Lecture 2: generated submodels, disjoint unions and bounded morphisms

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Expressive power in logic

A fact of life

Typically, the more expressive power a logical system has, the worse it behaves computationally.

- Second-order logic can describe the natural numbers up to isomorphism, is not recursively axiomatizable.
- First-order logic is recursively axiomatizable but admits non-standard models.

We are generally looking for *compromises* between good expressive power and good computational behavior (decidability, complete axiom systems etc.). So it is imperative to have a good understanding of the *expressive power of logical systems*!

One side of the equation is easy: a proof that property can be expressed in system $\mathfrak{L}$ just provides a formula that expresses it.

Proposition

FOL can express counting quantifiers, $\exists^{\geq n} x Px$.

Proof.

$$\exists x_1 \ldots \exists x_n (Px_1 \wedge \ldots \wedge Px_n \wedge \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j)$$

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Limits on expressive power

How do we prove that a property is *not* expressible in a logic? This is typically much more subtle!

Proposition

FOL can not express the infinite counting quantifier, $\exists^{\infty} x P x$.

Proof.

Compactness, ultra-products, EF-games... ∎

Example: the global modality

$$\mathfrak{M}, w \Vdash \mathsf{E}\varphi \;\Leftrightarrow\; \exists w' \in W : \mathfrak{M}, w' \Vdash \varphi$$

Shouldn't be definable in basic modal logic: not a "local" property!

But how to prove it?

Disjoint unions

Definition

Let $\{\mathfrak{M}_i\}_{i \in I}$ be a family of models of similarity type (O, τ). The *disjoint union*

$$\sum_{i \in I} \mathfrak{M}_i = (W', R', V')$$

is given by:

- $W' = \bigcup_{i \in I} W_i \times \{i\}$
- $R'_\Delta = \bigcup_{i \in I} \{\langle (u, i), (v_1, i), \ldots, (v_n, i) \rangle \mid \langle u, v_1, \ldots, v_n \rangle \in R^i_\Delta\}$
- $V'(p) = \bigcup_{i \in I} V_i(p) \times \{i\}$

Notation

Disjoint union of $\mathfrak{M}_1, \mathfrak{M}_2$ written as $\mathfrak{M}_1 + \mathfrak{M}_2$.

Or in plain words...

The disjoint union of models $\{\mathfrak{M}_i\}_{i \in I}$ is obtained by placing one copy of each $\mathfrak{M}_i$ side by side.

A first preservation result

Proposition

For formula φ, each $i \in I$ and each $u \in W_i$:

$$\mathfrak{M}_i, w \Vdash \varphi \quad \Leftrightarrow \quad \sum_{i \in I} \mathfrak{M}_i, (w, i) \Vdash \varphi$$

Proof.

Induction on φ. ∎

Notation

$$\mathfrak{M}_i, w \iff \sum_{i \in I} \mathfrak{M}_i, (w, i)$$

Proposition

The global modality is not definable in the modal language of any similarity type.

Proof.

...is now a piece of cake! ∎

Submodels

Definition

Let $\mathfrak{M} = (W, R, V)$ be any model. Then $\mathfrak{M}' = (W', R', V')$ is a *submodel* of $\mathfrak{M}$ if:

- $W' \subseteq W$
- $R'_\Delta = R_\Delta \cap W'^{n+1}$ $(\tau(\Delta) = n)$
- $V'(p) = V(p) \cap W'$

Submodels do *not* preserve satisfaction!

Generated submodels

Definition

Let $\mathfrak{M}'$ be a submodel of $\mathfrak{M}$. Then $\mathfrak{M}'$ is called a *generated submodel* of $\mathfrak{M}$ if the following "backwards" condition holds:

$$u \in W' \text{ and } R_\Delta u v_1 \ldots v_n \text{ implies } v_1 \ldots v_n \in W'$$

Notation

$$\mathfrak{M}' \rightarrowtail \mathfrak{M}$$

Proposition

If $\mathfrak{M}' \rightarrowtail \mathfrak{M}$ *and* $w \in W'$ *then:*

$$\mathfrak{M}', w \;\leftrightsquigarrow\; \mathfrak{M}, w$$

Proof.

Induction on formulas. ∎

Point-generated submodels

Definition

The smallest generated submodel of $\mathfrak{M}$ containing w is called a *point-generated* submodel of $\mathfrak{M}$ and is *generated by* w.

Example

Sub-trees of a tree are generated by their roots.

Homomorphisms of models

Definition

A map $f : W \to W'$ is called a *homomorphism* from $\mathfrak{M}$ to $\mathfrak{M}'$ if:

- $R_\Delta w v_1 \ldots v_n \quad \Rightarrow R'_\Delta f(w) f(v_1) \ldots f(v_n)$
- $w \in V(p) \quad \Rightarrow f(w) \in V'(p)$

Model homomorphisms do *not* preserve satisfaction!

...but they are appropriate for a certain fragment of modal logic, the *positive-existential formulas*. Cf. Lyndon's theorem in model theory.

Strong homomorphisms

Again, a certain "backwards" condition is missing. A first attempt at a fix:

Definition

The map f is said to be a *strong homomorphism* if:

- $R_\Delta w v_1 \dots v_n \quad \Leftrightarrow \quad R'_\Delta f(w) f(v_1) \dots f(v_n)$
- $w \in V(p) \quad \Leftrightarrow \quad f(w) \in V'(p)$

Too strong!

Example

Compare a *reflexive point* with $(\mathbb{N}, \text{successor})$...

Isomorphism and embedding

Definition

Let $f : \mathfrak{M} \rightarrow \mathfrak{M}'$ be a strong model homomorphism. Then f is said to be:

- an *embedding* if it is injective,
- an *isomorphism* if it is injective and surjective.

Bounded morphisms

... a.k.a. *p-morphisms*, for "pseudo-epimorphism".

Definition

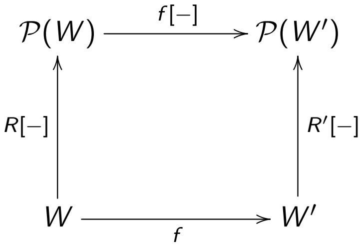
The map f is said to be a *bounded morphism* if:

- $R_\Delta w v_1 \ldots v_n \quad \Rightarrow \quad R'_\Delta f(w) f(v_1) \ldots f(v_n)$
- (Back condition:) if $R'_\Delta f(w) v'_1 \ldots v'_n$ then there exist $v_1, \ldots, v_n$ such that:
    1. $R_\Delta w v_1 \ldots v_n$ and
    2. $f(v_i) = v'_i$
- $w \in V(p) \quad \Leftrightarrow \quad f(w) \in V'(p)$

For basic modal language:

The map f is a bounded morphism if, for all $u \in W$:

- $u \in V(p)$ iff $f(u) \in V(p)$ for all p, and
- $R'[f(u)] = f[R[u]]$



Definition

If there is a *surjective* bounded morphism $f : \mathfrak{M} \to \mathfrak{M}'$, we say that $\mathfrak{M}'$ is a *bounded morphic image* of $\mathfrak{M}$ and write $\mathfrak{M} \twoheadrightarrow \mathfrak{M}'$.

Proposition

Let $f : \mathfrak{M} \to \mathfrak{M}'$ *be a bounded morphism and* $w \in W$*. Then:*

$$\mathfrak{M}, w \quad \leftrightsquigarrow \quad \mathfrak{M}', f(w)$$

An important corollary is that modal logic has the *tree-like model property*.