EXERCISE CLASS 18-11-2016: EVEN MORE ON SOUNDNESS AND COMPLETENESS

(1) Prove that

(a) The logic $\mathbf{S4} = \mathbf{K} + (\Box p \to p, \Box p \to \Box\Box p)$ is sound and complete with respect to the class of pre-ordered, reflexive and transitive, frames.

(b) The logic $\mathbf{KD} = \mathbf{K} + (\Diamond \top)$ is sound and complete with respect to the class of serial frames, i.e. frames (W, R) satisfying $\forall w \exists v\, wRv$.

(c) The logic $\mathbf{K4.2} = \mathbf{K4} + (\Diamond\Box p \to \Box\Diamond p)$ is sound and complete with respect to the class of transitive directed frames. Recall that a frame $\mathcal{F} = (W, R)$ is directed if it satisfies

$$\forall w, v, v' \in W (wRv \,\&\, wRv' \implies \exists u \in W (vRu \,\&\, v'Ru).$$

(*Hint. You might find it useful to use, together with the definition of R in terms of $\Box$'s the one in terms of $\Diamond$'s.*)

(d) The logic $\mathbf{Den} = \mathbf{K} + (\Box\Box p \to \Box p)$ is sound and complete with respect to the class of dense frames.

You are not allowed to use Sahlqvist completeness theorem.

(2) (BdRV 4.4.5) Call $Log(\mathcal{C})$ the set of valid formulas on some class of frames $\mathcal{C}$. Call $Fr(L)$ the class of frames on which the formulas in L are valid. Show that the two operations form a Galois connection, namely: for all $\mathcal{C}$ and L

$$L \subseteq Log(\mathcal{C}) \quad \text{iff} \quad \mathcal{C} \subseteq Fr(L).$$

Conclude that

$$L \subseteq Log(Fr(L)) \quad \text{and} \quad \mathcal{C} \subseteq Fr(Log(\mathcal{C})),$$

for any normal modal logic L and any class of Kripke frames $\mathcal{C}$ and therefore that any normal modal logic L will be sound with respect to the class of Kripke frames $Fr(L)$.

1. ADDITIONAL EXERCISES

(3) (BdRV 4.4.3) Show that for any consistent normal modal logic L in the language of basic modal logic $Fr(L) \neq \emptyset$. Conclude that any such consistent normal modal logic is sound with respect to some non-empty class of frames. (*Hint: It might be helpful to show that the normal modal logics $Log(\circ) = \mathbf{K} + p \leftrightarrow \Box p$ and $Log(\bullet) = \mathbf{K} + \Box\bot$ are maximal among normal modal logics in the language of basic modal logic. Use this to show that either $L \subseteq Log(\bullet)$ or $L \subseteq Log(\circ)$*)