Introduction to Modal Logic. Exercise class 3

18 September 2017

Exercise 1. Which of the following frame properties are preserved (reflected) by the operations of forming generated subframes, p-morphic images, disjoint unions?

(1) reflexivity;

(2) transitivity;

(3) irreflexivity;

(4) converse seriality ($\forall x \exists y\, Ryx$);

(5) having cardinality at least n, for some natural number n;

(6) having cardinality at most n, for some natural number n.

Exercise 2. Show that the following frame properties cannot be defined in the basic modal language:

(1) converse seriality;

(2) having cardinality at least n, for some natural number n;

(3) having cardinality at most n, for some natural number n;

(4) acyclicity: 'there is no finite path (of non-zero length) from any point to itself'.

Exercise 3. Prove the Filtration Theorem for the *largest* filtration.

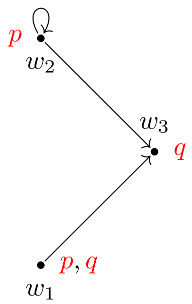
That is, let $\mathbb{M} = (W, R, V)$ be a Kripke model, and let Σ be a set of formulas that is closed under taking subformulas. Let $\mathbb{M}^l = (W^f, R^l, V^f)$ be the largest filtration of $\mathbb{M}$, where W^f, V^f and R^l are defined as in section 2.3 of [BdRV]. Prove that

$$\mathbb{M}, s \Vdash \phi \text{ iff } \mathbb{M}^l, \overline{s} \Vdash \phi,$$

for every $s \in W$ and every $\phi \in \Sigma$.

Exercise 4. Compute both the largest and the smallest filtration of the model below through each of the following sets

$$\Sigma_1 := \{p, q\} \quad \Sigma_2 := \{p, q, \Diamond p\} \quad \Sigma_3 := \{q\}.$$



Exercise 5 (BdRV, Ex. 2.3.5)**.** This exercise is about the so-called transitive (or Lemmon) filtration.

(1) Show that not every filtration of a transitive model is transitive.

(2) Proof Lemma 2. 24 of [BdRV]. That is, show that the relation R^t defined there is indeed a filtration, and that any filtration of a transitive model that makes use of the relation R^t is guranteed to be transitive.

Exercise 6. Which of the following properties of frames are preserved by taking suitable filtrations of Kripke models?

(1) reflexivity

(2) symmetry

(3) seriality

(4) directedness: $\forall x_0 x_1 x_2\,((Rx_0x_1 \wedge Rx_1x_2) \rightarrow \exists y(Rx_1y \wedge Rx_2y))$

(5) density: $\forall x_1 x_2 (x_1 R x_2 \rightarrow \exists y\,(Rx_1y \wedge Ryx_2))$

(6) every world sees a reflexive world: $\forall x \exists y\,(Rxy \wedge Rxy)$

Exercise 7. $(*)$ Is it possible to find a pair of pointed Kripke models $(\mathbb{M}, s)$ and $(\mathbb{M}', s')$ one of which is finitely branching such that $\mathbb{M}, s \leftrightsquigarrow \mathbb{M}', s'$ but $\mathbb{M}, s \not\underline{\leftrightarrow} \mathbb{M}', s'$?