Box p \to \Diamond p$$

is the formula

$$\forall x \forall y (Rxy \to \exists z (Rxz \wedge Ryz)).$$

(b) Show that the modal logic

$$\mathbf{KO} = \mathbf{K} + (\Diamond\Box p \to \Diamond p)$$

is canonical. That is, given **KO**-MCSs Γ, Δ with $R^{\mathbf{KO}}(\Gamma, \Delta)$, you will need to find a **KO**-MCS Θ with $R^{\mathbf{KO}}(\Gamma, \Theta)$ and $R^{\mathbf{KO}}(\Delta, \Theta)$.

(Hint: You should find Lindenbaum's lemma useful.)

(c) Deduce that **KO** is sound and complete with respect to **KO**-frames.

You are not allowed to use the Sahlqvist completeness theorem.

(3) (20pt)

(a) Show, using filtration, that **KO** (see Exercise 2) has the finite model property.

(b) Deduce that **KO** is decidable.

You can assume the facts stated in Exercise 2.

(4) (20pt)

(a) Define regular frames for **PDL**.

(b) Let $(\omega{-}^*)$ be the following rule:

$$\text{If} \vdash \varphi \to [\pi]^n\psi \text{ for each } n \in \mathbb{N}, \text{ then} \vdash \varphi \to [\pi^*]\psi.$$

Recall that $[\pi]^0 p = p$ and $[\pi]^{n+1} = [\pi][\pi]^n p$.

We say that $(\omega{-}^*)$ is *valid* on a frame (W, R_π, R_{π^*}) if for any valuation V

$$(W, R_\pi, R_{\pi^*}, V) \Vdash \varphi \to [\pi]^n\psi \text{ for each } n \in \mathbb{N} \text{ implies}$$
$$(W, R_\pi, R_{\pi^*}, V) \Vdash \varphi \to [\pi^*]\psi.$$

Let (W, R_π, R_{π^*}) be a (not necessarily regular) frame. Show that we have $R_{\pi^*} \subseteq (R_\pi)^*$ iff $(\omega{-}^*)$ is valid on (W, R_π, R_{π^*}).

(5) (20pt) Consider the Kripke frame (W, R), where

$$W = \{u, v, w\} \cup \{v_n, w_n : n \in \mathbb{N}\}$$

and R is defined as follows:

$$Ruv,\ Ruw,\ Rvv_n \text{ and } Rww_n \text{ (for all } n \in \mathbb{N});$$

see the figure below. Let A be the collection of all finite and co-finite subsets of W. Then (W, R, A) is a general frame. Show that

(a) $(W, R), u \not\Vdash \Diamond\Box p \to \Box\Diamond p$,

(b) $(W, R, A), u \Vdash \Diamond\Box p \to \Box\Diamond p$.