

# Modal and temporal logic

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## Part II

- ① Soundness and completeness. Canonical models. 3 lectures.
- ② Finite model property. Filtrations. 2 lectures.
- ③ Decidability. 2 lecture.
- ④ Modal  $\mu$ -calculus. 2 lectures.

## Syntactic approach

Let us fix a class  $\mathcal{C}$  of frames. We need to have, whenever possible, an effective criterion (algorithm) deducing whether a formula  $A$  is valid in  $\mathcal{C}$ .

If  $\mathcal{C}$  is infinite then going through all the frames might take infinite time. Even if  $\mathcal{C}$  is finite, but contains an infinite frame, the procedure might still take infinite time.

In order to overcome this difficulty we will develop a syntactic (axiomatic) approach to modal logic.

The idea of this approach is to find a small (possibly finite) number of formulas (axioms of our logic) and set some rules of inference which enable us to derive other formulas (theorems of our logic).

## Idea

The axioms are ‘given truths’. They should be valid (perhaps over some given class of frames).

The rules allow us to derive new truths from old. If the formulas above the line in a rule are already derived, the rule allows us to derive the formula underneath the line.

Rules should be chosen so as not to lead us from truth into falsehood. Formulas derived from valid formulas should also be valid. Here, we mean ‘valid in Kripke semantics’.

Axiomatic approach is simple, powerful, flexible, and in common use.

## Logics of classes of frames

In Part I we defined when a formula  $A$  is valid in a class  $\mathcal{C}$  of frames.

A formula  $A$  is *valid in a class  $\mathcal{C}$  of frames* if it is valid in every frame in  $\mathcal{C}$ . That is,  $A$  is true in **every** element of **every** frame in  $\mathcal{C}$  under **every** assignment.

Given a class  $\mathcal{C}$  of frames we can consider the set of all formulas valid in  $\mathcal{C}$ .

This set is called the *logic of  $\mathcal{C}$*  and will be denoted by  $Log(\mathcal{C})$ .

In formal terms

$$Log(\mathcal{C}) = \{A : A \text{ is a formula and } \forall \mathcal{F} \in \mathcal{C}, A \text{ is valid in } \mathcal{F}\}.$$

Let  $\mathcal{C}$  be the class of all frames and let  $\mathcal{C}_{\text{ref}}$  be the class of all reflexive frames. That is, such frames  $\mathcal{F} = (W, R)$  that for each  $x \in W$  we have  $R(x, x)$ .

What can we say about  $\text{Log}(\mathcal{C})$  and  $\text{Log}(\mathcal{C}_{\text{ref}})$ ? Is any of these two sets included in the other?

### Proposition 1

*Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be classes of frames. If  $\mathcal{C}_1 \subseteq \mathcal{C}_2$ , then  $\text{Log}(\mathcal{C}_2) \subseteq \text{Log}(\mathcal{C}_1)$ .*

### Proof.

Let  $A \in \text{Log}(\mathcal{C}_2)$ . Then  $A$  is valid in every frame in  $\mathcal{C}_2$ . But since every frame in  $\mathcal{C}_1$  is in  $\mathcal{C}_2$ , the formula  $A$  is also valid in every frame in  $\mathcal{C}_1$ . This means that  $A \in \text{Log}(\mathcal{C}_1)$ . □

From the above proposition we deduce that  $Log(\mathcal{C}) \subseteq Log(\mathcal{C}_{ref})$ .

To show that the inverse inclusion does not hold we note that the formula  $\Box A \rightarrow A$  is valid in  $\mathcal{C}_{ref}$ , but is *not* valid in  $\mathcal{C}$ .

Therefore, the formula  $\Box A \rightarrow A$  belongs to  $Log(\mathcal{C}_{ref})$ , but does not belong to  $Log(\mathcal{C})$ . Thus, these two logics are different.

## Exercise 2

*Show that the logics of the classes of reflexive, transitive, serial and symmetric frames, respectively, are all different.*

If a class  $\mathcal{C}$  consists of a single frame  $\mathcal{F}$ , we denote the logic of  $\mathcal{C}$  by  $Log(\mathcal{F})$  instead of  $Log(\{\mathcal{F}\})$ .

## Exercise 3

- 1 Show that if a frame  $\mathcal{F}$  is a  $p$ -morphic image of a frame  $\mathcal{G}$ , then  $\text{Log}(\mathcal{G}) \subseteq \text{Log}(\mathcal{F})$ .
- 2 Show that  $\text{Log}(\mathbb{N})$  is contained in the logic of a single reflexive point.
- 3 Recall that a frame  $(W, R)$  is serial if for each  $s \in W$  there exists  $t \in W$  such that  $R(s, t)$ . Show that the logic of any class  $\mathcal{C}$  of serial frames is contained in the logic of a single reflexive point.



Note that two different frame classes can have the same logic. For example, let  $\mathcal{C}$  and  $\mathcal{C}_{\text{irref}}$  be the classes of all frames and all irreflexive frames, respectively.

As we know from Part I (Lemmas 32 and 33) every frame is a  $p$ -morphic image of an irreflexive frame. Therefore, a formula is valid in the class of all frames iff it is valid in the class of all irreflexive frames.

This means that  $\text{Log}(\mathcal{C}) = \text{Log}(\mathcal{C}_{\text{irref}})$ . (This is just a different way of formulating Theorem 34 of Part I.)

## Maximal logics

There exists a characterization of the ‘maximal’ logics.

### Theorem 4

*Let  $\mathcal{C}$  be a non-empty class of frames. Then  $\text{Log}(\mathcal{C})$  is contained in the logic of a single reflexive point or  $\text{Log}(\mathcal{C})$  is contained in the logic of a single irreflexive point.*

### Proof.

Exercise. Use (3) of Exercise 3. □

This theorem is known as the *Makinson theorem*.



David Makinson (LSE)

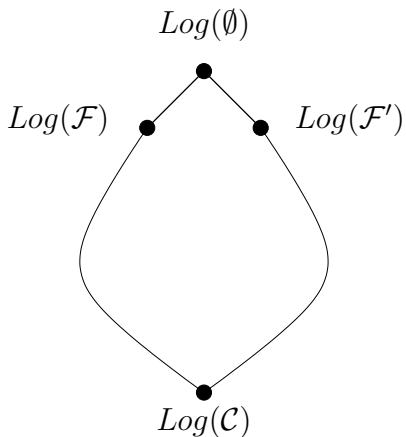


Figure: The lattice of logics

$\mathcal{C}$  is the class of all frames.  $\mathcal{F}$  is one reflexive point,  $\mathcal{F}'$  is one irreflexive point.

We have seen in Part I that for each  $A, B$  the formula  $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$  is valid in every Kripke frame.

## Exercise 5

- 1 Recall the proof of this fact.
- 2 Prove that if formulas  $A$  and  $A \rightarrow B$  are valid in a frame  $\mathcal{F}$ , then the formula  $B$  is also valid in  $\mathcal{F}$ .
- 3 Prove that if a formula  $A$  is valid in a frame  $\mathcal{F}$ , then the formula  $\Box A$  is also valid in  $\mathcal{F}$ .

Now we will give a formal syntactic definition of (normal) modal logics.

By a *propositional tautology* in the modal language we will mean any ‘instance’ of valid propositional formulas (tautologies in the propositional language).

E.g.,  $p \rightarrow (q \rightarrow p)$  is a tautology in the propositional language. Therefore,  $A \rightarrow (B \rightarrow A)$  is a propositional tautology in the modal language (for each modal formulas  $A, B$ ).

E.g.,  $p \vee \neg p$  is a tautology in the propositional language. So,  $A \vee \neg A$  is a propositional tautology in the modal language (for each modal formula  $A$ ).

## Exercise 6

*Are  $\Box p \vee \neg \Box p$  and  $\Box p \vee \Box \neg p$  propositional tautologies in the modal language?*

# Normal modal logics

## Definition 7

A *normal modal logic* (if it is clear from the context we will drop the words ‘normal’ and ‘modal’)  $L$  is a set of formulas that contains all propositional tautologies, the so called **K**-axioms:

$$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$$

and is closed under the rules of

- modus ponens (MP):  $\frac{A, A \rightarrow B}{B}$
- necessitation (N):  $\frac{A}{\Box A}$

This means that  $L$  is a set of formulas such that for each  $A, B$  the formula  $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \in L$  and

(1) if  $A \in L$  and  $A \rightarrow B \in L$ , then  $B \in L$

(2) if  $A \in L$ , then  $\Box A \in L$ .

Let  $L$  be a modal logic. Instead of  $A \in L$  we often write  $\vdash_L A$  and will read ‘ $A$  is a theorem of  $L$ ’.

The collection of all formulas is called the *inconsistent logic*.



## Proposition 8

*For each class  $\mathcal{C}$  of frames,  $Log(\mathcal{C})$  is a normal modal logic.*

## Proof.

Exercise. □

The inconsistent logic is equal to  $Log(\emptyset)$ .

Let  $\mathbf{K}$  be the set of all formulas that we can generate starting from the propositional tautologies and  $\mathbf{K}$ -axioms by applying (MP) and (N). We can show that  $\mathbf{K}$  be the smallest normal modal logic (exercise).

This means that  $A \in \mathbf{K}$  ( $\vdash_{\mathbf{K}} A$ ) iff if there are formulas  $A_1, \dots, A_n$  such that

- $A_n = A$
- each  $A_i$  ( $1 \leq i \leq n$ ) is either
  - a propositional tautology
  - a  $\mathbf{K}$ -axiom
  - or is obtained from some of  $A_1, \dots, A_{i-1}$  by (MP) or (N).

## Example

Let us prove  $\vdash_{\mathbf{K}} \Box(A \wedge B) \rightarrow \Box A$ .

$\vdash_{\mathbf{K}} A \wedge B \rightarrow A$ , (Propositional tautology)

$\vdash_{\mathbf{K}} \Box(A \wedge B \rightarrow A)$ , (N)

$\vdash_{\mathbf{K}} \Box(A \wedge B \rightarrow A) \rightarrow (\Box(A \wedge B) \rightarrow \Box A)$ , (**K**-axiom)

$\vdash_{\mathbf{K}} \Box(A \wedge B) \rightarrow \Box A$  (MP).

### Exercise 9

- 1 Prove that  $\vdash_{\mathbf{K}} A \rightarrow B$  implies  $\vdash_{\mathbf{K}} \Box A \rightarrow \Box B$ .
- 2 Prove that  $\vdash_{\mathbf{K}} A \rightarrow B$  implies  $\vdash_{\mathbf{K}} \Diamond A \rightarrow \Diamond B$ .
- 3 Prove that  $\vdash_{\mathbf{K}} \Box(A \wedge B) \leftrightarrow \Box A \wedge \Box B$ .
- 4 Prove that  $\vdash_{\mathbf{K}} \Diamond(A \vee B) \leftrightarrow \Diamond A \vee \Diamond B$ .

We will be using the following simple proposition throughout.

### Proposition 10

- 1 If a modal formula  $A \rightarrow B$  is a propositional tautology, then  $\vdash_{\mathbf{K}} A$  implies  $\vdash_{\mathbf{K}} B$ .
- 2  $\vdash_{\mathbf{K}} A$  and  $\vdash_{\mathbf{K}} B$  imply  $\vdash_{\mathbf{K}} A \wedge B$ .
- 3 If  $\vdash_{\mathbf{K}} A \rightarrow B$  and  $\vdash_{\mathbf{K}} B \rightarrow C$ , then  $\vdash_{\mathbf{K}} A \rightarrow C$ .
- 4  $\vdash_{\mathbf{K}} A \rightarrow B$  and  $\vdash_{\mathbf{K}} C \rightarrow D$ , then  $\vdash_{\mathbf{K}} (A \wedge C) \rightarrow (B \wedge D)$ .

### Proof.

We only prove (2) and (3). Proofs for (1) and (4) are similar.

$\vdash_{\mathbf{K}} A \rightarrow (B \rightarrow (A \wedge B))$ , (Propositional tautology)

$\vdash_{\mathbf{K}} A \wedge B$ , from  $\vdash_{\mathbf{K}} A$  and  $\vdash_{\mathbf{K}} B$  by applying (MP) twice.

$\vdash_{\mathbf{K}} [(A \rightarrow B) \wedge (B \rightarrow C)] \rightarrow (A \rightarrow C)$ , (Propositional tautology)

$\vdash_{\mathbf{K}} A \rightarrow C$ , by  $\vdash_{\mathbf{K}} A \rightarrow B$ ,  $\vdash_{\mathbf{K}} B \rightarrow C$ , (2) and (MP).  $\square$

Recall that  $\mathcal{C}_{\text{ref}}$  is the class of all *reflexive frames*. We want to find an axiomatic system that as we will see later characterizes  $\text{Log}(\mathcal{C}_{\text{ref}})$ .

Let **KT** be the set of formulas that we can generate starting from **K**-axioms, the *reflexivity axiom*  $\Box A \rightarrow A$  and applying the rules of (MP) and (N).

This we denote by  $\mathbf{KT} = \mathbf{K} + (\Box A \rightarrow A)$ .

Then **KT** is the smallest normal modal logic is the smallest extension of **K** that contains the reflexivity axiom (exercise).

In the same way we can define the logics

$\mathbf{K4} = \mathbf{K} + (\Box A \rightarrow \Box \Box A)$ ,  $\mathbf{S4} = \mathbf{K} + (\Box A \rightarrow A, \Box A \rightarrow \Box \Box A)$   
(‘**KT4**’) If you also add  $A \rightarrow \Box \Diamond A$ , you get **S5**.

See Chagrov and Zakharyashev p. 116 or Blackburn et al. p. 193 for a longer list.

## Definition 11

We say that a logic  $L$  is *sound with respect to a frame class  $\mathcal{C}$*  if  $L \subseteq \text{Log}(\mathcal{C})$ . That is, if every formula in  $L$  is valid in every frame in  $\mathcal{C}$ .

We say that a logic  $L$  is *complete with respect to a frame class  $\mathcal{C}$*  if  $\text{Log}(\mathcal{C}) \subseteq L$ . That is, if every formula valid in every frame in  $\mathcal{C}$  belongs to  $L$ .

Next we will discuss the *canonical model construction*. This construction is used for proving completeness of various normal modal logics and is certainly one of the most well-known and well-applied methods of modal logic.

The method of **canonical models** was introduced in 1966 by D. Scott and J. Lemmon (1930-1966) and independently by Makinson (1966), Creswell (1967) and Schütte (1968).



Dana Scott

# Soundness and completeness for $\mathbf{K}$

We are going to show that

## Theorem 12

*A formula is a theorem of  $\mathbf{K}$  iff it is valid in every Kripke frame. That is,  $\text{Log}(\mathcal{C}) = \mathbf{K}$ , where  $\mathcal{C}$  is the class of all frames.*

Later, we will prove the same result for other normal modal logics.

‘ $\Rightarrow$ ’ is called soundness (proofs are sound: they do not prove false things).

‘ $\Leftarrow$ ’ is called completeness (the proof system can prove *all* valid formulas).

So we get a syntactic ‘handle’ on truth.



## Soundness of $\mathbf{K}$ (easy)

### Proposition 13

*Any theorem of  $\mathbf{K}$  is valid in every Kripke frame.*

### Proof.

Follows from Exercise 5



Completeness is harder...

# Consistency

Let  $\Gamma$  (Gamma) be a **set** of formulas (maybe infinite!). The following definition is good for any normal modal logic  $L$ .

## Definition 14

If  $A$  is a formula,  $\Gamma \vdash_L A$  means:  $\vdash_L (B_1 \wedge \dots \wedge B_n) \rightarrow A$  for some formulas  $B_1, \dots, B_n \in \Gamma$ .

Idea:  $\Gamma \vdash_L A$  says that  $A$  follows (is provable) from assumptions in  $\Gamma$ . These assumptions need not be valid.

**Example:**  $\{p, p \rightarrow q\} \vdash_{\mathbf{K}} q$ ,

because  $(p \wedge (p \rightarrow q)) \rightarrow q$  is a propositional tautology, so that  $\vdash_{\mathbf{K}} (p \wedge (p \rightarrow q)) \rightarrow q$ .

## Definition 15

$\Gamma$  is said to be *L-consistent* if  $\Gamma \not\vdash_L \perp$ .

For now, ‘consistent’ will mean **K**-consistent.

Consistency is one of our most vital notions.

# Satisfiable sets are consistent

## Definition 16

For a world  $t$  of a model  $\mathcal{M}$ , we write  $\mathcal{M}, t \models \Gamma$  if  $\mathcal{M}, t \models B$  for all  $B \in \Gamma$ .

$\Gamma$  is said to be *satisfiable* if  $\mathcal{M}, t \models \Gamma$  for some model  $\mathcal{M}$  and some world  $t$  of  $\mathcal{M}$ .

This just generalises ‘satisfiability’ from formulas to (possibly infinite) sets of formulas.

## Exercise 17

*Show that if a set  $\Gamma$  of formulas is satisfiable then it is consistent.*

## Proving completeness

To show completeness it is (more than) enough to show the converse of Exercise 17.

### Theorem 18 (completeness for $\mathbf{K}$ )

*Any consistent set is satisfiable.*

For, if  $A$  is valid, the set  $\{\neg A\}$  isn't satisfiable. By the theorem, it must be inconsistent:  $\vdash_{\mathbf{K}} (\neg A \rightarrow \perp)$ . So  $\vdash_{\mathbf{K}} \neg\neg A$ .

But  $\neg\neg A \rightarrow A$  is a propositional tautology, so is a  $\mathbf{K}$ -axiom. So by (MP) (a rule of  $\mathbf{K}$ ), we obtain that  $\vdash_{\mathbf{K}} A$ , and  $A$  is a theorem of  $\mathbf{K}$ , as required.

So let  $\Gamma_0$  be a ( $\mathbf{K}$ -)consistent set of formulas. We have to build a model of  $\Gamma_0$ .

What will the model be made of?

**Answer:** syntactic objects, namely ‘maximal consistent’ sets of formulas.

## The idea

Let  $\mathcal{M} = (W, R, h)$  be any model, and  $t \in W$  a world of it.

As we said (Exercise 17), the set (say  $\Gamma_t$ ) of all formulas true at  $t$  in  $\mathcal{M}$  is consistent.

And it contains  $A$  or  $\neg A$ , for any formula  $A$ . So (exercise) no larger set is consistent. It is *maximal consistent*.

We try to represent  $t$  by this set  $\Gamma_t$ . (It doesn't matter what  $t$  actually is, after all.)

Replace each  $t$  in  $\mathcal{M}$  by its  $\Gamma_t$ . We get a new set of worlds:  
 $W' = \{\Gamma_t : t \in W\}$ .

$W'$  is made of (maximal consistent) sets of formulas! It is syntactic.

## Recovering $h$

Suppose we forget which  $t$  each  $\Gamma_t$  came from, and just take  $W'$  as a set of sets of formulas. Can we recover  $h$  and  $R$ ?

First,  $h$ . Let  $p$  be an atom. We know, for any  $t \in W$ :

$$t \in h(p) \iff \mathcal{M}, t \models p \iff p \in \Gamma_t.$$

So we can let  $h'(p) = \{\Gamma \in W' : p \in \Gamma\}$ . Then  $p$  is true at  $\Gamma$  iff it's in  $\Gamma$ !

This is the  $h'$  we wanted, as  $t \in h(p)$  iff  $\Gamma_t \in h'(p)$  (check it!)



## Recovering $R$

Now for  $R$ . In  $\mathcal{M}$ , we know if  $R(t, u)$  and  $\mathcal{M}, t \models \Box A$  then  $\mathcal{M}, u \models A$ .

That is, if  $R(t, u)$  and  $\Box A \in \Gamma_t$  then  $A \in \Gamma_u$ .

So why not try: for  $\Gamma, \Delta \in W'$ , let  $R'(\Gamma, \Delta)$  iff for any  $A$ , if  $\Box A \in \Gamma$  then  $A \in \Delta$  ('delta').

Here, we only get  $R(t, u) \Rightarrow R'(\Gamma_t, \Gamma_u)$ , so this is perhaps 'bigger' than the  $R'$  we really wanted.

Anyway, we obtain a model  $\mathcal{M}' = (W', R', h')$ .

$\mathcal{M}'$  is well-behaved

We may not expect  $\Box A \in \Gamma_t \Rightarrow \mathcal{M}', \Gamma_t \models \Box A$  —  $R'$  may be too big.

But we get it anyway! This is because  $R'$  is just small enough to give:

### Exercise 19

*For any formula  $A$  and  $\Gamma_t \in W'$ , show  $\mathcal{M}', \Gamma_t \models A$  iff  $A \in \Gamma_t$ .*

(Hint.) The proof goes via an easy induction, using that for all  $\Gamma_t, \Delta_u \in W'$  and all formulas  $A, B$ :

- $\top \in \Gamma_t$
- $A \in \Gamma_t$  and  $B \in \Gamma_t$  iff  $A \wedge B \in \Gamma_t$
- $\neg A \in \Gamma_t$  iff  $A \notin \Gamma_t$
- if  $\Box A \in \Gamma_t$  and  $R'(\Gamma_t, \Delta_u)$  then  $A \in \Delta_u$
- if  $\Box A \notin \Gamma_t$  then there is  $\Delta_u$  with  $R'(\Gamma_t, \Delta_u)$  and  $A \notin \Delta_u$ .

These properties are easily seen to be true (work out the details).

## The real story

This is all very well, but we needed a model  $\mathcal{M}$  to start with. We're given only a consistent set  $\Gamma_0$ , and we have to make a model of it.

But this  $\mathcal{M}'$  gives us a clue: why not take *all maximal consistent sets* as the worlds!

Set up  $h', R'$  as above.

Establish the 5 properties just listed (dry but necessary). Then we can prove  $\mathcal{M}', \Gamma \models A$  iff  $A \in \Gamma$ .

Next, show  $\Gamma_0$  is *contained* in some maximal consistent set  $\Gamma$ . Then every formula of  $\Gamma_0$  will be true at  $\Gamma$ ! We will have our model.

# Maximal consistent sets

## Definition 20

A set  $\Gamma$  of formulas is said to be *maximal consistent* (a ‘MCS’) if it is consistent but no larger set is consistent (always relative to  $\mathbf{K}$  here).

## Proposition 21 (Lindenbaum’s lemma)

*Any consistent set  $\Gamma_0$  extends to a MCS.*

## Proof.

Enumerate all  $L$ -formulas, as  $A_0, A_1, \dots$ . We define a chain  $\Gamma_0 \subseteq \Gamma_1 \subseteq \dots$  of consistent sets.  $\Gamma_0$  is given, and is consistent. Inductively, define

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{A_n\} & \text{if this is consistent} \\ \Gamma_n & \text{otherwise.} \end{cases}$$

Then each  $\Gamma_n$  is consistent. Their union,  $\Gamma$ , is therefore consistent too (for a proof of  $\perp$  from  $\Gamma$  would be a proof from some  $\Gamma_n$ ).

$\Gamma$  is **maximal** consistent. For, let  $A$  be a formula with  $A \notin \Gamma$ . We know  $A = A_n$  for some  $n$ . By construction,  $\Gamma_n \cup \{A\}$  is inconsistent (else  $\Gamma_{n+1} = \Gamma_n \cup \{A\} \subseteq \Gamma$ ). So  $\Gamma \cup \{A\}$  is inconsistent. This means that no set bigger than  $\Gamma$  is consistent. And  $\Gamma$  contains  $\Gamma_0$ . □

# Properties of MCSs

Next we discuss some important properties of MCSs.

## Lemma 22

- 1 If  $\Gamma$  is consistent and  $A$  is a formula, then at least one of  $\Gamma \cup \{A\}$ ,  $\Gamma \cup \{\neg A\}$  is also consistent.
- 2 If  $\Gamma$  is MCS, then  $A \in \Gamma$  or  $\neg A \in \Gamma$ .

## Proof.

(1) Otherwise, there are  $B_1, \dots, B_n, C_1, \dots, C_m \in \Gamma$  with  $\vdash_{\mathbf{K}} (B_1 \wedge \dots \wedge B_n \wedge A) \rightarrow \perp$  and  $\vdash_{\mathbf{K}} (C_1 \wedge \dots \wedge C_m \wedge \neg A) \rightarrow \perp$ .

So  $\vdash_{\mathbf{K}} (B_1 \wedge \dots \wedge B_n) \rightarrow \neg A$  and  $\vdash_{\mathbf{K}} (C_1 \wedge \dots \wedge C_m) \rightarrow A$  (instance of propositional tautology).

Thus,  $\vdash_{\mathbf{K}} (B_1 \wedge \dots \wedge B_n \wedge C_1 \wedge \dots \wedge C_m) \rightarrow (A \wedge \neg A)$  (instance of propositional tautology), and therefore,

$\vdash_{\mathbf{K}} (B_1 \wedge \dots \wedge B_n \wedge C_1 \wedge \dots \wedge C_m) \rightarrow \perp$ .

So  $\Gamma$  is inconsistent: contradiction.

(2) If  $A, \neg A \in \Gamma$  then as  $\vdash_{\mathbf{K}} (A \wedge \neg A) \rightarrow \perp$ ,  $\Gamma \vdash_{\mathbf{K}} \perp$ , so  $\Gamma$  is inconsistent. So at most one of them is in  $\Gamma$ .

By (1),  $\Gamma \cup \{A\}$  or  $\Gamma \cup \{\neg A\}$  is consistent. By maximality,  $\Gamma = \Gamma \cup \{A\}$  or  $\Gamma = \Gamma \cup \{\neg A\}$ . So  $A \in \Gamma$  or  $\neg A \in \Gamma$ . □



# Properties of MCSs

## Lemma 23

*Let  $\Gamma$  be a MCS. Then for all formulas  $A, B$  the following holds:*

- 1  $\Gamma \vdash_{\mathbf{K}} A$  iff  $A \in \Gamma$ .
- 2  $A, B \in \Gamma$  iff  $A \wedge B \in \Gamma$

## Proof.

(1) If  $A \in \Gamma$ , then as  $\vdash_{\mathbf{K}} (A \rightarrow A)$ . So we have  $\Gamma \vdash_{\mathbf{K}} A$  by definition.

Conversely, suppose  $\Gamma \vdash_{\mathbf{K}} A$ . So there exist  $B_1, \dots, B_n \in \Gamma$  such that  $\vdash_{\mathbf{K}} (B_1 \wedge \dots \wedge B_n) \rightarrow A$ . This means that  $\vdash_{\mathbf{K}} (B_1 \wedge \dots \wedge B_n \wedge \neg A) \rightarrow \perp$ .

By Lemma 22(2), we have that  $A \in \Gamma$  or  $\neg A \in \Gamma$ . Suppose  $A \notin \Gamma$ . Then  $\neg A \in \Gamma$ . But this implies that  $\Gamma$  is inconsistent — contradiction. Thus,  $A \in \Gamma$ .

(2) If  $A \wedge B \in \Gamma$ , then as  $\vdash_{\mathbf{K}} (A \wedge B \rightarrow A)$ , we have  $\Gamma \vdash_{\mathbf{K}} A$ . By 1st part,  $A \in \Gamma$ . Similarly,  $B \in \Gamma$ .

If  $A, B \in \Gamma$  then as  $\vdash_{\mathbf{K}} (A \wedge B) \rightarrow (A \wedge B)$ ,  $\Gamma \vdash_{\mathbf{K}} A \wedge B$ , so  $A \wedge B \in \Gamma$ . □

## Canonical model

Now let  $W = \{\text{all MCSs}\}$ .

We write  $\Gamma, \Delta$  for elements of  $W$ .

Define  $R(\Gamma, \Delta)$  iff for any formula  $A$ ,  
if  $\Box A \in \Gamma$  then  $A \in \Delta$ .

For any atom  $p \in L$ , define

$$h(p) = \{\Gamma : p \in \Gamma\}.$$

This is just as before.

We now have a model  $\mathcal{M} = (W, R, h)$ . It is called the *canonical model for  $\mathbf{K}$* .

It is very important and we will study it at some length.

## Lemma 24 (truth lemma)

*For each  $\Gamma \in W$  and each formula  $A$ :*

$$\mathcal{M}, \Gamma \models A \text{ iff } A \in \Gamma$$

## Proof.

By induction on  $A$ . For  $\top$ , we have  $\mathcal{M}, \Gamma \models \top$  and (exercise)  $\top \in \Gamma$ . The case for  $A$  an atom is by definition of  $h$ .

Inductively, assume the lemma for  $A, B$ .

For  $\neg A$ , we have  $\mathcal{M}, \Gamma \models \neg A$  iff  $\mathcal{M}, \Gamma \not\models A$ , iff (inductive hypothesis)  $A \notin \Gamma$ , iff (by lemma 22, as  $\Gamma$  is maximal consistent)  $\neg A \in \Gamma$ .

And  $\mathcal{M}, \Gamma \models A \wedge B$  iff  $\mathcal{M}, \Gamma \models A$  and  $\mathcal{M}, \Gamma \models B$ , iff (inductively)  $A \in \Gamma$  and  $B \in \Gamma$ , iff (lemma 23)  $A \wedge B \in \Gamma$ .

Now the big case:  $\Box A$ .

Suppose  $\Box A \in \Gamma$ . By definition of  $R$ , for any  $\Delta \in W$  with  $R(\Gamma, \Delta)$ , we have  $A \in \Delta$ . So (inductively)  $\mathcal{M}, \Delta \models A$ . By definition of ' $\models$ ', we see that  $\mathcal{M}, \Gamma \models \Box A$ .

## continuation.

Conversely, if  $\Box A \notin \Gamma$ , then consider the set

$$\Delta_0 = \{\neg A\} \cup \{B : \Box B \in \Gamma\}.$$

This is consistent.

For otherwise,  $\vdash_{\mathbf{K}} (B_1 \wedge \dots \wedge B_n \wedge \neg A) \rightarrow \perp$ , for some  $\Box B_1, \dots, \Box B_n \in \Gamma$ .

So  $\vdash_{\mathbf{K}} (B_1 \wedge \dots \wedge B_n) \rightarrow A$ .

Crucially, by (N),  $\vdash_{\mathbf{K}} \Box(B_1 \wedge \dots \wedge B_n \rightarrow A)$ . Using the normality axiom repeatedly (work out the details), we then get  $\vdash_{\mathbf{K}} (\Box B_1 \wedge \dots \wedge \Box B_n) \rightarrow \Box A$ . So  $\Gamma \vdash_{\mathbf{K}} \Box A$ . By lemma 23,  $\Box A \in \Gamma$ , contradicting our assumption.

## continuation.

By Lindenbaum's Lemma (proposition 21), there is a MCS  $\Delta \supseteq \Delta_0$ . Then  $\neg A \in \Delta$ . By lemma 23,  $A \notin \Delta$ . By the induction hypothesis,  $\mathcal{M}, \Delta \not\models A$ .

But as  $\Delta_0 \subseteq \Delta$ , we have  $\Box B \in \Gamma \Rightarrow B \in \Delta$ , and so  $R(\Gamma, \Delta)$ . So  $\mathcal{M}, \Gamma \not\models \Box A$ , as required.



## Completeness is proved

For if  $\Gamma_0$  is consistent, extend it to a MCS  $\Gamma$  (proposition 21).

By the truth lemma,  $\mathcal{M}, \Gamma \models A$  for every  $A \in \Gamma_0$ . So  $\Gamma_0$  has a model.



We showed any **K**-theorem is valid (= valid in the class of all frames), and vice versa: soundness and completeness for **K** over the class of all frames. **K** is ‘the’ modal logic of all frames.

We want to extend this: to find sound and complete axiomatisations for other classes of frames, such as

- the class of all **reflexive frames** (satisfying  $\forall x R(x, x)$ ).
- the class of all **transitive frames** (satisfying  $\forall xyz(R(x, y) \wedge R(y, z) \rightarrow R(x, z))$ ).
- the class of all **dense/discrete linear frames** (useful in temporal logic)
- etc.!

# Examples

Recall

- $\mathbf{KT} = \mathbf{K} + (\Box A \rightarrow A)$  (for all  $A$ )
- $\mathbf{K4} = \mathbf{K} + (\Box A \rightarrow \Box\Box A)$  (for all  $A$ )

## Proposition 25

1.  $\mathbf{K}$  is sound over the class of all frames.
2.  $\mathbf{KT}$  is sound over the class of all reflexive frames.
3.  $\mathbf{K4}$  is sound over the class of all transitive frames.

## Proof.

1. We know this.
2. We only need to show additionally that  $\Box A \rightarrow A$  is valid in all reflexive frames, which is easy.
3. Exercise!



## Completeness, canonical model

Let  $L$  be a normal modal logic.

**Question:** Can we generalise the proof of completeness for  $\mathbf{K}$  to  $L$ ?

All our completeness work goes through for  $L$ .

- The notion of (maximal) consistency works for  $L$ .
- The canonical model  $\mathcal{M}$  (of all maximal  $L$ -consistent sets) exists, and the truth lemma holds for it.
- And any  $L$ -consistent set extends to a MCS.

So given any  $L$ -consistent formula, we can build a model of it — the canonical  $L$ -model,  $\mathcal{M}$ .

But is  $\mathcal{M}$ 's frame a frame where all  $L$ -theorems are valid? If it is, we have completeness.

Eg: the class  $\mathcal{C}_{\text{ref}}$  of reflexive frames

We know that all formulas of the form  $\Box A \rightarrow A$  are valid in reflexive frames.

So  $\mathbf{KT} = \mathbf{K} + (\Box A \rightarrow A)$  is sound over  $\mathcal{C}_{\text{ref}}$ .

Is it complete for  $\mathcal{C}_{\text{ref}}$ ?

### Proposition 26

*The canonical model  $\mathcal{M}$  for  $\mathbf{KT}$  is based on a reflexive frame.  
So  $\mathbf{KT}$  is sound and complete over  $\mathcal{C}_{\text{ref}}$ .*

## Proof.

If  $\Gamma$  is a world of  $\mathcal{M}$ , we want  $R(\Gamma, \Gamma)$ . By definition of  $R$ , that means  $\Box A \in \Gamma$  must force  $A \in \Gamma$  (for any  $A$ ).

But  $\Gamma$  is a MCS. So for any  $A$ ,  $\Gamma \vdash_{\mathbf{KT}} A$  iff  $A \in \Gamma$  (lemma 23 for **KT**).

Now if  $\Gamma \vdash_{\mathbf{KT}} \Box A$ , then  $\Gamma \vdash_{\mathbf{KT}} (\Box A \rightarrow A)$  (axiom of **KT**), so  $\Gamma \vdash_{\mathbf{KT}} A$  and, thus,  $A \in \Gamma$ .

So  $R$  is indeed reflexive! □

## Exercise 27

*Prove using the canonical model construction that the logics **K4**, **S4** and **S5** are sound and complete (with respect to which frame classes?).*

## Sahlqvist's theorem (1975)

Probably the best generalisation of proposition 26 is by H. Sahlqvist.

Recall from Part I the definition of a Sahlqvist formula (Part I, page 38)

# Sahlqvist's theorem

## Theorem 28

*Let  $A$  be a Sahlqvist formula. Let  $L_A$  be the smallest modal logic that contains  $\mathbf{K}$ -axioms, axiom  $A$  and is closed under (MP) and (N). Then  $A$  is valid in the frame of the canonical model for  $L$ . (We say ' $A$  is canonical'. See Blackburn et al. page 204.)*

This together with the Sahlqvist correspondence theorem (Part I, page 41), give us that if  $A$  is a Sahlqvist formula, then  $L_A$  is sound and complete for the class  $\mathcal{C}_L$  of the frames with the corresponding first-order property! And, moreover, by the Sahlqvist algorithm this first-order property can be extracted effectively.

## Finite model property

Suppose we proved that a normal modal logic  $L$  is complete with respect to a class  $\mathcal{C}$  of Kripke frames; that is  $L = \text{Log}(\mathcal{C})$ .

The class  $\mathcal{C}$  might be huge. It might consist of infinitely many infinite frames. So this still does not give us a criterion for deciding whether a given formula  $A$  belongs to  $L$ .

However, if we manage to show a stronger version of completeness—a **completeness with respect to a class of finite frames**—then, as we will see below, we are one step closer to having such a criterion.



## Definition 29

We will say that a normal modal logic  $L$  has the *finite model property* if there exists a (not necessarily finite) class  $\mathcal{C}$  of finite frames such that  $L = \text{Log}(\mathcal{C})$ .

It would be more logical to call this property the *finite frame property*, as some authors do. But the finite model property is a fixed terminology by now. So we will stick to it.

There are a few methods for proving finite model property for modal logics.

The most popular and widely applied among those are *filtration* and *selective filtration*.

In what follows we will discuss the method of filtration and after that we will sketch the basic idea of selective filtration.



Krister Segerberg

## Subformulas

Subformulas are easier to recognize than to define!

Thinking of the formation tree of  $A$ , the subformulas are its subtrees.

(The proper definition is by induction on  $A$ ; see Blackburn et al.)

E.g., the subformulas of  $A = \neg\Box\neg(p \wedge \Box(\top \wedge p))$  are:  
 $p$ ,  $\top$ ,  $p$  (two distinct  $p$ s!),  $\top \wedge p$ ,  $\Box(\top \wedge p)$ ,  
 $p \wedge \Box(\top \wedge p)$ , and also:

$$\begin{aligned} &\neg(p \wedge \Box(\top \wedge p)), \\ &\Box\neg(p \wedge \Box(\top \wedge p)), \\ &\neg\Box\neg(p \wedge \Box(\top \wedge p)) = A. \end{aligned}$$

# Filtration

We will prove the **finite model property** for **K**:

## Theorem 30 (Filtration theorem)

*Let  $A$  be a satisfiable modal formula. Then  $A$  has a finite model — in fact, one with at most  $2^n$  worlds, where  $n$  is the number of subformulas of  $A$ .*

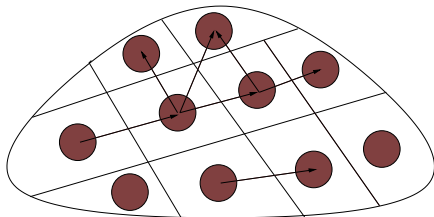
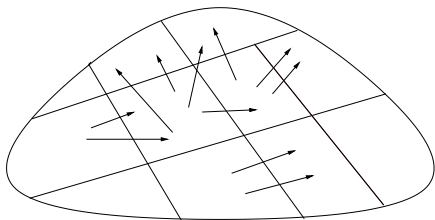
Why does this imply the finite model property for  $\mathbf{K}$ ?

Let  $\mathcal{C}_{\text{fin}}$  be the class of all finite frames. Then obviously  $\mathbf{K} \subseteq \text{Log}(\mathcal{C}_{\text{fin}})$ .

Now suppose  $A \notin \mathbf{K}$ . Then, by the completeness of  $\mathbf{K}$ , there exists a model (for example the canonical model),  $\mathcal{M}$  such that  $\neg A$  is satisfied in  $\mathcal{M}$ .

By theorem 30,  $\neg A$  is satisfied in a finite model  $\mathcal{M}'$ . This means that  $A$  is not valid in  $\mathcal{C}_{\text{fin}}$ , i.e.,  $A \notin \text{Log}(\mathcal{C}_{\text{fin}})$ . So  $\mathbf{K} = \text{Log}(\mathcal{C}_{\text{fin}})$ .

## Why is the finite model property true?



The formal proof may look frightening, but is just what we get from writing down the picture-idea in notation.

## Proof of finite model property

Fix a model  $\mathcal{M} = (W, R, h)$  satisfying  $A$ .

Define an equivalence relation  $\sim$  on  $W$  by

$$t \sim u \iff (\mathcal{M}, t \models B \Leftrightarrow \mathcal{M}, u \models B \\ \text{for all subformulas } B \text{ of } A).$$

Let  $W'$  be the set of  $\sim$ -equivalence classes. Then  $|W'| \leq 2^n$ . For, any two worlds making true the same subformulas of  $A$  are  $\sim$ -equivalent. So there are at most as many  $\sim$ -inequivalent worlds as there are sets of subformulas — that is, **at most  $2^n$** .



Define  $R'$  on  $W'$  by:  $R'(X, Y)$  iff  $R(t, u)$  for some  $t \in X, u \in Y$ .  
(Here,  $X, Y \in W'$ .)

For each atom occurring in  $A$  we let

$$h'(p) = \{X \in W' : \mathcal{M}, t \models p \text{ for some } t \in X\}.$$

We have a finite model  $\mathcal{M}' = (W', R', h')$ . It is called a *standard filtration* of  $\mathcal{M}$ .

But is it a model of  $A$ ?

### Lemma 31 (Filtration lemma)

For any subformula  $B$  of  $A$ ,

$$\mathcal{M}', X \models B \iff \mathcal{M}, t \models B$$

for any  $X \in W'$  and any  $t \in X$ .

### Proof.

By structural induction on  $B$ . Pick  $X \in W'$  and  $t \in X$ .

If  $B$  is an atom  $p$ ,  $\mathcal{M}', X \models p$  iff  $X \in h'(p)$  iff (by def.)  $\mathcal{M}, u \models p$  for some  $u \in X$ , iff  $\mathcal{M}, t \models p$  (because  $X$  is a  $\sim$ -class so all its elements 'agree' on  $p$ ).

If  $B = \top$ , it's clear —  $\mathcal{M}', X \models \top$  and  $\mathcal{M}, t \models \top$ .

## continuation.

The case  $\wedge$  is easy. Assume that  $B \wedge C$  is a subformula of  $A$ . Then  $B, C$  are also subformulas of  $A$ ; assume (inductively) the lemma for them. Then

$\mathcal{M}', X \models B \wedge C$  iff  $\mathcal{M}', X \models B$  and  $\mathcal{M}', X \models C$ , iff (by the inductive hypothesis)  $\mathcal{M}, t \models B$  and  $\mathcal{M}, t \models C$ , iff  $\mathcal{M}, t \models B \wedge C$ .

The case of  $\neg$  is similar (exercise).

## continuation.

Finally, suppose that  $\Box B$  is a subformula of  $A$ , and assume the lemma for  $B$  ( $B$  is also a subformula of  $A$ , so can use ind. hyp.).

Assume  $\mathcal{M}', X \models \Box B$ ; we prove  $\mathcal{M}, t \models \Box B$ .

Pick any  $u \in W$  with  $R(t, u)$ . Let  $Y$  be the  $\sim$ -class of  $u$ .

By definition of  $R'$  we have  $R'(X, Y)$ . By assumption,  $\mathcal{M}', X \models \Box B$ , so  $\mathcal{M}', Y \models B$ .

But  $u \in Y$ , so inductively,  $\mathcal{M}, u \models B$ .

This is true for all  $u$  with  $R(t, u)$ . So  $\mathcal{M}, t \models \Box B$ .

## continuation.

Now assume  $\mathcal{M}, t \models \Box B$ . Take  $Y \in W'$  with  $R'(X, Y)$ . We show  $\mathcal{M}', Y \models B$ .

By definition of  $R'$ , there are  $t' \in X$  and  $u \in Y$  with  $R(t', u)$ .

As  $t' \sim t$ , and  $\Box B$  is a subformula of  $A$ ,  $\mathcal{M}, t' \models \Box B$ . So  $\mathcal{M}, u \models B$ .

Inductively,  $\mathcal{M}', Y \models B$ . Because  $Y$  was arbitrary (with  $R'(X, Y)$ ), we get  $\mathcal{M}', X \models \Box B$ , as required.



It is obvious that the filtration lemma implies the filtration theorem (exercise).

Thus we proved that  $\mathbf{K}$  *has the finite model property*.

Can we do the same for other normal modal logics?

Sometimes yes, sometimes not.

In the general case we need to prove that the underlying frame of  $\mathcal{M}'$  validates the logic. If we can do it, then the logic has the finite model property.

## Reflexive frames

We first consider the logic **KT** of all reflexive frames.

Let  $A$  be satisfiable in a model  $\mathcal{M} = (W, R, h)$  based on a reflexive frame  $(W, R)$ . We **filtrate** it as before. The obtained finite model then satisfies  $A$ .

It is left to be shown that the underlying frame of  $\mathcal{M}'$  is reflexive.

Thus we need to show that for each element  $X$  of  $\mathcal{M}'$  we have  $R'(X, X)$ .

But this is obvious since for each  $s \in X$  we have  $R(s, s)$  (check the details).

We deduce that the logic **KT** has the *finite model property*.



## Transitive frames

Now let us consider the logic **K4** of all transitive frames.

We first note that if we filtrate a model based on a transitive frame, the relation  $R'$  may no longer be transitive.

### Exercise 32

*Construct a filtration of a model based on a transitive frame that is not transitive.*

So what do we do?

We will modify the definition of  $R'$ .

For each  $X, Y \in W'$  we set

$$XR^*Y \text{ iff } \mathcal{M}, s \models \Box B \text{ implies } \mathcal{M}, t \models \Box B \wedge B$$

for each  $s \in X$  and  $t \in Y$  and  $\Box B$  a subformula of  $A$ .

Let  $\mathcal{M}' = (W', R^*, h')$ , where  $R^*$  is as above and  $W'$  and  $h'$  as in the standard filtration (slide 65).  $\mathcal{M}'$  is called the *Lehmann Filtration*.

### Lemma 33 (Filtration lemma)

For any subformula  $B$  of  $A$ ,

$$\mathcal{M}', X \models B \iff \mathcal{M}, s \models B$$

for any  $X \in W'$  and any  $s \in X$ .

## Proof.

We first show that if  $s, t \in W$ ,  $s \in X$ ,  $t \in Y$  and  $R(s, t)$ , then  $R^*(X, Y)$ .

So suppose for a subformula  $\Box B$  of  $A$  we have  $\mathcal{M}, s \models \Box B$ . We need to show that  $\mathcal{M}, t \models \Box B \wedge B$ .

Since  $R(s, t)$ , we have that  $\mathcal{M}, t \models B$ .

Now if  $\mathcal{M}, t \not\models \Box B$ , then there exists  $u \in W$  such that  $R(t, u)$  and  $\mathcal{M}, u \not\models B$ . But since  $R$  is transitive we have  $R(s, u)$ . So  $\mathcal{M}, u \not\models B$  contradicts  $\mathcal{M}, s \models \Box B$ .

Thus,  $R(s, t)$  implies  $R^*(X, Y)$ .

## continuation.

Next we proceed by structural induction on  $B$ . The boolean cases are simple (check!).

Now let  $\mathcal{M}, s \models \Box B$ , where  $\Box B$  is a subformula of  $A$ . Then for each  $Y$  such that  $R^*(X, Y)$  we have  $\mathcal{M}, t \models \Box B \wedge B$  for  $t \in Y$ .

By the ind hyp  $\mathcal{M}, t \models B$  implies  $\mathcal{M}', Y \models B$  and thus  $\mathcal{M}', X \models \Box B$ .

Finally, assume  $\mathcal{M}, s \not\models \Box B$ . Then there exists  $t \in W$  such that  $R(s, t)$  and  $\mathcal{M}, t \not\models B$ . As we showed above,  $R(s, t)$  implies  $R^*(X, Y)$  for  $t \in Y$ . Moreover, by the ind hyp,  $\mathcal{M}', Y \not\models B$ . Thus,  $\mathcal{M}', X \not\models \Box B$ . □

Finally, we need to check that  $R^*$  is transitive.

Let  $R^*(X, Y)$  and  $R^*(Y, Z)$  and suppose  $\mathcal{M}, s \models \Box B$ , for  $s \in X$  and  $\Box B$  a subformula of  $A$ . Then  $\mathcal{M}, t \models \Box B \wedge B$ , for  $t \in Y$ . Then  $\mathcal{M}, t \models \Box B$  implies  $\mathcal{M}, u \models \Box B \wedge B$  for  $u \in Z$  and thus  $R^*(X, Z)$ .

Note that in the proof of transitivity of  $R^*$  we **have not used** that  $R$  is transitive. The transitivity of  $R$  was only used in the filtration lemma.

So we proved that **K4** has the *finite model property*.

### Exercise 34

*Prove using filtration that **S4** and **S5** have the finite model property.*

## Selective filtration

Next we sketch the basic idea of *selective filtration* – another powerful method for proving the finite model property.

If the main idea of filtration is to identify points (split a big model into finitely many blocks), the idea of selective filtration is to select finitely many points from a big model in such a way that by restricting the relation and assignment to this set of selected points we obtain a model still satisfying a given formula.

For example we might have a **huge** model  $\mathcal{M}$  satisfying the formula  $\diamond p$ . This means that there are points  $s$  and  $t$  in  $\mathcal{M}$  such that  $R(s, t)$  and  $\mathcal{M}, t \models p$ . Then obviously we can just select the points  $s$  and  $t$ , restrict the relation and assignment to  $\{s, t\}$  and the obtained 2-element model will satisfy  $\diamond p$ .



We will illustrate the selective filtration method on modal logic **S5** of all frames with an equivalence relation.

Let  $\mathcal{M} = (W, R, h)$  be a model with  $R$  being an equivalence relation and  $A$  a formula such that  $\mathcal{M}, s \models A$  for some  $s \in W$ .

For each  $\Box B$  subformula of  $A$  we select a point  $t \in W$  such that  $R(s, t)$  and  $\mathcal{M}, t \not\models B$  (if such a point exists).

Let  $S$  be the set of all selected points together with the point  $s$ . We let  $\mathcal{M}'$  be the model obtained by restricting  $h$  and  $R$  to  $S$ . Then  $|S| \leq n + 1$ , where  $n$  is the number of subformulas of  $A$  (why?).



We first show that for each  $u, v \in S$  we have  $R(u, v)$ . Indeed, if  $u, v \in S$ , then  $R(s, u)$  and  $R(s, v)$ . Since  $R$  is symmetric we have  $R(u, s)$  and by transitivity  $R(u, v)$ .

### Lemma 35 (truth lemma)

*For each subformula  $B$  of  $A$  and each  $u \in S$  we have:*

$$\mathcal{M}, u \models B \text{ iff } \mathcal{M}', u \models B.$$

### Proof.

By structural induction on  $B$ . The boolean cases are simple (check!).

Suppose  $\mathcal{M}', u \not\models \Box B$ . Then there exists  $t \in S$  such that  $R(u, t)$  and  $\mathcal{M}', t \not\models B$ . But, by ind hyp, this means that  $\mathcal{M}, t \not\models B$  and thus  $\mathcal{M}, u \not\models \Box B$ .

## continuation.

Now let  $\mathcal{M}, u \not\models \Box B$ , then there is  $t \in W$  such that  $R(u, t)$  and  $\mathcal{M}, t \not\models B$ .

But  $u \in S$  so we have  $R(s, u)$ . By the transitivity of  $R$  we get  $R(s, t)$ . The point  $t$  may not belong to  $S$ .

But by the construction of  $S$ , there is  $t' \in S$  such that  $R(s, t')$  and  $\mathcal{M}, t' \not\models B$ . By ind hyp, we have  $\mathcal{M}', t' \not\models B$ . Since for each  $u, v \in S$  we have  $R(u, v)$  we obtain that  $R(u, t')$ . This means that  $\mathcal{M}', u \not\models \Box B$ . □

To finish the proof of the finite model property of **S5** we only need to note that by restricting the equivalence relation  $R$  to  $S$  we obtain a frame with an equivalence relation. But this is fairly obvious. In fact,  $S$  consists of one equivalence class (why?).

Thus, we proved that **S5** has the *finite model property*.

Note that the size of the model satisfying the formula  $A$  obtained via selective filtration, is  $n + 1$  where  $n$  is the number of subformulas of  $A$ . Recall that the size of the model, satisfying  $A$  obtained via standard and Lemmon filtrations, is  $2^n$ .