Introduction to Modal Logic. Exercise class 6

9 October 2017

Lemma 1 (Key Lemma)**.** *Let* $\mathbb{M} = (S, R, V)$ *be a Kripke model. Then for every modal formula* ϕ*, and for every ultrafilter* $u \in \mathsf{Uf}(S)$ *we have*

$$\mathbb{M}^*, u \Vdash \phi \textit{ iff } [\![\phi]\!]^{\mathbb{M}} \in u. \tag{1}$$

Exercise 1. The key lemma is proved by induction on ϕ. In this exercise we focus on the hard part of the inductive case for the formula $\Diamond\phi$. That is, assume as inductive hypothesis that (1) holds for the formula ϕ.

(a) Show that $[\![\Diamond\phi]\!]^{\mathbb{M}} = \langle R \rangle [\![\phi]\!]^{\mathbb{M}}$.

(b) Suppose that $\langle R \rangle X \in u$, for some set $X \in \mathcal{P}(S)$. Consider the set

$$E := \{X\} \cup \{Y \in \mathcal{P}(S) \mid [R]Y \in u\}.$$

(b1) Show that E has the finite intersection property.

(b2) Let $v \in \mathsf{Uf}(S)$ be such that $E \subseteq v$. Show that R^*uv and $X \in v$.

(c) Suppose that $[\![\Diamond\phi]\!]^{\mathbb{M}} \in u$, and prove that $\mathbb{M}^*, u \Vdash \Diamond\phi$.

Exercise 2. Prove the key lemma.

Exercise 3. Let $\mathbb{M} = (S, R, V)$ be a Kripke model, let $u \in \mathsf{Uf}(S)$ be an ultrafilter and let Σ be a set of modal formulas. Assume that Σ is finitely satisfiable in the set $R^*(u)$ of successors of u, in the model $\mathbb{M}^*$. Define

$$H := \{[\![\phi]\!]^{\mathbb{M}} \mid \phi \in \Sigma\} \cup \{Y \in \mathcal{P}(S) \mid [R]Y \in u\}.$$

(a) Show that H has the finite intersection property.

(b) Let $v \in \mathsf{Uf}(S)$ be such that $H \subseteq v$. Show that R^*uv and that $\mathbb{M}^*, v \Vdash \phi$, for all $\phi \in \Sigma$.

(c) Show that $\mathbb{M}^*$ is m-saturated.

Exercise 4. The *ultrafilter extension* of a Kripke frame $\mathbb{F} = (S, R)$ is defined as the structure $\mathbb{F}^* := (\mathsf{Uf}(S), R^*)$.

(a) Show that if $\mathbb{F}^* \Vdash \phi$ then $\mathbb{F} \Vdash \phi$.

(b) Show that the frame property $\forall x \exists y (xRy \;\&\; yRy)$ is preserved under taking disjoint unions, generated subframes and p-morphic images[1], but is nevertheless not modally definable.

[1] That is images under p-morphisms also known as bounded morphisms.

(c)* Give a counterexample showing that the converse implication of (a) does not hold. (Hint: you need to find a formula which expresses a property which is not first-order definable.)

Exercise 5. Recall that a *partial order* on a set W is a binary relation $R \subseteq W^2$ which is reflexive, transitive and anti-symmetric.

(1) Are partial orders modally definable?

(2) Are partial orders modally definable within the class of finite Kripke frames?

Exercise 6. Compute the first order correspondents of the following closed formulas.

(1) $\Box\bot$;

(2) $\Box\Diamond\top \to \Diamond\Box\top$;

(3) $\Diamond\Box\bot \to \Box\Diamond\bot$;

(4) $\Box\Diamond\top \to \Box\bot$;

(5) $\Diamond_P(\Diamond_F\Box_P\bot \to \Box_F\bot)$;

(6) $[*](\Diamond\top \to \langle * \rangle\Box\bot)$;