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Lecture 1: relational structures, modal languages, frames and models

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What is modal logic?

An old definition:

Modal logic is the logic of necessity and possibility.

- $\Box p$: the proposition p is necessarily true
- $\Diamond p$: the propostion p might possibly be true

Paradoxes of material implication

- $p \rightarrow (q \rightarrow p)$
- $\neg p \rightarrow (p \rightarrow q)$
- $(p \rightarrow q) \vee (q \rightarrow p)$

Strict implication:

$$p \Rightarrow q \quad = \quad \Box(p \rightarrow q)$$

Modal logic in philosophy

Philosophy as *conceptual analysis*. Other readings of the box:

- p will be true some time in the future (temporal logic)
- p is known to be true (epistemic logic)
- p is believed to be true (doxastic logic)
- p ought to be true (deontic logic)

Intensional/non-truth functional connectives.

Semantics

The purpose of semantics is to assign *interpretations* of the logical symbols occurring in formulas. Early semantics of modal logics were *algebraic*.

A paradigm shift occurred when Saul Kripke introduced *relational semantics*, now called *Kripke semantics*. The origin of relational semantics comes from Leibniz' analysis of necessity as *truth across all possible worlds*.

What is modal logic?

A more up-to-date definition:

Modal logics are formal languages for describing and reasoning about relational structures, often combining simplicity and good computational properties with expressive power.

Applications:

- Philosophy (logics of necessity, knowledge, time, obligations...)
- Mathematics (provability logic, logics for topology, connections with set theory...)
- Computer science (temporal and dynamic logics, description logic...)
- Many more, and still expanding...

Relational structures

Definition

A *relational structure* is a structure $(W, \{R_i\}_{i \in I})$ where W is a set and $\{R_i\}_i \in I$ is a family of relations, each with a given *arity*.

Example

Familiar examples of relational structures are $(\mathbb{N}, <)$, $(\mathbb{Z}, <)$, $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$. We get another family of relational structures by putting $\leq$ in the place of $<$.

Example

$(\mathbb{N}, R)$ where:

$$R = \{\langle i, j, k\rangle \in \mathbb{N}^3 \mid i = j + k\}$$

$\langle 8, 3, 5\rangle \in R$, $\langle 3, 10, 4\rangle \notin R$...

Example

The *full binary tree* is the set of all finite words over $\{0, 1\}$. For example, 00101 is a member of the full binary tree, as is the empty word. It is a relational structure (T, succ) where succ is the *successor relation*.

Rabin's theorem, "mother of all decidability results".

Definition

A *tree* is a relational structure (W, R) with a distinguished *root* r such that, for all $w \in W$, there is a unique R-path from r to w.

Example

Labelled transition systems, used extensively in computer science, are relational structures $(W, \{R_a\}_{a \in A})$ where $R_a xy$ holds iff there is an edge from x to y labelled a.

Example

The Euclidean plane is a relational structure with a ternary relation C defined as: $C(x, y, z)$ iff x is closer to y than to z with respect to the Euclidean distance.

A quick repetition...

Definition

Let R be a binary relation on a set W. Then R is said to be:

- *reflexive* if $\forall x Rxx$
- *irreflexive* if $\forall x \neg Rxx$
- *transitive* if $\forall x, y, z(Rxy \wedge Ryz \to Rxz)$
- *symmetric* if $\forall x, y(Rxy \to Ryx)$
- *anti-symmetric* if $\forall x, y(Rxy \wedge Ryx \to x = y)$
- *connected* or *total* if $\forall xy(Rxy \vee Ryx)$

Definition

R is:

- A *pre-order* if it is reflexive and transitive
- A *partial order* if it is a pre-order and anti-symmetric
- A *total order* if it is a partial order and total.
- A *strict total order* if it is irreflexive, transitive and total.
- An *equivalence relation* if it is reflexive, transitive and symmetric.

A little representation theorem

Proposition

There is a one-to-one correspondence between the equivalence relations on any set W and the partitions of W.

Basic modal language

$$\varphi := p \mid \bot \mid \varphi \vee \varphi \mid \neg\varphi \mid \Diamond\varphi$$

- $\top = \neg\bot$
- $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$
- $\varphi \rightarrow \psi = \neg(\varphi \wedge \neg\psi)$
- $\Box\varphi = \neg\Diamond\neg\varphi$

Frames and models

Definition

A *frame* $\mathfrak{F}$ for the basic modal language is a relational structure (W, R) where R is a binary relation.

Definition

A *model* $\mathfrak{M}$ for the basic modal language is a triple (W, R, V) such that (W, R) is a frame and $V : \mathsf{Var} \to \mathcal{P}(W)$ a *valuation*.

Truth clauses:

- $\mathfrak{M}, w \Vdash \bot$ never,
- $\mathfrak{M}, w \Vdash p$ iff $w \in V(p)$,
- $\mathfrak{M}, w \Vdash \varphi \vee \psi$ iff $\mathfrak{M}, w \Vdash \varphi$ or $\mathfrak{M}, w \Vdash \psi$,
- $\mathfrak{M}, w \Vdash \neg\varphi$ iff $\mathfrak{M}, w \nVdash \varphi$,
- $\mathfrak{M}, w \Vdash \Diamond\varphi$ iff $\mathfrak{M}, v \Vdash \varphi$ for some v, wRv.

Crucial property of modal logic: formulas are evaluated “locally”.
Think of formulas as *automata*!

Definition

A formula is *globally* true on $\mathfrak{M}$, $\mathfrak{M} \Vdash \varphi$, if $\mathfrak{M}, w \Vdash \varphi$ for all $w \in W$. A formula is *valid* on $\mathfrak{F}$ if $\mathfrak{F}, V \Vdash \varphi$ for all valuations V.

Similarity types

Definition

A similarity type is a pair (O, τ) where O is a set of *operator symbols* and $\tau : O \to \omega$.

Modal language for (O, τ)

$$p \mid \bot \mid \varphi \vee \varphi \mid \neg\varphi \mid \Delta(\varphi_1, ..., \varphi_n)$$

where $\Delta \in O$, $\tau(\Delta) = n$.

$$\nabla(\varphi_1, ..., \varphi_n) = \neg\Delta(\neg\varphi_1, ..., \neg\varphi_n)$$

Definition

A frame for similarity type (O, τ) is a pair $(W, \{R_\Delta\}_{\Delta \in O})$ where each R_Δ is a $\tau(\Delta) + 1$-ary relation on W.

Definition

Model = frame plus valuation, same as before.

Truth clauses:

- $\mathfrak{M}, w \Vdash \bot$ never,
- $\mathfrak{M}, w \Vdash p$ iff $w \in V(p)$,
- $\mathfrak{M}, w \Vdash \varphi \vee \psi$ iff $\mathfrak{M}, w \Vdash \varphi$ or $\mathfrak{M}, w \Vdash \psi$,
- $\mathfrak{M}, w \Vdash \neg\varphi$ iff $\mathfrak{M}, w \nVdash \varphi$,
- $\mathfrak{M}, w \Vdash \Delta(\varphi_1, \ldots, \varphi_n)$ iff

$$\mathfrak{M}, v_1 \Vdash \varphi_1 \ \&\ldots\&\ \mathfrak{M}, v_n \Vdash \varphi_n$$

for some $v_1, \ldots, v_n$, $Rwv_1 \ldots v_n$.

Truth clause for ∇?

Basic temporal language

$O = \{\mathsf{F}, \mathsf{P}\}$, both unary. Duals: G, H.

- Will be the case in the Future.
- Was the case in the Past.
- Always Going to be the case.
- Always Has been the case.

Suitable frames for the basic temporal language

- $(\mathbb{N}, <)$: time is *discrete* and has a *beginning* but *no end*.
- $(\mathbb{Z}, <)$: time is *discrete* and *endless* in both directions.
- $(\mathbb{Q}, <)$: time is *dense*.
- $(\mathbb{R}, <)$: time is *Dedekind complete* (has no "gaps").

Temporal formulas:

- $p \to \mathrm{GP}p$ (all)
- $p \to \mathrm{HF}p$ (all)
- $\mathrm{H}p \to \mathrm{P}p$ (not $\mathbb{N}$)
- $\mathrm{F}p \to \mathrm{FF}p$ (not $\mathbb{N}$ or $\mathbb{Z}$)

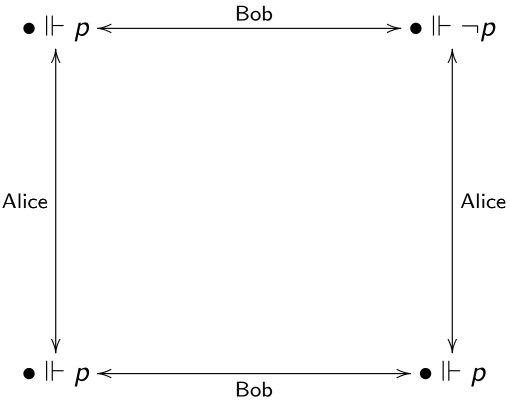
Epistemic logic

$O = \{\mathsf{K}_a\}_{a \in \mathcal{A}}$, all operators unary. $\mathsf{K}_a\varphi$ read "a knows that φ".

Traditional frames: *equivalence relations*.

Equivalently, *partitions* of spaces of possible situations.

A model of epistemic logic



Some epistemic formulas:

- $\mathsf{K}_a\varphi \to \varphi$ (valid)
- $\mathsf{K}_a\varphi \to \mathsf{K}_b\mathsf{K}_a\varphi$ (not valid)
- $\mathsf{K}_a\varphi \to \mathsf{K}_a\mathsf{K}_a\varphi$ (valid)
- $\neg\mathsf{K}_a\varphi \to \mathsf{K}_a\neg\mathsf{K}_a\varphi$ (valid)

There is room for philosophical discussion here! Which formulas *should* be valid on reasonable models of knowledge?

A problematic example

Traditional analysis:

Knowledge = true, justified belief.

Example

Bob is a member of the Flat Earth Society. It is not *true* that the Earth is flat and he is not *justified* in believing so. Hence he doesn't know it. But he doesn't believe that he doesn't know it, so he doesn't *know that he doesn't know* either!

Propositional dynamic logic of regular programs

O consists of terms in the language of *regular expressions* built over a set of atoms A:

$$a \mid \pi;\pi \mid \pi \cup \pi \mid \pi^*$$

Each term is viewed as a unary operator.

Diamond $\Diamond_\pi$ usually written as $\langle \pi \rangle$.

Reading of the box in PDL

Floyd-Hoare logic

$$\{\varphi\}\pi\{\psi\}$$

"If pre-condition φ holds, then after execution of the program π the post-condition ψ will hold".

In PDL:

$$\varphi \to [\pi]\psi$$

Important PDL formulas

The following should be valid according to the intended interpretation:

- $\langle \pi_1 ; \pi_2 \rangle p \leftrightarrow \langle \pi_1 \rangle \langle \pi_2 \rangle p$
- $\langle \pi_1 \cup \pi_2 \rangle p \leftrightarrow (\langle \pi_1 \rangle p \vee \langle \pi_2 \rangle p)$
- $\langle \pi^* \rangle p \leftrightarrow (p \vee \langle \pi ; \pi^* \rangle p)$ (fixed point formula)

Segerberg's induction axiom

$$[\pi^*](p \to [\pi]p) \to (p \to [\pi^*]p)$$

More program constructors

- Test ($\varphi?$): stay in the same state if φ is true, otherwise terminate with no output.
- Intersection: $(\pi_1 \cap \pi_2)$: run both programs π_1 and π_2 in parallell.

Examples:

$$(p?; a) \cup (\neg p?; b) \quad = \quad \texttt{if } p \texttt{ do } a \texttt{ else } b$$

$$(p?; a)^*; \neg p? \quad = \quad \texttt{while } p \texttt{ do } a$$

Arrow logic

Many structures involve objects that can be visualized as "arrows", that may have a *direction* in some sense and can be *composed*.

- Linear algebra (vectors, composition = vector addition)
- Category theory (morphisms, composition primitive)
- Formal language theory (words, composition = concatenation)
- Programs (composition = execution in sequence)

Language of arrow logic

$O = \{C, I, \otimes\}$: ternary, nullary, binary.

- $R_C uvw$: u is the composition of v and w
- $R_I u$: u is an "identity arrow" (zero vector, identity morphism, empty word, "skip" program...)
- $R_\otimes uv$: u is a converse of v

Some formulas of arrow logic

- $C(I, p) \to p$
- $C(p, I) \to p$
- $I \leftrightarrow \otimes I$

Simple frames are *Cartesian squares*.

Validity and frame constraints

Given a reading of the box and diamond operators, some principles are more reasonable than others. On the semantic side, some frames are compatible with the intended reading while others are not. These two aspects are tightly related, and sets us on the path to *correspondence theory*.

Temporal logic and bi-directional frames

$$\mathsf{P}p \to \mathsf{GP}p$$

$$\mathsf{F}p \to \mathsf{HF}p$$

Valid on frames satisfying:

$$\forall x \forall y (Rxy \leftrightarrow Ryx)$$

Other constraints on temporal frames

- $\mathsf{FF}p \to \mathsf{F}p$ ($\mathsf{PP}p \to \mathsf{P}p$). Valid on *transitive* temporal frames ("later than" and "earlier than" are transitive).
- $\mathsf{F}p \to \mathsf{FF}p$ ($\mathsf{P}p \to \mathsf{PP}p$). Valid on *dense* temporal frames.
- $(\mathsf{F}p \wedge \mathsf{F}q) \to (\mathsf{F}(p \wedge \mathsf{F}q)) \vee \mathsf{F}(q \wedge \mathsf{F}p) \vee \mathsf{F}(p \wedge q))$. Valid on temporal frames on which, for all x, y: $xR_{\mathsf{F}}y \vee yR_{\mathsf{F}}x \vee x = y$.

Note

We normally think of time as *continuous* and *deterministic*. But in computer science, "time" is frequently modelled as *discrete* and *branching*.

Regular frames for PDL

$\langle \pi_1 \cup \pi_2 \rangle p \leftrightarrow (\langle \pi_1 \rangle p \vee \langle \pi_2 \rangle p)$

$$R_{\pi_1 \cup \pi_2} = R_{\pi_1} \cup R_{\pi_2}$$

$\langle \pi_1; \pi_2 \rangle p \leftrightarrow (\langle \pi_1 \rangle \langle \pi_2 \rangle p)$

$$\forall x \forall y (R_{\pi_1;\pi_2} \leftrightarrow \exists z (R_{\pi_1} xz \wedge R_{\pi_2} zy))$$

Fixed point formula & induction axiom

$$R_{\pi^*} = R_\pi^*$$

Here, R^* is the *reflexive transitive closure* of R:

$$R^* = \bigcap\{R' \mid R' \text{ ref. trans. } \& \ R \subseteq R'\}$$

A *model* constraint for Test

$\langle \varphi ? \rangle p \leftrightarrow \varphi \wedge p$

$$R_{\varphi ?} = \{(w, w) \mid w \Vdash \varphi\}$$

We haven't mentioned *logic* or *consequence* yet...

Normal modal logics

Definition

A *normal modal logic* is a set $\mathfrak{L}$ of formulas closed under:

- Modus ponens: $\frac{\varphi \quad \varphi \to \psi}{\psi}$
- Uniform substitution: $\frac{\varphi}{\varphi[\sigma]}$
- Necessitation: $\frac{\varphi}{\Box\varphi}$

and containing the following formulas:

- The K-axiom: $\Box(p \to q) \to (\Box p \to \Box q)$
- Dual: $\Diamond p \leftrightarrow \neg\Box\neg p$

Proposition

For every class F *of frames, there is a normal modal logic* Λ_{F} *given by:*

$$\varphi \in \Lambda_{\mathsf{F}} \quad \Leftrightarrow \quad \forall \mathfrak{F} \in \mathsf{F} : \mathfrak{F} \Vdash \varphi$$

Observe that:

$$\mathsf{K}_1 \subseteq \mathsf{K}_2 \quad \Rightarrow \quad \Lambda_{\mathsf{K}_2} \subseteq \Lambda_{\mathsf{K}_1}$$

The minimal modal logic

The logic $\Lambda_{[\text{All frames}]}$ is the smallest normal modal logic, and denoted by **K**...

... after Saul **K**ripke.

Modal consequence

Definition

A formula φ is a *local consequence* of formulas Γ if, for any model $\mathfrak{M}$ and $w \in W$:

$$\mathfrak{M}, w \Vdash \Gamma \text{ implies } \mathfrak{M}, w \Vdash \varphi$$

Notation: $\Gamma \vDash_l \varphi$.

Definition

A formula φ is a *global consequence* of formulas Γ if, for any model $\mathfrak{M}$:

$$\mathfrak{M} \Vdash \Gamma \text{ implies } \mathfrak{M} \Vdash \varphi$$

Notation: $\Gamma \vDash_g \varphi$.