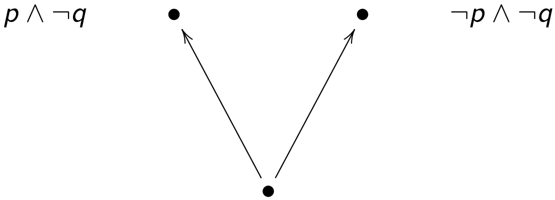
Lecture 4: finite models

September 20, 2016

Question

Is the following formula valid on all frames?

$$\Diamond(p \rightarrow q) \rightarrow (\Diamond p \rightarrow \Diamond q)$$


$p \wedge \neg q$
$\neg p \wedge \neg q$

Normal modal logics

Definition

A *normal modal logic* is a set $\mathfrak{L}$ of formulas closed under:

- Modus ponens: $\frac{\varphi \quad \varphi \to \psi}{\psi}$
- Uniform substitution: $\frac{\varphi}{\varphi[\sigma]}$
- Necessitation: $\frac{\varphi}{\Box\varphi}$

and containing the following formulas:

- The K-axiom: $\Box(p \to q) \to (\Box p \to \Box q)$
- Dual: $\Diamond p \leftrightarrow \neg\Box\neg p$

Proposition

For every class F *of frames, there is a normal modal logic* Λ_{F} *given by:*

$$\varphi \in \Lambda_{\mathsf{F}} \quad \Leftrightarrow \quad \forall \mathfrak{F} \in \mathsf{F} : \mathfrak{F} \Vdash \varphi$$

Observe that:

$$\mathsf{K}_1 \subseteq \mathsf{K}_2 \quad \Rightarrow \quad \Lambda_{\mathsf{K}_2} \subseteq \Lambda_{\mathsf{K}_1}$$

Question

Is the converse true?

$$\Lambda_{K_2} \subseteq \Lambda_{K_1} \quad \stackrel{?}{\Rightarrow} \quad K_1 \subseteq K_2$$

Hint

Last lecture gave the answer...

The minimal modal logic

The logic $\Lambda_{[\text{All frames}]}$ is the smallest normal modal logic, and denoted by **K**...

... after Saul **K**ripke.

We want to understand the logic Λ_K of a class of frames. Two sides to the equation:

- Which formulas *are* valid on K?
- Which formulas are *not* valid on K?

Proof theory

Find good, transparent systems of axioms and rules allowing us to *derive* valid formulas.

The satisfiability problem

Formula φ is *not* valid $\Leftrightarrow$ the formula $\neg\varphi$ is *satisfiable*.

Reformulation of our question:

- Which formulas *are* valid on K?
- Which formulas are *satisfiable* in K?

The finite model property for **K**

Theorem

Let φ be any modal formula. Then φ is satisfiable if, and only if, it is satisfiable on a finite model.

Two methods for finite models:

- Selection
- Filtration

n-bisimulations

Definition

An *n-bisimulation* between models $\mathfrak{M}, \mathfrak{M}'$ is a chain:

$$Z_0 \subseteq \ldots \subseteq Z_n$$

such that:

- (Atomic) wZ_iw' implies $w \in V(p)$ iff $w' \in V'(p)$
- (Forth) $wZ_{i+1}w'$ and wRv implies $\exists v'$ such that $w'R'v'$ and vZ_iv'
- (Back) $wZ_{i+1}w'$ and $w'R'v'$ implies $\exists v$ such that wRv and vZ_iv'

Notation

$$\mathfrak{M}, w \underline{\longleftrightarrow}_n \mathfrak{N}, v$$

(Optional)

Think of n-bisimulations in terms of n-round pebble games!

Modal depth

- $\mathsf{md}(p) = \mathsf{md}(\bot) = 0$
- $\mathsf{md}(\neg\varphi) = \mathsf{md}(\varphi)$
- $\mathsf{md}(\varphi \vee \psi) = \max(\mathsf{md}(\varphi), \mathsf{md}(\psi))$
- $\mathsf{md}(\Diamond\varphi) = \mathsf{md}(\varphi) + 1$

Proposition

There are, up to logical equivalence, only finitely many formulas of modal depth $\leq n$ built from finitely many variables P.

Proposition

If $\mathfrak{M}, w \underline{\leftrightarrow}_n \mathfrak{N}, v$ then $\mathfrak{M}, w$ and $\mathfrak{N}, v$ satisfy the same formulas of depth $\leq n$.

Example

Finite “fan” vs. “fan” with an added infinite branch.

Definition

Let $\mathfrak{M}$ be a model and $w \in W$. Then $\mathfrak{M}{\restriction}^k_w$ is the unique submodel of $\mathfrak{M}$ consisting of w together with all elements v of W such that the longest path from w to v is of length at most k.

Proposition

If $\mathfrak{M}$ is a tree rooted at w, then:

$$\mathfrak{M}, w \underline{\longleftrightarrow}_n \mathfrak{M}{\restriction}_w^n$$

Proposition

Every satisfiable formula of depth n is satisfiable on a tree of height $\leq n$.

Proof.

Unravel, restrict to height n and then find an n-bisimulation. ■

Selection

Proposition

Let $\mathfrak{M}, w$ be a rooted tree model of height $\leq n$. Then there is a finite tree model $\mathfrak{M}', w$ satisfying the same formulas of depth $\leq n$ as $\mathfrak{M}, w$

Proof.

By induction: select witnesses for all formulas $\Diamond\varphi$ of depth $\leq n$, replace them by finite trees and cut of all other successors. ∎

Definition

Let Σ be a finite set of formulas closed under subformulas. The equivalence relation $\leftrightsquigarrow_\Sigma$ on $\mathfrak{M}$ is defined by:

$$\forall \varphi \in \Sigma : \; \mathfrak{M}, u \Vdash \varphi \;\Leftrightarrow\; \mathfrak{M}, v \Vdash \varphi$$

The equivalence class of w is denoted by $[w]_\Sigma$.

Definition

Let $\mathfrak{M}$ be any Kripke model and let $\mathfrak{M}'$ be a model based on $\{[w]_\Sigma \mid w \in W\}$. Then $\mathfrak{M}'$ is a *filtration* of $\mathfrak{M}$ through Σ if:

- (Atomic) For $p \in \Sigma$, $w \in V(p)$ iff $[w]_\Sigma \in V'(p)$.
- (Forth) If uRv then $[u]_\Sigma R'[v]_\Sigma$.
- (Back) If $[u]_\Sigma R'[v]_\Sigma$ and $\Diamond\varphi \in \Sigma$ then $\mathfrak{M}, v \Vdash \varphi$ implies $\mathfrak{M}, u \Vdash \Diamond\varphi$.

Filtration lemma

Theorem

Let $\varphi \in \Sigma$ and let $\mathfrak{M}'$ be a filtration of $\mathfrak{M}$ through Σ. Then:

$$\mathfrak{M}, u \Vdash \varphi \;\Leftrightarrow\; \mathfrak{M}', [u]_\Sigma \Vdash \varphi$$

...But we don't yet know that filtrations always exist!

The smallest filtration

Definition

$$R^s = \{([u]_\Sigma, [v]_\Sigma) \mid uRv\}$$

The largest filtration

Definition

$$R^l = \{([u]_\Sigma, [v]_\Sigma) \mid \forall \Diamond\varphi \in \Sigma : v \Vdash \varphi \;\Rightarrow\; u \Vdash \Diamond\varphi\}$$

Small model theorem

Theorem

Let φ be any modal formula. Then φ is satisfiable if, and only if, φ is satisfiable on a model of size at most 2^k where k is the number of subformulas of φ.

Proof.

Filtrate through $\Sigma =$ the set of subformulas of φ. ■