Introduction to Modal Logic
Exercise class 7

October 16, 2017

Examples.

- In the formula $\neg p$, the proposition p appears *negatively* because it appears under the scope of a negation;
- In the formula $\neg\neg p$, the proposition p appears *positively* because it appears under the scope of an even number of negations;
- In the formula $\neg(p \vee \neg p)$, the left occurrence of proposition p appears *negatively*, while the right occurrence appears positively.

Exercise 1. Consider as primitive connectives $\vee$, $\neg$, $\bot$ and $\Diamond$. Let p be a propositional letter that occurs in φ. Define by induction on φ: The occurrence of p is positive (negative).

Definition 1. A formula φ is called positive (negative) in p if all occurrences of p are positive (negative).

A formula φ is called upward monotome (respectively downward monotone) in p if for every frame $\mathbb{F}$, every point w and every pair of assignments V and V' such that

$$\left.\begin{array}{ll} V(p) \subseteq V'(p) & \\ V(q) = V'(q) & \text{for } q \neq p \end{array}\right\}$$

it holds

$$(\mathbb{F}, V), w \vDash \varphi \Rightarrow (\mathbb{F}, V'), w \vDash \varphi$$
$$\big(\text{resp. } (\mathbb{F}, V'), w \vDash \varphi \Rightarrow (\mathbb{F}, V), w \vDash \varphi\big)$$

Exercise 2.

- Show that if φ is positive in p then it is upward monotone in p, and if it is negative in p then it is downward monotone in p.
- What about the converse? If φ upward (downward) monotone in p does it follow that φ is positive (negative) in p?

Exercise 3. Sahlqvist algorithm.
Compute the standard translation and a first order corespondent of (some of) the following formulas.

- $\Box\Box p \to \Box p$
- $\Box p \wedge p \to \Diamond\Diamond p$
- $\Diamond\Box p \to \Box\Diamond p$
- $\Diamond\Box p \to \Box\Diamond\Diamond p$
- $\Box(\Box p \to p)$
- $\Box((\Box p \to p) \vee (\Diamond p \to \Box\Box p))$
- $\Box(\Box p \to q) \vee \Box(\Box q \to p)$
- $\Diamond p \wedge \Diamond q \to (\Diamond(p \wedge q) \vee \Diamond(p \wedge \Diamond q) \vee \Diamond(q \wedge \Diamond p))$
- $q \to \Box_F \Box_P q$ (a temporal example; can you think how to do it?)

Exercise 4.

- Show that the frame property "R is the identity relation" is modally definable, but the property "R is the complement of the identity relation" is **not** modally definable.
- Show that if a first-order definable class $\mathcal{K}$ of Kripke frames is closed under disjoint unions, p-morphic images and subframes, then it is modally definable.
- Show that the frame property of being cyclic is reflected by ultrafilter extensions. That is, if every state in the ultrafilter extension $\mathbb{F}^*$ lies on a cycle then so does every state in $\mathbb{F}$. Why does this not contradict Exercise 1(b) of Homework 2?