Exercise class 4-11-2016:
Soundness and Hilbert systems

(1) Let $\mathcal{C}$ be a class of frames. Show that

$$\mathsf{Log}(\mathcal{C}) := \{\varphi \colon \forall \mathfrak{F} \in \mathcal{C}\ (\mathfrak{F} \Vdash \varphi)\}$$

is a normal modal logic. Conclude that **K** is sound w.r.t. the class of all frames.

Let $\mathcal{C}_{mod}$ be a class of Kripke models. Is the set of formulas

$$\mathsf{Th}(\mathcal{C}_{mod}) := \{\phi \colon \forall \mathfrak{M} \in \mathcal{C}_{mod}\ (\mathfrak{M} \Vdash \phi)\}$$

a normal modal logic?

(2) Let Σ be a set of formulas. Show that:

- If $\varphi \to \psi$ is a propositional tautology, then $\vdash_\Sigma \varphi$ implies $\vdash_\Sigma \psi$.
- If $\vdash_\Sigma \varphi$ and $\vdash_\Sigma \psi$ then $\vdash_\Sigma \varphi \wedge \psi$.
- If $\vdash_\Sigma \varphi \to \psi$ and $\vdash_\Sigma \psi \to \chi$ then $\vdash_\Sigma \varphi \to \chi$.
- If $\vdash_\Sigma \varphi \to \psi$ and $\vdash_\Sigma \varphi' \to \psi'$ then $\vdash_\Sigma (\varphi \wedge \varphi') \to (\psi \wedge \psi')$

(3) Let Σ be a set of formulas. Prove that

- $\vdash_\Sigma \varphi \to \psi$ implies $\vdash_\Sigma \Box\varphi \to \Box\psi$
- $\vdash_\Sigma \varphi \to \psi$ implies $\vdash_\Sigma \Diamond\varphi \to \Diamond\psi$
- $\vdash_\Sigma \Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$
- $\vdash_\Sigma \Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$

(4) (Equivalent replacement). Let $\varphi[\psi]$ be a formula that contains ψ as a subformula. Let $\varphi[\chi]$ denote the formula where ψ in $\varphi[\psi]$ is replaced with the formula χ. Let Σ be a set of formulas. Show that

$$\vdash_\Sigma \psi \leftrightarrow \chi \quad \text{implies} \quad \vdash_\Sigma \varphi[\psi] \leftrightarrow \varphi[\chi].$$

(5) Show that $\not\vdash_{\mathbf{S4}} p \to \Box\Diamond p$ and that $\not\vdash_{\mathbf{K}} \Box p \vee \Box\neg p$.

(6) A normal modal logic L is *Halldén complete* if for every pair of formulas ϕ and ψ with no common variables we have that

$$\vdash_L \phi \vee \psi \quad \text{implies} \quad \vdash_L \phi \,\text{or} \vdash_L \psi.$$

Is the normal modal logic **K** Halldén complete? Give proof or counterexample.

(7) This exercise is for those of you who like syntactical manipulations.

$$\mathbf{GL} = \mathbf{K} + (\Box(\Box p \to p) \to \Box p).$$

Try to find a **GL**–proof of $\Box p \to \Box\Box p$, i.e., show that $\vdash_{\mathbf{GL}} \Box p \to \Box\Box p$ (As we will see later this can also be shown semantically.)

Hint: Don't try to hard.