INTRODUCTION TO MODAL LOGIC.
FINAL EXAM

15 DECEMBER 2014
UNIVERSITY OF AMSTERDAM

Name:

UvA Student ID:

General comments.

1. The time for this exam is 3 hours (180 minutes).
2. There are 100 points in the exam.
3. Please mark the answers to the questions in Exercise 1 on this sheet by crosses.
4. Each multiple-choice question is worth 2pt only if answered correctly, no negative points will be given.
5. You can choose between Exercises 6 and 7. Each of these exercises is worth 20pt. If you solve both exercises you will get best of the two marks.
6. Make sure that you have your name and student ID on each of the sheets you are handing in.
7. If you have any questions, please indicate this silently and someone will come to you. Answers to questions that are relevant for anyone will be announced publicly.
8. No talking during the exam.
9. Cell phones must be switched off and stowed.

	Maximal Points	Your points
Exercise 1	20	
Exercise 2	20	
Exercise 3	10	
Exercise 4	10	
Exercise 5	20	
Exercise 6 or 7	20	
Total	100	

(1) (20pt) Please mark the answers to this question by crosses.

(a) One of the following statements is true. Which one? The basic modal logic **K** is sound and complete with respect to the class of

☐ symmetric frames
☐ directed frames
☐ rooted frames
☐ a frame consisting of the set of natural numbers with strict order

(b) One of the following statements is true. Which one? A modal formula φ is valid in a frame (W, R) if

☐ for each valuation V there is a point $w \in W$ with $(W, R, V), w \models \varphi$
☐ there is a valuation V and there is a point $w \in W$ with $(W, R, V), w \models \varphi$
☐ for each valuation V and for each point $w \in W$, $(W, R, V), w \models \varphi$
☐ there is a valuation V such that for each point $w \in W$, $(W, R, V), w \models \varphi$

(c) Let $\mathcal{C}$ be a non-empty class of Kripke frames. Then $\mathcal{C}$ is modally definable if one of the following statements is true. Which one?

☐ $\mathcal{C} \supseteq Fr(Log(\mathcal{C}))$
☐ $\mathcal{C} \subseteq Fr(Log(\mathcal{C}))$
☐ $Log(\mathcal{C}) = Log(Fr(Log(\mathcal{C})))$
☐ $Fr(Log(\mathcal{C})) \neq \emptyset$

(d) Ler $\mathfrak{F}$ and $\mathfrak{G}$ be arbitrary Kripke frames and $f : \mathfrak{F} \to \mathfrak{G}$ a surjective bounded morphism. One of the following statements is true. Which one? For any modal formula φ

☐ $\mathfrak{G} \models \varphi \Rightarrow \mathfrak{F} \models \varphi$
☐ $\mathfrak{F} \models \varphi \Rightarrow \mathfrak{G} \models \varphi$
☐ $\mathfrak{G} \models \neg\varphi \Rightarrow \mathfrak{F} \models \neg\varphi$
☐ $\mathfrak{F}$ is infinite, then $\mathfrak{G}$ is infinite

(e) Let Γ be a maximal consistent set of formulas. One of the following statements is false. Which one?

☐ Γ is infinite
☐ For each formula φ we have $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$
☐ For each formula φ we have $\varphi \in \Gamma$ and $\neg\varphi \in \Gamma$
☐ For each formula φ and ψ, $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$

(f) One of the following statements is true. Which one? The class $\mathcal{C}$ is not modally definable in the basic modal language if $\mathcal{C}$ consists of

☐ transitive frames with no infinite descending chain
☐ transitive frames with no infinite ascending chain
☐ symmetric frames
☐ transitive frames

(g) One of the following statements is false. Which one?

- ☐ The formula $(\Diamond p \wedge \Box p) \to p$ is a Sahlqvist formula
- ☐ The formula $\Box(\Box p \to p)$ is a Sahlqvist formula
- ☐ The formula $(\Diamond\Box p \wedge \Diamond q) \to \Box\Diamond p$ is a Sahlqvist formula
- ☐ The formula $\Box\Diamond p \to q$ is a Sahlqvist formula

(h) One of the following statements is false. Which one?

- ☐ **K4** has the finite model property
- ☐ **PDL** has the finite model property
- ☐ **K** has the finite model property
- ☐ $\mathbf{K_t ThoM}$ has the finite model property

(i) One of the following statements is true. Which one?

- ☐ Every canonical modal logic is complete
- ☐ Every complete modal logic is canonical
- ☐ Every sound and complete modal logic has the finite model property
- ☐ Every finitely axiomatizable modal logic is decidable

(j) One of the following statements is false. Which one?

- ☐ $\vdash_{\mathbf{K}} \Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$
- ☐ $\vdash_{\mathbf{K}} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- ☐ $\vdash_{\mathbf{K}} \Box(\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$
- ☐ $\vdash_{\mathbf{K}} \Diamond(\varphi \wedge \psi) \to (\Diamond\varphi \wedge \Diamond\psi)$

(2) (20pt) Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be two models.

(a) Define when is a binary relation $Z \subseteq W \times W'$ a bisimulation. Define when points $w \in W$ and $w' \in W'$ are bisimilar.

(b) One can prove by induction on the complexity of modal formulas that if w and w' are bisimilar, then they satisfy the same modal formulas. Prove the $\Diamond$-step of this induction.

(c) Formulate the Hennessy-Milner Theorem and give its proof.

(3) (10pt) Use the Sahlqvist algorithm to compute the first-order correspondent of the formula:

$$\Box\Box p \to \Diamond p.$$

(4) (10pt) Prove that for any modal formulas φ and ψ we have

$$\vdash_{\mathbf{K}} \Box\varphi \vee \Box\psi \text{ implies } \vdash_{\mathbf{K}} \varphi \text{ or } \vdash_{\mathbf{K}} \psi.$$

(Hint: use completeness of **K** with respect to Kripke frames.)

(5) (20pt)

(a) Define regular frames for **PDL**.

(b) Show that **PDL** is not compact. That is, show that there exists an infinite set of formulas Σ such that every finite subset of Σ is satisfiable on a regular frame and Σ itself is not satisfiable on a regular frame.

You can choose between Exercises 6 and 7. Each of these exercises is worth 20pt. If you solve both exercises you will get best of the two marks.

(6) (20pt) In the following exercise you can use that the canonical model for **S4.3** is reflexive and transitive.

(a) Show that the canonical model for the modal logic

$$\mathbf{S4.3} = \mathbf{S4} + \Box(\Box p \to q) \vee \Box(\Box q \to p)$$

has no branching to the right. Recall that a reflexive Kripke frame has no branching to the right if

$$\forall x \forall y \forall z((Rxy \wedge Rxz) \to (Ryz \vee Rzy)).$$

You are not allowed to use Sahlqvist completeness theorem.

(b) Deduce that **S4.3** is sound and complete with respect to reflexive transitive frames with no branching to the right. (You can assume soundness of **S4.3** with respect to these frames.)

(7) (20pt)

(a) Let Σ be a finite subformula closed set. Let $\mathfrak{M} = (W, R, V)$ be a model such that (W, R) is a rooted transitive reflexive frame with no branching to the right. Show that a transitive filtration of $\mathfrak{M}$ through Σ is a rooted reflexive transitive frame with no branching to the right. See (6) for the definition of no branching to the right. (Hint: start by showing that if r is a root of $\mathfrak{M}$, then $[r]$ is a root of the filtrated model $\mathfrak{M}_\Sigma$.)

(b) Deduce that **S4.3** has the finite model property. See (6) for the definition of **S4.3**.

(c) Deduce that **S4.3** is decidable.