SOME PARTIAL SOLUTIONS TO SOME OF THE EXERCISE FROM THE TUTORIAL FRIDAY 4 NOVEMBER 2016

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This is small note providing some sketches of solutions to some of the exercises from last Fridays tutorial. The usual caveats apply. In particular there are quite a few lacunae which you should fill in yourselves. Consequently, this is by no means a model solution which can be used as a guide for your homework solutions[1].

In case there is something which is not clear—or maybe even wrong—then feel free to come by my office (F2.23) with your questions.

1. Exercise 1

To show that the set

$$\mathrm{Log}(\mathcal{C}) := \{\varphi\colon \forall \mathfrak{F} \in \mathcal{C}\ (\mathfrak{F} \Vdash \varphi)\}$$

is a normal modal logic, for any class of Kripke frames $\mathcal{C}$, one has simply to verify that

(i) Every propositional tautology belongs to $\mathrm{Log}(\mathcal{C})$;
(ii) The (K) axiom $\Box(p \to q) \to \Box p \to \Box q$ belongs to $\mathrm{Log}(\mathcal{C})$;
(iii) The (Dual) axiom $\Diamond p \leftrightarrow \neg\Box\neg p$ belongs to $\mathrm{Log}(\mathcal{C})$;
(iv) The set $\mathrm{Log}(\mathcal{C})$ is closed under applications of the Modus Ponens rule;
(v) The set $\mathrm{Log}(\mathcal{C})$ is closed under applications of the Necessitation rule;
(vi) The set $\mathrm{Log}(\mathcal{C})$ is closed under uniform substitution.

Verifying items (i)–(v) is straightforward; you just check that the relevant formulas (rules) are valid (admissible) on all frames—and so in particular on any frame in $\mathcal{C}$. For (i) note that the *colour* $c(w) := \{p \in \mathsf{Prop}\colon \mathfrak{M}, w \Vdash p\}$ is a model of classical propositional logic for any model $\mathfrak{M}$ and any $w \in |\mathfrak{M}|$.

Finally, for item (vi) let $\varphi \in \mathrm{Log}(\mathcal{C})$ be given and let σ be a substitution. We must show that $\varphi[\sigma] \in \mathrm{Log}(\mathcal{C})$ as well. Therefore, let $\mathfrak{F}$ be a Kripke frame from $\mathcal{C}$ and let V be a valuation on $\mathfrak{F}$. To see that $\mathfrak{F}, V, w \Vdash \varphi[\sigma]$ for any world $w \in |\mathfrak{F}|$ define a valuation V^σ on $\mathfrak{F}$ by

$$V^\sigma(p) = \{w' \in |\mathfrak{F}|\colon \mathfrak{F}, V, w' \Vdash \sigma(p)\},$$

and prove by induction of the complexity of the formula φ that

$$\mathfrak{F}, V, w \Vdash \varphi[\sigma] \quad \text{iff} \quad \mathfrak{F}, V^\sigma, w \Vdash \varphi.$$

Since $\varphi \in \mathrm{Log}(\mathcal{C})$ and $\mathfrak{F} \in \mathcal{C}$ the right-hand side obtains and so, as $\mathfrak{F} \in \mathcal{C}$, the valuation V on $\mathfrak{F}$ and $w \in |\mathfrak{F}|$ were arbitrary, we may conclude that $\varphi[\sigma] \in \mathrm{Log}(\mathcal{C})$.

To conclude that the normal modal logic $\mathbf{K}$ is sound with respect to the class of all Kripke frames $\mathcal{K}$ we simply have to argue that

$$\mathbf{K} \subseteq \mathrm{Log}(\mathcal{K}).$$

But since $\mathbf{K}$ is the least normal modal logic and we have just shown that for any class of Kripke frames $\mathcal{C}$ the set $\mathrm{Log}(\mathcal{C})$ is a normal modal logic, in particular $\mathrm{Log}(\mathcal{K})$ is a normal modal logic, and so the above inclusion follows immediately from the minimality of $\mathbf{K}$.

To see that $\mathrm{Th}(\mathcal{C}_{mod}) := \{\varphi\colon \forall \mathfrak{M} \in \mathcal{C}_{mod}\ (\mathfrak{M} \Vdash \varphi)\}$ is not necessarily a normal modal logic, simply convince yourself that this set will not necessarily be closed under uniform substitution. In fact you may take $\mathcal{C}_{mod}$ to consist of a single model to obtain a counter example.

[1] Unless, of course, the presentation of your homework solutions are usually more sloppy then the presentation at hand.

2. Exercise 2

We only provide a solution to item 2 as the the rest of the items are similar.

Let $\Sigma \cup \{\varphi, \psi\}$ be a set of formulas in the language of basic modal logic and suppose that we have $\vdash_\Sigma \varphi$ and $\vdash_\Sigma \psi$ then we must show that $\vdash_\Sigma \varphi \wedge \psi$.

1.	$\vdash_\Sigma p \to (q \to (p \wedge q))$	Prop. tautology;
2.	$\vdash_\Sigma \varphi \to (\psi \to (\varphi \wedge \psi))$	Uniform substitution w. 1;
3.	$\vdash_\Sigma \varphi$	Given by assumption;
4.	$\vdash_\Sigma \psi \to (\varphi \wedge \psi)$	(MP) w. 1 and 3;
5.	$\vdash_\Sigma \psi$	Given by assumption;
4.	$\vdash_\Sigma \varphi \wedge \psi$	(MP) w. 5 and 4;

Thus the rule

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}$$

is *admissible* for any normal modal logic.

3. Exercise 3

Again we provide a solution for item 2 only as the rest are similar. Given a set $\Sigma \cup \{\varphi, \psi\}$ of formulas in the language of basic modal logic such that $\vdash_\Sigma \varphi \to \psi$ we must show that also $\vdash_\Sigma \Diamond\varphi \to \Diamond\psi$.

1.	$\vdash_\Sigma \varphi \to \psi$	Given by assumption;
2.	$\vdash_\Sigma (p \to q) \to (\neg q \to \neg p))$	Prop. tautology;
3.	$\vdash_\Sigma (\varphi \to \psi) \to (\neg\psi \to \neg\varphi))$	Uni. subst. w. 3;
4.	$\vdash_\Sigma (\neg\psi \to \neg\varphi)$	(MP) w. 1 and 3;
5.	$\vdash_\Sigma \Box(\neg\psi \to \neg\varphi)$	(N) to 4;
6.	$\vdash_\Sigma \Box(\neg\psi \to \neg\varphi)) \to (\Box\neg\psi \to \Box\neg\varphi)$	Uni. subst. of the (K)-axiom;
7.	$\vdash_\Sigma \Box\neg\psi \to \Box\neg\varphi$	(MP) w. 6 and 5;
8.	$\vdash_\Sigma (\Box\neg\psi \to \Box\neg\varphi)) \to (\neg\Box\neg\varphi \to \neg\Box\neg\psi))$	Uni. subst. on 2;
9.	$\vdash_\Sigma (\neg\Box\neg\varphi \to \neg\Box\neg\psi)$	(MP) w. 8 and 7;
10.	$\vdash_\Sigma \Diamond\varphi \to \neg\Box\neg\varphi$	Uni. subst. of (Dual)-axiom and prop. reasoning;
11.	$\vdash_\Sigma (\Diamond\varphi \to \neg\Box\neg\varphi) \to ((\neg\Box\neg\varphi \to \neg\Box\neg\psi) \to (\Diamond\varphi \to \neg\Box\neg\psi))$	Uni. subst. instance of a propositional tautology;
12.	$\vdash_\Sigma \Diamond\varphi \to \neg\Box\neg\psi$	(MP) twice: w. 11 and 10 and then w. 7;
13.	$\vdash_\Sigma \neg\Box\neg\psi \to \Diamond\psi$	Uni. subst. on (Dual)-axiom and prop. reasoning;
14.	$\vdash_\Sigma (\Diamond\varphi \to \neg\Box\neg\psi) \to ((\neg\Box\neg\psi \to \Diamond\psi) \to (\Diamond\varphi \to \Diamond\psi)$	Uni. subst. instance of a prop. taut.;
15.	$\vdash_\Sigma \Diamond\varphi \to \Diamond\psi$	(MP) twice: w. 14 and 12 and then w. 13;

In the above we used—among other things—that $(p \to q) \to ((q \to r) \to (p \to r))$ is a propositional tautology.

Please note that the above is technically not a derivation in the Hilbert style derivation system $\vdash_\Sigma$. The above does, however, provided a scaffolding from which a correct derivation may be constructed.

4. Exercise 4

The trick is here to show that all the connectives of the language of basic modal logic are *congruent* in the sense that the rules

$$\frac{\varphi \leftrightarrow \psi \qquad \varphi' \leftrightarrow \psi'}{\varphi * \varphi' \leftrightarrow \psi * \psi'}$$

for $* \in \{\wedge, \vee, \to\}$ and

$$\frac{\varphi \leftrightarrow \psi}{\bullet\varphi \leftrightarrow \bullet\psi}$$

for $\bullet \in \{\neg, \Box\}$, are all admissible for every normal modal logic[2]. Having established this, showing the Equivalent Replacement Lemma, i.e, that

$$\vdash_\Sigma \psi \leftrightarrow \chi \quad \text{implies} \quad \vdash_\Sigma \varphi[\psi] \leftrightarrow \varphi[\chi],$$

for any set $\Sigma \cup \{\varphi, \psi, \chi\}$ of formulas in the language of basic modal logic, is a straightforward induction on the complexity of φ.

5. Exercise 5

To show that $\nvdash_{\mathbf{S4}} p \to \Box\Diamond p$ it suffices to show that the normal modal logic **S4** is sound with respect to some class of Kripke frames on which the formulas $p \to \Box\Diamond p$ is not valid. Recall that **S4** is $\mathbf{K} + (\Box p \to p) + (\Box p \to \Box\Box p)$. Now prove that these two axioms are valid on all frames where the relation is both transitive and reflexive, i.e., a *pre-order*. From this you may conclude that the normal modal logic **S4** is sound with respect to the class of all pre-ordered Kripke frames. Finally, find a pre-ordered Kripke frames which does not validate the formula $p \to \Box\Diamond p$ and conclude that $\nvdash_{\mathbf{S4}} p \to \Box\Diamond p$.

To show that $\nvdash_{\mathbf{K}} \Box p \vee \Box\neg p$ you may follow a similar strategy—only we already know a class of Kripke frames with respect to which the normal modal logic **K** is sound.

6. Exercise 6

The normal modal logic **K** is not Halldén complete. To see this note that since $\Box\bot \vee \neg\Box\bot$ is a substitution instance of the propositional tautology $p \vee \neg p$ we have that $\vdash_{\mathbf{K}} \Box\bot \vee \neg\Box\bot$. However, you can easily find Kripke frames $\mathfrak{F}_1$ and $\mathfrak{F}_2$ such that $\mathfrak{F}_1 \nVdash \Box\bot$ and $\mathfrak{F}_2 \nVdash \neg\Box\bot$, showing that $\nvdash_{\mathbf{K}} \Box\bot$ and $\nvdash_{\mathbf{K}} \neg\Box\bot$ since **K** is sound with respect to the class of all Kripke frames.

7. Exercise 7

This is rather tricky but also quite a bit of fun. One way to proceed is to observe that $p \to ((q \wedge r) \to (r \wedge p))$ is a propositional tautology. You then have to come up with a clever substitution instance of this propositional tautology in which—after a bit of manipulation—you can recognise a substitution instance of the Löb axiom $\Box(\Box p \to p) \to \Box p$. From there onwards it is all fairly straightforward.

[2]In fact, since some of the connectives are inter-definable in terms of each other you can be a bit clever about how you chose the basic connectives.