EXERCISE CLASS 1-11-2017: NORMAL MODAL LOGICS AND HILBERT STYLE DERIVIATIONS

(1) Let $\mathcal{C}$ be a class of frames and let $\mathcal{M}$ be a class of models.
   (a) Show that
   $$\mathsf{Log}(\mathcal{C}) := \{\varphi\colon \forall \mathbb{F} \in \mathcal{C}\ (\mathbb{F} \Vdash \varphi)\}$$
   is a normal modal logic. Conclude that **K** is sound w.r.t. the class of all frames.
   (b) Is the set of formulas
   $$\mathsf{Th}(\mathcal{M}) := \{\varphi\colon \forall \mathbb{M} \in \mathcal{M}\ (\mathbb{M} \Vdash \varphi)\}$$
   a normal modal logic?

(2) Let Λ be a normal modal logic and let φ, φ', ψ and ψ' be formulas in the language of basic modal logic. Show that:
   (a) If $\varphi \to \psi$ is (a substitution instance of) a propositional tautology, then $\vdash_\Lambda \varphi$ implies $\vdash_\Lambda \psi$.
   (b) If $\vdash_\Lambda \varphi$ and $\vdash_\Lambda \psi$ then $\vdash_\Sigma \varphi \wedge \psi$.
   (c) If $\vdash_\Lambda \varphi \to \psi$ and $\vdash_\Lambda \psi \to \chi$ then $\vdash_\Lambda \varphi \to \chi$.
   (d) If $\vdash_\Lambda \varphi \to \psi$ and $\vdash_\Lambda \varphi' \to \psi'$ then $\vdash_\Lambda (\varphi \wedge \varphi') \to (\psi \wedge \psi')$

(3) Let Λ be a normal modal logic and let φ and ψ be formulas in the language of basic modal logic. Prove that
   (a) $\vdash_\Lambda \varphi \to \psi$ implies $\vdash_\Lambda \Box\varphi \to \Box\psi$
   (b) $\vdash_\Lambda \varphi \to \psi$ implies $\vdash_\Lambda \Diamond\varphi \to \Diamond\psi$
   (c) $\vdash_\Lambda \Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$
   (d) $\vdash_\Lambda \Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$

(4) (Equivalent replacement). Let $\varphi[\psi]$ be a formula that contains ψ as a subformula. Let $\varphi[\chi]$ denote the formula where ψ in $\varphi[\psi]$ is replaced with the formula χ. Show that
$$\vdash_\Lambda \psi \leftrightarrow \chi \quad \text{implies} \quad \vdash_\Lambda \varphi[\psi] \leftrightarrow \varphi[\chi],$$
for any normal modal logic Λ.

(5) Show that $\nvdash_{\mathbf{S4}} p \to \Box\Diamond p$ and that $\nvdash_{\mathbf{K}} \Box p \vee \Box\neg p$. Recall that $\mathbf{S4} = \mathbf{K} + (\Box p \to p) + (\Box p \to \Box\Box p)$.

(6) A normal modal logic Λ is *Halldén complete* if for every pair of formulas φ and ψ with no common variables we have that
$$\vdash_\Lambda \varphi \vee \psi \quad \text{implies} \quad \vdash_\Lambda \varphi \text{ or } \vdash_\Lambda \psi.$$
Is the normal modal logic **K** Halldén complete? Give proof or counter-example.