INTRODUCTION TO MODAL LOGIC.
FINAL EXAM

17 DECEMBER 2015
UNIVERSITY OF AMSTERDAM

Name:

UvA Student ID:

General comments.

1. The time for this exam is 3 hours (180 minutes).
2. There are 100 points in the exam.
3. Please mark the answers to the questions in Exercise 1 on this sheet by crosses.
4. Each multiple-choice question is worth 2pt only if answered correctly, no negative points will be given.
5. You can choose between Exercises 5 and $5'$. Each of these exercises is worth 20pt. If you solve both exercises you will get best of the two marks.
6. Make sure that you have your name and student ID on each of the sheets you are handing in.
7. If you have any questions, please indicate this silently and someone will come to you. Answers to questions that are relevant for anyone will be announced publicly.
8. No talking during the exam.
9. Cell phones must be switched off and stowed.

	Maximal Points	Your points
Exercise 1	20	
Exercise 2	10	
Exercise 3	20	
Exercise 4	20	
Exercise 5 or $5'$	20	
Exercise 6	10	
Total	100	

(1) (20pt) Please mark the answers to this question by crosses.

(a) One of the following statements is false. Which one?

☐ A generated subframe of a transitive frame is transitive
☐ A generated subframe of a converse well-founded frame is converse well-founded
☐ A generated subframe of a directed frame is directed
☐ A generated subframe of a rooted frame is rooted

(b) Suppose Σ is a set of **PDL**-formulas. One of the following statements is false. Which one?

☐ 1. $\langle\pi;\sigma\rangle\varphi \in \neg FL(\Sigma) \Rightarrow \langle\pi\rangle\langle\sigma\rangle\varphi \in \neg FL(\Sigma)$
☐ 2. $\varphi \in \neg FL(\Sigma) \Rightarrow \neg\varphi \in \neg FL(\Sigma)$
☐ 3. $\neg\varphi \in \neg FL(\Sigma) \Rightarrow \varphi \in \neg FL(\Sigma)$
☐ 4. $\langle\pi^*\rangle\varphi \in \neg FL(\Sigma) \Rightarrow \langle\pi\rangle\langle\pi^*\rangle\varphi \in \neg FL(\Sigma)$

(c) Ler $\mathfrak{F}$ and $\mathfrak{G}$ be arbitrary Kripke frames and $f : \mathfrak{F} \to \mathfrak{G}$ a surjective bounded morphism. One of the following statements is false. Which one?

☐ If $\mathfrak{F}$ is symmetric, then $\mathfrak{G}$ is symmetric
☐ If $\mathfrak{F}$ is reflexive, then $\mathfrak{G}$ is reflexive
☐ if $\mathfrak{F}$ is rooted, then $\mathfrak{G}$ is rooted
☐ If $\mathfrak{F}$ is irreflexive, then $\mathfrak{G}$ is irreflexive

(d) Ler $\mathfrak{F}$ and $\mathfrak{G}$ be arbitrary Kripke frames and $f : \mathfrak{F} \to \mathfrak{G}$ a surjective bounded morphism. One of the following statements is true. Which one?

☐ $Log(\mathfrak{F}) \subseteq Log(\mathfrak{G})$
☐ $Log(\mathfrak{G}) \subseteq Log(\mathfrak{F})$
☐ $Log(\mathfrak{F}) \neq Log(\mathfrak{G})$
☐ $Log(\mathfrak{F}) \subsetneq Log(\mathfrak{G})$

(e) One of the following statements is true. Which one? Lindenbaum's lemma states that

☐ Every set of formulas can be extended to a maximal consistent set
☐ Every set of formulas can be extended to a consistent set
☐ Every consistent set of formulas can be extended to a maximal consistent set
☐ An inconsistent set of formulas can be extended to a consistent set

(f) Recall that $\mathbf{KD} = \mathbf{K} + \Diamond\top$. One of the following statements is true. Which one?

☐ 1. $\vdash_{\mathbf{KD}} \Diamond\Box\bot$
☐ 2. $\vdash_{\mathbf{KD}} \Diamond p \to \Box p$
☐ 3. $\vdash_{\mathbf{KD}} \Box p \to \Diamond p$
☐ 4. $\vdash_{\mathbf{KD}} \Box(\Box p \to p)$

(g) Let $\mathfrak{F} = (W, R)$ be Kripke frame and $\mathfrak{F}_A = (W, R, A)$ a general frame with $A \subseteq \mathcal{P}(W)$. One of the following statements is false. Which one?

- ☐ If a modal formula φ is valid in $\mathfrak{F}$, then φ is valid in $\mathfrak{F}_A$
- ☐ If a modal formula φ is valid in $\mathfrak{F}_A$, then φ is valid in $\mathfrak{F}$
- ☐ $Log(\mathfrak{F})$ is a normal modal logic
- ☐ $Log(\mathfrak{F}_A)$ is a normal modal logic

(h) One of the following statements is false. Which one?

- ☐ **S4.3** is complete
- ☐ $\mathbf{K_t ThoM}$ is complete
- ☐ **K4.2** is complete
- ☐ **S5** is complete

(i) One of the following statements is true. Which one?

- ☐ Every modal logic with the finite model property is canonical
- ☐ Every modal logic with the finite model property is complete
- ☐ Every modal logic with the finite model property is decidable
- ☐ Every modal logic is finitely axiomatizable

(j) One of the following formulas is not valid on frames that are reflexive and transitive. Which one?

- ☐ 1. $\Diamond\Diamond p \to (\Diamond p \vee p)$
- ☐ 2. $\Box\Diamond p \to \Diamond p$
- ☐ 3. $\Diamond\Box p \to \Diamond p$
- ☐ 4. $\Diamond p \to \Box\Diamond p$

(2) (10pt) A Kripke frame (W, R) is 2-*cyclic* if for each $x \in W$ there exists $y \in W$ such that $x \neq y$ and xRy and yRx. Show that the class of 2-cyclic frames is not modally definable.

(3) (20pt) Use the Sahlqvist algorithm to compute the first-order correspondent of the formula:

$$\Diamond\Box p \to \Box\Box p.$$

(4) (20pt) Let $\mathcal{M} = (W, \{R_\pi\}_{\pi\in\Pi}, V)$ and $\mathcal{M}' = (W', \{R'_\pi\}_{\pi\in\Pi}, V')$ be two regular models of **PDL**. Let $Z \subseteq W \times W'$ be a bisimulation for each R_a and R'_a, where a is a basic program. Show by induction on the complexity of programs that then Z is a bisimulation for each R_π and R'_π, where $\pi \in \Pi$ is any program.

You can choose between Exercises 5 and 5′. Each of these exercises is worth 20pt. If you solve both exercises you will get best of the two marks.

(5) (20pt)

(a) Show that the canonical model for the modal logic

$$\mathbf{KF} = \mathbf{K} + (\Diamond p \to \Box p)$$

is functional. Recall that a Kripke frame is *functional* if

$$\forall x\forall y\forall z((Rxy \wedge Rxz) \to (y = z)).$$

You are not allowed to use Sahlqvist completeness theorem.

(b) Using (a) show that **KF** is sound and complete with respect to functional frames.

(5′) (20pt)

(a) Show, using filtration, that $\mathbf{KB} = \mathbf{K} + (\Diamond\Box p \to p)$ has the finite model property. You can assume that **KB** is sound and complete with respect to its Kripke frames.

(b) Deduce that **KB** is decidable.

(6) (10pt) Assume that modal formulas are built from propositional variables and $\bot$ by using $\vee$, $\neg$ and $\Diamond$. Suppose a modal formula φ is satisfiable in a model $\mathfrak{M} = (W, R, V)$ such that (W, R) is an **S5**-frame (i.e., R is an equivalence relation). Show that then φ is satisfiable in a model $\mathfrak{N} = (W', R', V')$ such that (W', R') is an **S5**-frame and the cardinality of W' is bounded by $|\varphi| + 1$, where $|\varphi|$ is the number of $\Diamond$'s occurring in φ.