EXERCISE CLASS 11-11-2016: CANONICAL MODELS

(1) Prove that if Γ is satisfiable then it is consistent.

(2) Let L be a normal modal logic. Given a world w in an L-model $\mathfrak{M}$, show that the set $\{\varphi\colon M, w \Vdash \varphi\}$ is an L-MCS.

(3) Let L be a normal modal logic and define a relation R'' on the canonical model for L by

$$R''(\Gamma, \Delta) \quad \text{iff} \quad \forall\varphi(\varphi \in \Delta \implies \Diamond\varphi \in \Gamma)$$

where Γ and Δ are L-MCSs. Show that $R'' = R'$, where R' is the relation

$$R'(\Gamma, \Delta) \quad \text{iff} \quad \forall\varphi(\Box\varphi \in \Gamma \implies \varphi \in \Delta),$$

where Γ and Δ are L-MCSs. Thus we may define the canonical relation R^L for any L as either R' or R''.

(4) Let $\Gamma := \{p, q, p \wedge q, \Box p, \Box q, \Box(p \wedge q)\}$, $\Delta := \{p, \neg q, \Box p\}$, and $\Delta' := \{\Box p, \Box q, \Box(p \wedge q)\}$ be sets of formulas.
  (a) Are these sets maximal consistent (in some language)?
  (b) Let the relation R' on $\{\Gamma, \Delta, \Delta'\}$ and the assignment h' on $\{\Gamma, \Delta, \Delta'\}$ be as defined on the canonical model. Draw the resulting Kripke model.

(5) Show that in the canonical model for **K** (or any other normal modal logic L) there exist (L-)MCSs Γ and Δ that are incomparable (i.e., we have neither $R^L(\Gamma, \Delta)$ nor $R^L(\Delta, \Gamma)$).

(6) Let L be a normal modal logic and let Γ be an L-MCS. Show that
  (i) If $\varphi \in \Gamma$ and $\varphi \to \psi \in \Gamma$ then $\psi \in \Gamma$;
  (ii) $L \subseteq \Gamma$;
  (iii) For every formula φ either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$;
  (iv) For every pair of formulas φ and ψ we have that $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$;

Additional exercises

(7) Let $\mathfrak{M} = (W, R, V)$ be a Kripke model we say that
  (i) The Kripke model $\mathfrak{M}$ is *tight* if

$$\forall w, w' \in W((\{\varphi\colon \mathfrak{M}, w \Vdash \Box\varphi\} \subseteq \{\varphi\colon \mathfrak{M}, w' \Vdash \varphi\}) \implies wRw');$$

  (ii) The Kripke model $\mathfrak{M}$ is *differentiated* if

$$\forall w, w' \in W((\{\varphi\colon \mathfrak{M}, w \Vdash \varphi\} = \{\varphi\colon \mathfrak{M}, w' \Vdash \varphi\}) \implies w = w');$$

  (iii) The Kripke model $\mathfrak{M}$ is *compact* if for every set of formulas Σ have that

$$\exists w(\mathfrak{M}, w \Vdash \Sigma) \quad \text{iff} \quad \forall \Sigma_0 \subseteq_\omega \Sigma \exists w(\mathfrak{M}, w \Vdash \Sigma_0)$$

  (iv) The Kripke model $\mathfrak{M}$ is *refined* if it is both tight and differentiated.

Let L be a normal modal logic show that the canonical model $\mathfrak{M}^L$ for L is a refined and compact Kripke model.

(8) (*For those that know a bit of topology:*) Let L be a normal modal logic and let $\mathfrak{M}^L$ be the canonical model for L. Show that the collection of set

$$V^L(\varphi) = \{\Gamma \in W^L : \mathfrak{M}^L, \Gamma \Vdash \varphi\},$$

with φ ranging over the set of formulas in the language of basic modal logic, generates a topology on the set W^L which is compact, Hausdorff and zero-dimensional.