EXERCISE CLASS 2-12-2016: FMP AND DECIDABILITY, GENERAL FRAMES, PDL

(1) Recall that **Den** is the normal modal logic $\mathbf{K} + (\Box\Box p \to \Box p)$
   (a) Prove that the logic **Den** is sound and complete with respect to the class of dense Kripke frames.
   (b) Show that the logic **Den** has the finite model property and is decidable.

(2) Let φ be your favourite formula in the language of basic modal logic, not equivalent to falsum. Construct a general frame $\mathfrak{f} = (\mathfrak{F}, A)$ such that $\mathfrak{F} \not\Vdash \varphi$ but $\mathfrak{f} \Vdash \varphi$.

(3) Completeness with respect to general frames
   (a) Let $\mathfrak{M} = (\mathfrak{F}, V)$ be Kripke model and let $A_{\mathfrak{M}} := \{V(\varphi) \colon \varphi \in Form(\tau, \Phi)\}$. Show that $\mathfrak{f}_{\mathfrak{M}} := (\mathfrak{F}, A_{\mathfrak{M}})$ is a general frame.
   (b) Let L be a consistent normal modal logic and let $\mathfrak{f}^L$ denote the general frame $\mathfrak{f}_{\mathfrak{M}^L}$, where $\mathfrak{M}^L$ is the canonical model for L. Show that $\mathfrak{f}^L \Vdash L$.
   (c) Conclude that any consistent normal modal logic is sound and (strongly) complete with respect to some class of general frames.

(4) Show that
   (a) $\langle \pi_1; \pi_2 \rangle p \leftrightarrow \langle \pi_1 \rangle \langle \pi_2 \rangle p$ is valid on a frame $(W, R_{\pi_1}, R_{\pi_2})$ iff $R_{\pi_1;\pi_2} = R_{\pi_1} \circ R_{\pi_2}$.
   (b) $\langle \pi_1 \cup \pi_2 \rangle p \leftrightarrow \langle \pi_1 \rangle p \vee \langle \pi_2 \rangle p$ is valid on a frame $(W, R_{\pi_1}, R_{\pi_2})$ iff $R_{\pi_1 \cup \pi_2} = R_{\pi_1} \cup R_{\pi_2}$.
   (c) $p \vee \langle \pi \rangle \langle \pi^* \rangle p \to \langle \pi^* \rangle p$ is valid on a frame (W, R_π, R_{π^*}) iff $(R_\pi)^* \subseteq R_{\pi^*}$.

1. Additional exercise

(1) A logic L is *compact for the class $\mathcal{C}$ of Kripke frames* if the following condition is met: for every set of formulas Σ, if every finite subset of Σ is satisfiable in a model based on a frame in $\mathcal{C}$, then Σ itself can be satisfied in a model based on a frame in $\mathcal{C}$. Show that **PDL** is not compact for the class of regular frames.

(2) Prove the properties of atoms listed in Lemma 4.81.

(3) Prove that $At(\Sigma) = \{\Gamma \cap \neg FL(\Sigma) \mid \Gamma \text{ is MCS}\}$.

(4) Let $\Sigma = \{\varphi\}$ be a singleton set (of formulas in the language of **PDL**). Show that $\neg\mathrm{FL}(\Sigma)$ is finite. *Hint: This is not so easy.*