INTRODUCTION TO MODAL LOGIC 2016
HOMEWORK 3

- Deadline: October 18 — at the **beginning** of class.
- Grading is from 0 to 100 points.
- Success!

(1) (30pt) Suppose that $\gamma(x), \delta(x)$ are first–order formulas. We say that $\gamma(x)$ *semantically entails* $\delta(x)$ (Notation: $\gamma(x) \models \delta(x)$) provided that for any model $\mathfrak{M} = (W, R, V)$ and any $w \in W$,

$$\mathfrak{M} \models \gamma(x)[w] \text{ implies } \mathfrak{M} \models \delta(x)[w].$$

We say that $\gamma(x)$ *entails* $\delta(x)$ *along bisimulation* provided that for any models $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$, and any $w \in W$, $w' \in W'$,

$$(\mathfrak{M}, w \leftrightarrows \mathfrak{M}', w' \text{ and } \mathfrak{M} \vDash \gamma(x)[w]) \text{ implies } \mathfrak{M}' \vDash \delta(x)[w'].$$

A modal formula φ is called a *modal interpolant* of $(\gamma(x), \delta(x))$ provided that

$$\gamma(x) \vDash \mathsf{ST}_x(\varphi) \text{ and } \mathsf{ST}_x(\varphi) \vDash \delta(x).$$

In the following let $\alpha(x)$ and $\beta(x)$ be first–order formulas.

(a) Show that if $\alpha(x)$ entails $\beta(x)$ along bisimulation, then $\alpha(x) \models \beta(x)$. (**BONUS!**) (10pt) Show that the converse does not hold.

(b) Suppose that the pair $(\alpha(x), \beta(x))$ has a modal interpolant. Show that $\alpha(x)$ entails $\beta(x)$ along bisimulation.

(c) In fact, the converse of (b) is true (but non-trivial to show), i.e. if $\alpha(x)$ entails $\beta(x)$ along bisimulation then the pair $(\alpha(x), \beta(x))$ has a modal interpolant. Show that this implies (the non-obvious direction of) the van Benthem Characterization Theorem.

(2) (40pt) Show that Grzegorczyk's formula

$$\Box(\Box(p \to \Box p) \to p) \to p$$

characterizes the class of frames $\mathfrak{F} = (W, R)$ satisfying (i) R is reflexive, (ii) R is transitive and (iii) there are no infinite paths $x_0 R x_1 R x_2 R ...$ such that for all i we have $x_i \neq x_{i+1}$.

(3) (30pt) Show that the following properties of frames are not modally definable,:

(a) antisymmetry $\forall xy(Rxy \wedge Ryx \to x = y))$,

(b) acyclicity (there is no finite path from any x to itself).