Introduction to Modal Logic. Exercise class 5

2 October 2017

Definition 1. Let S be a non-empty set. A *filter* over S is a set $F \subseteq \mathcal{P}(S)$ such that (1) $S \in F$, (2) if $X \in F$ and $X \subseteq Y \subseteq S$ then $Y \in F$, and (3) if $X, Y \in F$ then $X \cap Y \in F$.

A filter F is *proper* if $F \neq \mathcal{P}(S)$, or equivalently, $\varnothing \notin F$. An *ultrafilter* is a proper filter u such that for all $X \in \mathcal{P}(S)$: $X \in u$ iff $S \setminus X \notin u$. The collection of ultrafilters over a set S is denoted as $\mathsf{Uf}(S)$.

Given an element $s \in S$, we define the *principal ultrafilter* $\pi_s := \{X \in S \mid s \in X\}$.

Exercise 1. Let S be a non-empty set.

(a) Verify that $\mathcal{P}(S)$ and $\{S\}$ are filters.

(b) Given a subset $X \subseteq S$, verify that $\uparrow X := \{Y \in \mathcal{P}(S) \mid X \subseteq Y\}$ is a filter.

(c) Verify that $\{X \in \mathcal{P}(S) \mid S \setminus X \text{ is finite }\}$ is a filter, if S is infinite.

(d) Given an element $s \in S$, verify that π_s is indeed an ultrafilter.

Exercise 2. Let S be some non-empty set, and let u be an ultrafilter over S.

(a) Show that for every pair $X, Y \in \mathcal{P}(S)$: $X \cup Y \in u$ iff $X \in u$ or $Y \in u$.

(b) Show that u is principal if S is finite.

Definition 2. Let S be some non-empty set. A collection $E \subseteq \mathcal{P}(S)$ has the *finite intersection property* if the intersection $\bigcap E'$ of any finite subcollection $E' \subseteq E$ is nonempty.

Theorem 1 (Ultrafilter Theorem)**.** *Let S be some non-empty set. If the collection $E \subseteq \mathcal{P}(S)$ has the finite intersection property then there is an ultrafilter $u \in \mathsf{Uf}(S)$ such that $E \subseteq u$.*

Exercise 3. Show that every infinite set has a non-principal ultrafilter.

Definition 3. Given a binary relation $R \subseteq S \times S$ we define the following operations[1] $\langle R\rangle, [R]$ on the power set of S:

$$\begin{array}{lcl} \langle R\rangle(X) & := & \{s \in S \mid Rsx \text{ for some } x \in X\}, \\ [R](X) & := & \{s \in S \mid Rsx \text{ implies } x \in X, \text{ for all } x \in S\}. \end{array}$$

Clearly the operations $\langle R\rangle$ and $[R]$ encode the semantics of the $\Diamond$ and $\Box$ modality (see for instance Exercise 5).

Definition 4. Given a Kripke model $\mathbb{M} = (S, R, V)$, we define its *ultrafilter extension*[2] as the Kripke model $\mathbb{M}^* = (S^*, R^*, V^*)$, where

$$\begin{array}{lcl} W^* & := & \mathsf{Uf}(S) \\ R^* & := & \{(u, v) \in \mathsf{Uf}(S) \times \mathsf{Uf}(S) \mid \langle R\rangle X \in u \text{ for all } X \in v\} \\ V^*(p) & := & \{u \in \mathsf{Uf}(S) \mid V(p) \in u\} \end{array}$$

[1] In [BdRV] these operations are denoted as m_R and l_R, respectively.
[2] In [BdRV] the ultrafilter extension of a model $\mathfrak{M}$ is denoted as $\mathfrak{ue}\mathfrak{M}$.

Exercise 4. Given a Kripke model $\mathbb{M} = (S, R, V)$, show that R^*uv iff for all $X \in \mathcal{P}(S)$: $[R]X \in u$ implies $X \in v$.

Definition 5. Given a Kripke model $\mathbb{M} = (S, R, V)$, we define the *extension*[3] of a formula ϕ as the set

$$[\![\phi]\!]^{\mathbb{M}} := \{s \in S \mid \mathbb{M}, s \Vdash \phi\}.$$

We may write $[\![\phi]\!]$ instead of $[\![\phi]\!]^{\mathbb{M}}$ if $\mathbb{M}$ is clear from context.

Lemma 1 (Key Lemma)**.** *Let $\mathbb{M} = (S, R, V)$ be a Kripke model. Then for every modal formula ϕ, and for every ultrafilter $u \in \mathsf{Uf}(S)$ we have*

$$\mathbb{M}^*, u \Vdash \phi \textit{ iff } [\![\phi]\!]^{\mathbb{M}} \in u. \tag{1}$$

Exercise 5. The key lemma is proved by induction on ϕ. In this exercise we focus on the hard part of the inductive case for the formula $\Diamond\phi$. That is, assume as inductive hypothesis that (1) holds for the formula ϕ.

(a) Show that $[\![\Diamond\phi]\!]^{\mathbb{M}} = \langle R\rangle[\![\phi]\!]^{\mathbb{M}}$.

(b) Suppose that $\langle R\rangle X \in u$, for some set $X \in \mathcal{P}(S)$. Consider the set

$$E := \{X\} \cup \{Y \in \mathcal{P}(S) \mid [R]Y \in u\}.$$

- (b1) Show that E has the finite intersection property.
- (b2) Let $v \in \mathsf{Uf}(S)$ be such that $E \subseteq v$. Show that R^*uv and $X \in v$.

(c) Suppose that $[\![\Diamond\phi]\!]^{\mathbb{M}} \in u$, and prove that $\mathbb{M}^*, u \Vdash \Diamond\phi$.

Exercise 6. Prove the key lemma.

Exercise 7. Let $\mathbb{M} = (S, R, V)$ be a Kripke model, let $u \in \mathsf{Uf}(S)$ be an ultrafilter and let Σ be a set of modal formulas. Assume that Σ is finitely satisfiable in the set $R^*(u)$ of successors of u, in the model $\mathbb{M}^*$. Define

$$H := \{[\![\phi]\!]^{\mathbb{M}} \mid \phi \in \Sigma\} \cup \{Y \in \mathcal{P}(S) \mid [R]Y \in u\}.$$

(a) Show that H has the finite intersection property.

(b) Let $v \in \mathsf{Uf}(S)$ be such that $H \subseteq v$. Show that R^*uv and that $\mathbb{M}^*, v \Vdash \phi$, for all $\phi \in \Sigma$.

(c) Show that $\mathbb{M}^*$ is m-saturated.

Exercise 8. The *ultrafilter extension* of a Kripke frame $\mathbb{F} = (S, R)$ is defined as the structure $\mathbb{F}^* := (\mathsf{Uf}(S), R^*)$.

(a) Show that if $\mathbb{F}^* \Vdash \phi$ then $\mathbb{F} \Vdash \phi$.

(b) Show that the frame property $\forall x \exists y(xRy \;\&\; yRy)$ is preserved under taking disjoint unions, generated subframes and p-morphic images[4], but is nevertheless not modally definable.

(c)* Give a counterexample showing that the converse implication of (a) does not hold. (Hint: you need to find a formula which expresses a property which is not first-order definable.)

[3]In [BdRV] the extension of ϕ is denoted as $V(\phi)$.
[4]That is images under p-morphisms also known as bounded morphisms.