INTRODUCTION TO MODAL LOGIC 2016
HOMEWORK 4

- Deadline: November 15 — at the **beginning** of class.
- Grading is from 0 to 100 points.
- Success!

(1) (20pt) Compute, via the Sahlqvist algorithm, the global first-order correspondents of the following modal formulas:

(a) $\Diamond p \to \Box p$,
(b) $p \wedge \Box p \wedge \Box\Box p \to \Diamond p$,
(c) $\Diamond\Diamond\Box p \to \Box\Box\Diamond p$,
(d) $\Diamond_1\Diamond_2 p \to \Diamond_2\Diamond_1 p$.

Simplify the first-order correspondents as much as you can.

(2) (20pt) Let M denote the *McKinsey formula* $\Box\Diamond p \to \Diamond\Box p$. Show that for a transitive frame $\mathfrak{F} = (W, R)$ we have that

$$\mathfrak{F} \models \forall x \exists y (Rxy \wedge \forall z (Ryz \to z = y)) \text{ implies } \mathfrak{F} \models M.$$

You don't have to show it, but in fact, the converse is also true (see p. 168 of the book). So there exist formulas that are **not** Sahlqvist (e.g., $M \wedge \Diamond\Diamond p \to \Diamond p$), but still have global first-order correspondents.

(3) (20pt) Let φ and ψ be formulas in the language of basic modal logic. Prove

(a) $\vdash_{\mathbf{K}} \Box\varphi \to \Box(\psi \to \varphi)$

(b) $\vdash_{\mathbf{K}} (\Diamond\varphi \wedge \Box(\varphi \to \psi)) \to \Diamond\psi$

You may find it helpful to note that the following are propositional tautologies:

- $p \to (q \to p)$
- $p \to (p \vee q)$
- $(p \to q) \to (\neg q \to \neg p)$
- $(p \to (q \to r)) \leftrightarrow ((p \wedge q) \to r)$
- $p \to (q \to (p \wedge q))$

(4) (20pt) Recall that $\mathbf{S5} = \mathbf{K} + (\Box p \to p) + (\Box p \to \Box\Box p) + (p \to \Box\Diamond p)$. Show:

(a) $\vdash_{\mathbf{S5}} \Diamond p \to \Box\Diamond p$

(b) $\nvdash_{\mathbf{S5}} \Diamond p \to \Box p$ (You may use that **S5** is sound with respect to the class of frames (W, R), where R is an equivalence relation)

(5) (20pt)

(a) Show that if a frame $\mathfrak{F}$ is a bounded morphic image of a frame $\mathfrak{G}$, then

$$Log(\mathfrak{G}) \subseteq Log(\mathfrak{F}).$$

(b) Let $\mathcal{C}$ be a non-empty class of frames. Prove that $Log(\mathcal{C})$ is contained in the logic of a single reflexive point or $Log(\mathcal{C})$ is contained in the logic of a single irreflexive point.