THE VARPAKHOVSKII CALCULUS AND MARKOV ARITHMETIC

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Abstract. The study of propositional realizability logic was initiated in the 50th of the last century. Unfortunately, no description of the class of realizable propositional formulas is found up to now. Nevertheless some attempts of such a description were made. In 1974 the author proved that every known realizable propositional formula has the property that every one of its closed arithmetical instances is deducible in the system obtained by adding Extended Church Thesis and Markov Principle as axiom schemes to Intuitionistic Arithmetic. A. Visser calles this system Markov Arithmetic. In 1990 another attempt of describing the class of realizable propositional formulas was made by F. L. Varpakhovskii who proposed a calculus in an extended propositional language and proved that all known realizable propositional formulas are deducible in this calculus. In this paper we prove that every propositional formula deducible in Varpakhovskii's calculus has the property that every one of its closed arithmetical instances is deducible in Markov Arithmetic.

Contents

§1. Introduction. Recursive realizability as a semantics of arithmetical sentences was introduced by S. C. Kleene [2]. It can be considered as a more precise definition of the informal intuitionistic semantics. From the point of view of intuitionists, a sentence is true if it is proved. Thus the truth of a sentence is connected with its proof. In order to avoid any confusion with formal proofs, we shall use the term "a verification" instead of "a proof". For every true sentence A we can consider its verification as a text justifying A. Such understanding of the meaning of sentences leads to an original interpretation of logical connectives and quantifiers. In particular, if A and B are sentences, then a verification of an implication $A \supset B$ is a text describing a general effective operation for obtaining

Received by the editors February, 2009.
†Partially supported by INTAS (grant 05-1000008-8144) and RFBR (grants 08-01-0039a and 06-01-72554-NCNIL_a)

a verification of B from every verification of A. If $A(x)$ is a predicate with a parameter x over a domain M given in an appropriate way, then a verification of an universal sentence $\forall x\, A(x)$ is a text describing a general effective operation which gives a verification of $A(m)$ for every given $m \in M$. The main idea of Kleene was coding verifications by natural numbers and using the exact notion of a partial recursive function instead of the vague intuitionistic concept of an effective operation. Partial recursive functions are coded by natural numbers by means of the Gödel enumeration. A code of a verification of an arithmetical sentence is called a realization of the sentence. An arithmetical formula is called realizable iff its universal closure has a realization. It was proved by D. Nelson [4] that every formula deducible in intuitionistic arithmetic HA is realizable.

Kleene's semantics was the main base of the constructive approach to mathematics worked out by A. A. Markov and his school. In connection with the development of constructive mathematics it is of interest to study logical principles acceptable from the constructive point of view. In mathematical logic logical principles are expressed by logical formulas. The simplest logical formulas are propositional ones.

There are various notions of a realizable propositional formula. It would be rather natural to define a realizable propositional formula as a formula A such that every one of its arithmetical instances is realizable. But this definition admits several variants. First we can distinguish arbitrary arithmetical instances and only closed ones. Second we can distinguish constructive and nonconstructive treatment of the definition depending on the existence of an algorithm giving a realization for every arithmetical instance. Thus we have a priori four variants of realizability for propositional formulas.

1) A propositional formula is called weakly realizable if every one of its closed arithmetical instances is realizable. It is easy to prove that a propositional formula is weakly realizable iff it is a classical tautology. In particular, the law of excluded middle $p \vee \neg p$ is weakly realizable. Thus this notion of realizability for propositional formulas is not interesting.

2) A propositional formula is called irrefutable if every one of its (not necessary closed) arithmetical instances is realizable. It can be proved that the formula $p \vee \neg p$ is not irrefutable.

3) A propositional formula A is called effectively realizable if there exists an algorithm that for every closed arithmetical instance of A produces a realization for it. Obviously, every effectively realizable formula is irrefutable. Any example of an irrefutable propositional formula which is not effectively realizable is not found until now. On the other hand, it is not known whether these two notions coincide.

4) The fourth notion of realizability for propositional formulas means that for any (not necessary closed) arithmetical instance of a propositional formula we can effectively find a number realizing the universal closure of this instance. It is rather obvious that this notion coincides with the notion of an effectively realizable propositional formula.

It follows from Nelson's results [4] that for every propositional formula A deducible in Intuitionistic Propositional Calculus IPC there exists a common realization for all closed arithmetical instances of A. This observation leads to the following notion.

5) A propositional formula A is called uniformly realizable if there exists a natural number realizing every closed arithmetical instance of A. Evidently, every uniformly realizable formula is effectively realizable, hence irrefutable. No example of effectively realizable propositional formula which is not uniformly realizable is known. However it is open whether these two notions coincide.

A detailed survey of propositional realizability logic can be found in [6].

Every realizable (in any sense) propositional formula can be considered as a constructive logical principle. Thus studying propositional realizability logic was at first important for foundations of mathematics. Later the applications of constructive mathematics in computer science also increased the interest in the subject. The problem of describing the class of realizable propositional formulas is still open. However some attempts of a description of propositional realizability logic were made. In this paper we consider two of them.

The first one was undertaken by the author. Note that the definition of recursive realizability can be formulated as a translation from the language of arithmetic into itself. Namely for any arithmetical formula A, a formula $x \,\mathsf{r}\, A$ with an additional parameter x is defined in such a way that $x \,\mathsf{r}\, A$ means that x is a realization of A (see [2], [4]). The formula $x \,\mathsf{r}\, A$ can be constructed so that it does not contain $\vee$ and contains $\exists$ only in the subformulas of the form $\exists x_1 \ldots \exists x_n \, B$, where B is a quantifier-free formula. Formulas of this kind are called almost negative. The author [5] considered a formal system S obtained from HA by adding the axioms $A \equiv \exists x \, x \,\mathsf{r}\, A$ for any arithmetical formula A and the axiom scheme

$$\forall x \, (A(x) \vee \neg A(x)) \,\&\, \neg\neg\exists x \, A(x) \supset \exists x \, A(x)$$

called Markov Principle MP. It was proved that every known realizable propositional formula has the property that every one of its arithmetical instances is deducible in S.

Extended Church's Thesis ECT is the following scheme of axioms, where $A(x)$ is an almost negative formula, $B(x, y)$ is an arbitrary arithmetical formula, $\{e\}$ is a partial recursive fuction with the Gödel number e:

$$\forall x \, (A(x) \supset \exists y \, B(x, y)) \supset \exists e \, \forall x \, (A(x) \supset \exists y \, (\{e\}(x) = y \,\&\, B(x, y))).$$

The system S is equivalent to the system of "Russian" or "traditional" constructivism (see [1]) HA+MP+ECT. A. Visser [9] calles this system Markov Arithmetic and denotes it by MA.

Another attempt of a description of the realizable propositional formulas was made by F. L. Varpakhovskii [8] who proposed an axiomatic class of uniformly realizable formulas in an extended propositional language. Namely additional connectives called strong implication and conditional disjunction are added to the propositional and arithmetical languages. The notion of recursive realizability is generalized to the new connectives. The notion of an uniformly realizable propositional formula in the extended language is defined in an obvious way. F. L. Varpakhovskii proposed a propositional calculus in the extended language and has proved that any propositional formula deducible in this calculus is uniformly realizable. Moreover, all the known realizable propositional formulas are deducible. F. L. Varpakhovskii observes that his calculus gives an uniform formalization of the principles used in proving realizability of propositional formulas. The problem of completeness of Varpakhovskii's calculus is still open.

In this paper we prove that every propositional formula deducible in the Varpakhovskii calculus has the property that every one of its closed arithmetical instances is deducible in Markov Arithmetic.

§2. Kleene's recursive realizability. We shall consider the first-order language of formal arithmetic containing the constant 0, the function symbol s and symbols for all the primitive recursive functions, the predicate symbols $=$ and $\leq$, the propositional connectives &, $\vee$, $\supset$, and $\neg$, and the quantifier symbols $\forall$ and $\exists$. It will be assumed that each of the function symbols codes the way of obtaining the corresponding primitive recursive function from the basic functions by means of superposition and primitive recursion.

If A and B are formulas, then $A \equiv B$ is an abbreviation for the formula $(A \supset B) \,\&\, (B \supset A)$. By writing formulas, we shall use bounded quantifiers $(\forall x \leq t)$ and $(\exists x \leq t)$, where t is a term without the variable x, considering $(\forall x \leq t)\, A$ and $(\exists x \leq t)\, A$ as abbreviations for the formulas $\forall x\, (x \leq t \supset A)$ and $\exists x\, (x \leq t \,\&\, A)$ respectively. The closed arithmetical formulas will be called arithmetical sentences. Logical length of an arithmetical formula A is the number of logical connectives and quantifiers in A.

Intuitionistic arithmetic HA is a formal system in the language of arithmetic based on the intuitionistic predicate calculus IQC and containing Peano axioms, defining axioms for the primitive recursive functions, and natural axioms for the relation $\leq$.

Elements of the theory of partial recursive functions can be found in [3]. The total partial recursive functions are called general recursive. An indexing of the partial recursive functions is described in [3]. The unary function whose index is x will be denoted by $\{x\}$. Note that every natural number is an index of an unary partial recursive function.

The relation "*a natural number e realizes an arithmetical sentence A*" is denoted $e\,\mathbf{r}\,A$ and is defined by induction on the logical length of A.

- If A is an atomic sentence $t_1 = t_2$, then $e\,\mathbf{r}\,A \rightleftharpoons [e = 0$ and A is true$]$.

Let A and B be arithmetical sentences. Then

- $e\,\mathbf{r}\,(A \,\&\, B) \rightleftharpoons [e$ is of the form $2^a \cdot 3^b$ and $a\,\mathbf{r}\,A$, $b\,\mathbf{r}\,B]$;
- $e\,\mathbf{r}\,(A \vee B) \rightleftharpoons [e$ is of the form $2^0 \cdot 3^a$ and $a\,\mathbf{r}\,A$ or e is of the form $2^1 \cdot 3^b$ and $b\,\mathbf{r}\,B]$;
- $e\,\mathbf{r}\,(A \supset B) \rightleftharpoons [e$ is an index of a partial recursive function ϕ such that for any a, if $a\,\mathbf{r}\,A$, then $\phi(a)\,\mathbf{r}\,B]$;
- $e\,\mathbf{r}\,\neg A \rightleftharpoons [e\,\mathbf{r}\,(A \supset 0 = 1)]$.

Let $A(x)$ be an arithmetical formula with the only parameter x. Then

- $e\,\mathbf{r}\,\forall x\, A(x) \rightleftharpoons [e$ is an index of a general recursive function f such that $f(n)\,\mathbf{r}\,A(n)$ for every $n]$;
- $e\,\mathbf{r}\,\exists x\, A(x) \rightleftharpoons [e$ is of the form $2^n \cdot 3^a$ and $a\,\mathbf{r}\,A(n)]$.

An arithmetical sentence A is called *realizable* if there exists e such that $e\,\mathbf{r}\,A$. An arithmetical formula $A(x_1,\dots,x_m)$ with the only parameters $x_1,\dots,x_m$ is called realizable if the sentence $\forall x_1\dots\forall x_m\,A(x_1,\dots,x_m)$ is realizable. Obviously, a formula $A(x_1,\dots,x_m)$ is realizable iff there exists an m-ary recursive function f such that $f(k_1,\dots,k_m)\,\mathbf{r}\,A(k_1,\dots,k_m)$ for every $k_1,\dots,k_m$. In this case, we say that the function f realizes the formula $A(x_1,\dots,x_m)$.

The following theorem is proved by D. Nelson.

THEOREM 2.1. *Every formula deducible in intuitionistic arithmetic* HA *is realizable.*

PROOF. See [4]. ⊣

§3. Formalizing recursive functions in arithmetic. The notion of realizability can be formulated as a translation from the language of arithmetic into itself. Obviously, we need a kind of a formalization of the theory of recursive functions in the language of arithmetic. For this purpose we shall use specifications of recursive fuctions by Σ-formulas. Note that one can define a binary primitive recursive function j providing a one-to-one correspondence between the naturals and the pairs of naturals and the corresponding primitive recursive inverse functions l and r in such a way that the formulas $\mathsf{j}(\mathsf{l}(x),\mathsf{r}(x))=x$, $\mathsf{l}(\mathsf{j}(x,y))=x$, $\mathsf{r}(\mathsf{j}(x,y))=y$ are deducible in HA.

Futher, an one-to-one enumeration ν_n of the n-tuples of natural numbers for $n\geq 2$ can be defined inductively:

$$\nu_2(x_1,x_2)=\mathsf{j}(x_1,x_2),$$
$$\nu_{n+1}(x_0,x_1,\dots,x_n)=\mathsf{j}(x_0,\nu_n(x_1,\dots,x_n)).$$

In this case, there exist inverse functions δ^n_i with the provable in HA property

$$\delta^n_i(\nu_{n+1}(x_0,x_1,\dots,x_n))=x_i,$$

where $n>0$, $0\leq i\leq n$. Let $\langle x_0,x_1,\dots,x_n\rangle$ be $\nu_{n+1}(x_0,x_1,\dots,x_n)$.

Arithmetical Δ_0-formulas are inductively defined as follows:

- every atomic formula is a Δ_0-formula;
- if A and B are Δ_0-formulas, then $\neg A$, $(A\vee B)$, $(A\,\&\,B)$, $(A\supset B)$ are Δ_0-formulas;
- if x is a variable, t is a term not containing x, A is a Δ_0-formula, then $(\forall x\leq t)\,A$ and $(\exists x\leq t)\,A$ are Δ_0-formulas.

Every Δ_0-formula is equivalent in HA to an atomic formula of the form $t_1=t_2$. It follows that if A is a Δ_0-formula, then the formula $A\vee\neg A$ is deducible in HA.

Now we inductively define Σ-formulas:

- every Δ_0-formula is a Σ-formula;
- if A and B are Σ-formulas, then $(A\vee B)$ and $(A\,\&\,B)$ are Σ-formulas;
- if x is a variable, t is a term not containing x, A is a Σ-formula, then $(\forall x\leq t)\,A$, $(\exists x\leq t)\,A$, and $\exists x\,A$ are Σ-formulas.

Every Σ-formula is equivalent in HA to a formula of the form $\exists x\,A$, where A is an atomic formula.

Futher, we shall use the notion of an almost negative arithmetical formula defined inductively as follows:

- every Σ-formula is an almost negative formula;
- if A and B are almost negative formulas, then $(A \supset B)$, $(A \,\&\, B)$, and $\neg A$ are almost negative formulas;
- if x is a variable, A is a an almost negative formula, then $\forall x\, A$ is an almost negative formula.

Recall that there is a truth definition for the Σ-formulas. Namely one can construct a Σ-formula $Tr_\Sigma(i,x)$ with two free variables i and x such that for every Σ-formula $A(x_1,\dots,x_n,y_1,\dots,y_m)$ with the parameters $x_1,\dots,x_n,y_1,\dots,y_m$ there exists a term $t_A(y_1,\dots,y_m)$ such that the formula

$$A(x_1,\dots,x_n,y_1,\dots,y_m) \equiv Tr_\Sigma(t_A(y_1,\dots,y_m),\langle x_1,\dots,x_n\rangle)$$

is deducible in HA (see [7] for details). We may assume that the formula $Tr_\Sigma(i,x)$ is of the form $\exists u\, S(i,x,u)$, where $S(i,x,u)$ is a Δ_0-formula.

Let us remember (see [3]) that every partial recursive function is represented in HA by a Σ-formula in the following sense: if f is an n-ary partial recursive function, then there is a Σ-formula $A_f(x_1,\dots,x_n,y)$ with the parameters $x_1,\dots,x_n,y$ such that for every natural numbers $k_1,\dots,k_n,k$

$$f(k_1,\dots,k_n)=k \Leftrightarrow \mathsf{HA} \vdash A_f(k_1,\dots,k_n,k).$$

On the other hand, a partial recursive function can be associated with every Σ-formula in the following way. Let a Σ-formula $F(x_1,\dots,x_n,y)$ of the form $\exists u\, A(x_1,\dots,x_n,y,u)$ be given, where $A(x_1,\dots,x_n,y,u)$ is a Δ_0-formula. Consider another Σ-formula

$$\exists u\,(A(x_1,\dots,x_n,y,u) \,\&\, (\forall v < \mathsf{j}(y,u))\neg A(x_1,\dots,x_n,\mathsf{l}(v),\mathsf{r}(v))).$$

Denote it by $F^{U_y}(x_1,\dots,x_n,y)$. We shall say that F^{U_y} is a uniformization of the formula F relative to the parameter y. The formula F^{U_y} has the property that for every natural $k_1,\dots,k_n$ the sentence $F^{U_y}(k_1,\dots,k_n,k)$ is true and deducible in HA for at most one k. Thus $F^{U_y}(x_1,\dots,x_n,y)$ represents an n-ary partial recursive function f defined as follows:

$$f(k_1,\dots,k_n)=k \Leftrightarrow \mathsf{HA} \vdash F^{U_y}(k_1,\dots,k_n,k).$$

We shall say that the function f is defined by the formula $F(x_1,\dots,x_n,y)$.

We say that a Σ-formula $F(x_1,\dots,x_n,y)$ is uniformized relative to the parameter y if the formula $F(x_1,\dots,x_n,y) \,\&\, F(x_1,\dots,x_n,z) \supset y=z$ is decucible in HA. In this case, F and F^{U_y} are equivalent in HA.

Let $F(x_1,\dots,x_n,y)$ be a Σ-formula with the parameters $x_1,\dots,x_n,y$. Using the property of the formula $Tr_\Sigma(i,x)$ with an empty list $y_1,\dots,y_m$, we can find a closed term (i.e., a natural number) t such that the formula

$$Tr_\Sigma(t,\langle x_1,\dots,x_n,y\rangle) \equiv F(x_1,\dots,x_n,y)$$

is deducible in HA. Let $\{t\}$ denote a partial recursive function f defined by $Tr_\Sigma(t,\langle x_1,\dots,x_n,y\rangle)$ (and, of course, by $F(x_1,\dots,x_n,y)$). The value of the term t can be considered as a code of the function f. On the other hand, for a given natural number e let f be an n-ary function defined by the formula $Tr_\Sigma(e,\langle x_1,\dots,x_n,y\rangle)$. Then $\{e\}$ is just the function f. Thus for any $n \geq 1$, every natural number is a code of an n-ary partial recursive function. It is rather obvious that such an enumeration of the partial recursive functions is equivalent to the Kleene enumeration but is more convenient for an arithmetical formalization.

Let us summarize our conventions on representing the partial recursive functions in the language of arithmetic.

- The expression $\{e\}(x_1,\dots,x_n)=y$ means the formula
$$(Tr_\Sigma(e,\langle x_1,\dots,x_n,y\rangle))^{U_y}.$$
- For every Σ-formula $F(x_1,\dots,x_n,y)$ one can effectively construct a natural number e such that the formula
$$\{e\}(x_1,\dots,x_n)=y\equiv F^{U_y}(x_1,\dots,x_n,y)$$
is deducible in HA.
- If a Σ-formula $F(x_1,\dots,x_n,y)$ is uniformized relative to the parameter y, then one can effectively construct a natural number e such that the formula
$$\{e\}(x_1,\dots,x_n)=y\equiv F^{U_y}(x_1,\dots,x_n,y)$$
is deducible in HA.

If $s,t_1,\dots,t_n,t$ are terms, then $\{s\}(t_1,\dots,t_n)=t$ is the formula obtained by substituting $s,t_1,\dots,t_n,t$ for $i,x_1,\dots,x_n,y$ into $(Tr_\Sigma(i,\langle x_1,\dots,x_n,y\rangle))^{U_y}$. Using the properties of the formula $Tr_\Sigma(i,x)$, we obtain that for every Σ-formula $F(x_1,\dots,x_n,y,y_1,\dots,y_m)$ one can effectively construct a term $t(y_1,\dots,y_m)$ such that the formula
$$\{t(y_1,\dots,y_m)\}(x_1,\dots,x_n)=y\equiv F^{U_y}(x_1,\dots,x_n,y,y_1,\dots,y_m)$$
is deducible in HA. This property is an analogue of the well-known s-m-n-theorem in the recursion theory. Note that if $F(x_1,\dots,x_n,y,y_1,\dots,y_m)$ is uniformized relative to the parameter y, then the formula
$$\{t(y_1,\dots,y_m)\}(x_1,\dots,x_n)=y\equiv F^{U_y}(x_1,\dots,x_n,y,y_1,\dots,y_m)$$
is deducible in HA for an appropriate term $t(y_1,\dots,y_m)$.

The following proposition is a result of the preceding consideration.

PROPOSITION 3.1. *Let $F(x_1,\dots,x_n,y,y_1,\dots,y_m)$ be a Σ-formula uniformized relative to y. Then one can effectively construct a term $t(y_1,\dots,y_m)$ such that the formula*
$$\{t(y_1,\dots,y_m)\}(x_1,\dots,x_n)=y\equiv F(x_1,\dots,x_n,y,y_1,\dots,y_m)$$
is deducible in HA.

§4. Formalizing recursive realizability. For the purpose of formalizing recursive realizability we use a slight modification of Kleene's definition. In the original definition only the numbers of a special form could be realizations of atomic sentences, conjunctions, disjunctions, and existential sentences. We omit this restriction.

For every arithmetical formula A we define a formula $e\,\mathbf{r}\,A$ by induction on the logical length of A.

- $e\,\mathbf{r}\,(t_1=t_2)$ is $t_1=t_2$ for arbitrary terms t_1,t_2.
- $e\,\mathbf{r}\,(A\,\&\,B)$ is $\mathsf{l}(e)\,\mathbf{r}\,A\,\&\,\mathsf{r}(e)\,\mathbf{r}\,B$.

- $e\,\mathbf{r}\,(A \vee B)$ is $(\mathsf{l}(e) = 0 \,\&\, \mathsf{r}(e)\mathbf{r}\,A) \vee (\mathsf{l}(e) \neq 0 \,\&\, \mathsf{r}(e)\mathbf{r}\,B)$.
- $e\,\mathbf{r}\,(A \supset B)$ is $\forall a\,(a\,\mathbf{r}\,A \supset \exists y\,(\{e\}(a) = y \,\&\, y\,\mathbf{r}\,B))$.
- $e\,\mathbf{r}\,\neg\,A$ is $\forall a\,\neg\,a\,\mathbf{r}\,A$.
- $e\,\mathbf{r}\,\forall x\,A(x)$ is $\forall x \exists y(\{e\}(x) = y \,\&\, y\,\mathbf{r}\,A(x))$.
- $e\,\mathbf{r}\,\exists x\,A(x)$ is $\mathsf{r}(e)\,\mathbf{r}\,A(\mathsf{l}(e))$.

Note that the formula $e\,\mathbf{r}\,(A \vee B)$ is equivalent in HA to the formula

$$(\mathsf{l}(e) = 0 \supset \mathsf{r}(e)\,\mathbf{r}\,A) \,\&\, (\mathsf{l}(e) \neq 0 \supset \mathsf{r}(e)\,\mathbf{r}\,B), \tag{1}$$

the formula $e\,\mathbf{r}\,(A \supset B)$ is equivalent in HA to the formula

$$\forall a\,(a\,\mathbf{r}\,A \supset \exists y\,\{e\}(a) = y \,\&\, \forall y(\{e\}(a) = y \supset y\,\mathbf{r}\,B)), \tag{2}$$

and the formula $e\,\mathbf{r}\,\forall x\,A(x)$ is equivalent in HA to the formula

$$\forall x(\exists y\,\{e\}(x) = y \,\&\, \forall y\,(\{e\}(a) = y \supset y\,\mathbf{r}\,A(x))). \tag{3}$$

Notice that the formulas (1), (2), (3) and the formulas $e\,\mathbf{r}\,(t_1 = t_2)$, $e\,\mathbf{r}\,(A \,\&\, B)$, and $e\,\mathbf{r}\,\exists x\,A(x)$ are almost negative provided that $e\,\mathbf{r}\,A$ and $e\,\mathbf{r}\,B$ are almost negative. Thus for every formula A, the formula $e\,\mathbf{r}\,A$ is equivalent in HA to an almost negative formula. In what follows we shall use various equivalent variants of the formula $e\,\mathbf{r}\,A$.

§5. Markov's arithmetic. Consider a formal system of arithmetic obtained by adding to HA the axiom schemes

ECT: $\forall x\,(A(x) \supset \exists y\,B(x,y)) \supset \exists e\,\forall x\,(A(x) \supset \exists y\,(\{e\}(x) = y \,\&\, B(x,y)))$,

where $A(x)$ is an almost negative formula, and

MP: $\forall x\,(A(x \vee \neg\,A(x)) \,\&\, \neg\neg\,\exists x\,A(x) \supset \exists x\,A(x)$.

The scheme ECT is called extended Church's thesis and the scheme MP is called Markov's principle. This system plays an important role in the formalization of constructive mathematics and is considered as a system of "Russian" or "traditional" constructivism (see [1]). A. Visser [9] calles it Markov's arithmetic and denotes it by MA. The following fact is rather obvious but very important for our considerations.

PROPOSITION 5.1. *In the system* MA

- *the formula* $\exists e\,e\,\mathbf{r}\,(A \supset B)$ *is equivalent to the formula*

$$\forall a\,(a\,\mathbf{r}\,A \supset \exists b\,b\,\mathbf{r}\,B)$$

  and to the formula

$$\exists a\,a\,\mathbf{r}\,A \supset \exists b\,b\,\mathbf{r}\,B;$$

- *the formula* $\exists e\,e\,\mathbf{r}\,(A \,\&\, B)$ *is equivalent to the formula*

$$\exists a\,a\,\mathbf{r}\,A \,\&\, \exists b\,b\,\mathbf{r}\,B.$$

Markov's principle plays the main role in the proof of the following proposition.

PROPOSITION 5.2. *If A is an almost negative arithmetical formula, then*

$$\mathsf{MA} \vdash \neg\neg\,A \equiv A.$$

PROOF. Induction on the definition of an almost negative arithmetical formula. ⊣

Let T be an axiomatic arithmetical theory. The propositional logic of T is the set $\mathcal{PL}(\mathsf{T})$ of propositional formulas such that every one of their arithmetical instances is deducible in T. In [5] a formal system S obtained from HA by adding the axiom scheme $A \equiv \exists a\, a\,\mathbf{r}\, A$ and MP was considered. It was proved that all the known realizable propositional formulas are in the logic $\mathcal{PL}(\mathsf{S})$. The system S is equivalent to the system MA. Thus the result just mentioned means that any known realizable propositional formula is in the logic $\mathcal{PL}(\mathsf{MA})$.

§6. Varpakhovskii's calculus. F. L. Varpakhovskii [8] has proposed an axiomatic class of realizable formulas in an extended propositional language. Namely two clauses are added to the inductive definitions of propositional and arithmetical formulas:

- if A and B are formulas, then $A \Rightarrow B$ is a formula (this formula is called a *strong implication* from A to B);
- if $B_1, \dots, B_n$ are formulas, θ is a list of formulas $A_1, \dots, A_m$, and for every $i \in \{1, \dots, n\}$, θ_i is a sublist $A_{(i,1)}, \dots, A_{(i,m_i)}$ of the list θ, then $(\theta(\theta_1 B_1 \nabla \dots \nabla \theta_n B_n))$ is a formula (this formula is called a *conditional disjunction* of the formulas $B_1, \dots, B_n$ with the conditions $\theta, \theta_1, \dots, \theta_n$ and is denoted by $(\theta(\nabla_{i=1}^n \theta_i B_i)))$.

The notion of recursive realizability is generalized to the new connectives in the following way.

Let A and B be closed arithmetical formulas. Then $e\,\mathbf{r}\,(A \Rightarrow B)$ iff for any a such that $a\,\mathbf{r}\,A$ the value $\{e\}(a)$ is defined and for any a, if $\{e\}(a)$ is defined, then $\{e\}(a)\,\mathbf{r}\,B$. This definition can be written by the following arithmetical formula denoted by $e\,\mathbf{r}\,(A \Rightarrow B)$:

$$\forall a\, ((a\,\mathbf{r}\,A \supset \exists y\, \{e\}(a) = y) \,\&\, (\forall y\, (\{e\}(a) = y \supset y\,\mathbf{r}\,B)).$$

Obviously, this formula is almost negative.

The definition of realizability for conditional disjunction is more complicated. We consider a slight modification of Varpakhovskii's definition.

Let $A_1, \dots, A_m, B_1, \dots, B_n$ be closed arithmetical formulas, $\theta, \theta_1, \dots, \theta_n$ be the same as in the definition of the formula $(\theta\, (\nabla_{i=1}^n \theta_i B_i))$. Let $\mathbf{a}$ be a list of distinct variables $a_1, \dots, a_m$ and $\mathbf{a}_i$ for $i = 1, \dots, n$ be its sublist $a_{(i,1)}, \dots, a_{(i,m_i)}$. Then $\mathbf{a}\,\mathbf{r}\,\theta$ means the formula $\bigwedge_{j=1}^{m} a_j\,\mathbf{r}\,A_j$. The formula $e\,\mathbf{r}\,(\theta\,(\nabla_{i=1}^n \theta_i B_i))$ is defined as a conjunction of the following two formulas:

$$\forall \mathbf{a}\, (\mathbf{a}\,\mathbf{r}\,\theta \supset \exists y\, \{\delta_0^m(e)\}(\mathbf{a}) = y \,\&\, \forall y\, (\{\delta_0^m(e)\}(\mathbf{a}) = y \supset \bigvee_{i=1}^{n} y = i \,\&$$

$$\& \bigwedge_{i=1}^{n} (y = i \supset \exists z\, \{\delta_i^n(e)\}(\mathbf{a}_i) = z)));$$

$$\forall \mathbf{a} \bigwedge_{i=1}^{n} (\mathbf{a}_i\,\mathbf{r}\,\theta_i \supset \forall y\, (\{\delta_i^n(e)\}(\mathbf{a}_i) = y \supset y\,\mathbf{r}\,B_i)).$$

Note that these formulas are almost negative.

The notion of an uniformly realizable propositional formula in the extended language is defined in an obvious way. F. L. Varpakhovskii proposed a propositional calculus in the extended language obtained from IPC by adding the following axiom schemes, where Ss, As, and Bs are arbitrary propositional formulas, θ is an arbitrary list of propositional formulas $A_1, \dots, A_m$, and $\theta_1, \theta_2, \dots$ are its sublists:

1. $\neg\neg S \supset ((S \Rightarrow A) \supset A)$;
2. $(S_1 \Rightarrow A_1) \,\&\, (S_2 \Rightarrow A_2) \supset ((S_1 \vee S_2) \Rightarrow (A_1 \vee A_2))$;
3. $(S \Rightarrow A_1) \,\&\, (A_1 \supset A_2) \supset (S \Rightarrow A_2)$;
4. $(S \Rightarrow A_1) \,\&\, (S \Rightarrow A_2) \supset (S \Rightarrow (A_1 \,\&\, A_2))$;
5. $(A \supset B) \equiv (A(AB))$;
6. $(\theta(\dots \nabla \theta_i B_i \nabla \theta_{i+1} B_{i+1} \nabla \dots)) \equiv (\theta(\dots \nabla \theta_{i+1} B_{i+1} \nabla \theta_i B_i \nabla \dots))$;
7. $(\theta(\theta_1(B' \vee B'')\nabla \dots)) \supset (\theta(\theta_1 B' \nabla \theta_1 B'' \nabla \dots))$;
8. $(\theta(\theta_1(A \supset B)\nabla \dots)) \supset (\theta^*(\theta_1^* B \nabla \dots))$,
   where θ^* and θ_1^* are obtained by adding A to θ and θ_1 as the last element;
9. $(A \supset A_p) \,\&\, (\theta\,(\nabla_{i=1}^n \theta_i B_i)) \supset (\theta'\,(\nabla_{i=1}^n \theta_i' B_i))$,
   where $1 \le p \le m$, θ' and θ_i' are obtained from θ and θ_i by replacing A_p by A;
10. $(B_1 \supset B) \,\&\, (\theta(\theta_1 B_1 \nabla \dots)) \supset (\theta(\theta_1 B \nabla \dots))$;
11. $\theta\left((\nabla_{i=1}^r \theta_i B_i)\,\nabla\left(\nabla_{i=r+1}^n \theta_i(\neg S)\right)\right) \supset (S \Rightarrow \bigvee_{i=1}^{r} (\neg\neg T^i \supset B_i))$,
    where $1 \le r \le n$; all the elements of θ are of the form $\neg S \supset \bigvee_{t=1}^{l} \neg\neg T_t$ or $\neg T_1 \supset S$ with a fixed propositional formula S; T^i is a conjunction of formulas T_t chosen in one subformula from every element of the list θ_i.

F. L. Varpakhovskii proves that any propositional formula deducible in this calculus is uniformly realizable. Moreover, all the known realizable propositional formulas are deducible. F. L. Varpakhovskii observes that his calculus gives an uniform formalization of the principles used in proving realizability of propositional formulas. The problem of completeness of Varpakhovskii's calculus is still open.

§7. Varpakhovskii's calculus and MA. In this section we prove the main result of the paper, that every propositional formula (in the ordinary propositional language) is in the logic $\mathcal{PL}(\mathsf{MA})$. The scheme of the proof is following. If Φ is an arithmetical formula in the extended language, let $\mathbf{r}\,\Phi$ denote the formula $\exists e\, e\, \mathbf{r}\, \Phi$. Note that $\mathbf{r}\,\Phi$ is an ordinary arithmetical formula. We prove that if a propositional formula $A(p_1, \dots, p_n)$ in the extended language is deducible in Varpakhovskii's calculus and $\Phi_1, \dots, \Phi_n$ are closed arithmetical formulas in the extended language, then the formula $\mathbf{r}\, A(\mathbf{r}\,\Phi_1, \dots, \mathbf{r}\,\Phi_n)$ is deducible in MA. In particular, this is valid for an ordinary propositional formula $A(p_1, \dots, p_n)$ and ordinary arithmetical formulas $\Phi_1, \dots, \Phi_n$. Finally note that for an ordinary arithmetical formula Φ, the formula $\Phi \equiv \mathbf{r}\,\Phi$ is deducible in the system

HA+ECT (see [1, part 2, 5.8]), consequently,

$$\mathsf{MA} \vdash \Phi \equiv \mathbf{r}\,\Phi. \tag{4}$$

This implies that if an ordinary propositional formula is deducible in the Varpakhovskii calculus, then every one of its closed instances in the ordinary language of arithmetic is deducible in MA. Indeed, if $A(p_1, \ldots, p_n)$ is an ordinary propositional formula deducible in Varpakhovskii's calculus and $\Phi_1, \ldots, \Phi_n$ are ordinary arithmetical formulas, then

$$\mathsf{MA} \vdash \mathbf{r}\,A(\mathbf{r}\,\Phi_1, \ldots, \mathbf{r}\,\Phi_n).$$

It follows from (4) that $\mathsf{MA} \vdash A(\mathbf{r}\,\Phi_1, \ldots, \mathbf{r}\,\Phi_n)$. Using (4) again, we obtain $\mathsf{MA} \vdash A(\Phi_1, \ldots, \Phi_n)$. This yields the main result of the paper.

THEOREM 7.1. *If a propositional formula $A(p_1, \ldots, p_n)$ in the extended language is deducible in Varpakhovskii's calculus and $\Phi_1, \ldots, \Phi_n$ are closed formulas in the extended language of arithmetic, then $\mathbf{r}\,A(\mathbf{r}\,\Phi_1, \ldots, \mathbf{r}\,\Phi_n)$ is deducible in Markov's arithmetic* MA.

PROOF. It is enough to prove the theorem only in the case when A is an axiom of Varpakhovskii's calculus, because modus ponens preserves the property under proof. This follows from the fact that the formula

$$\mathbf{r}\,(A(\mathbf{r}\,\Phi_1, \ldots, \mathbf{r}\,\Phi_n) \supset B(\mathbf{r}\,\Phi_1, \ldots, \mathbf{r}\,\Phi_n))$$

is equivalent in MA to the formula

$$\mathbf{r}\,A(\mathbf{r}\,\Phi_1, \ldots, \mathbf{r}\,\Phi_n) \supset \mathbf{r}\,B(\mathbf{r}\,\Phi_1, \ldots, \mathbf{r}\,\Phi_n)$$

(see Proposition 5.1).

The statement is rather clear if $A(p_1, \ldots, p_n)$ is an axiom of IPC. In this case, if $\Phi_1, \ldots, \Phi_n$ are sentences in the extended language, then $\mathsf{MA} \vdash A(\mathbf{r}\,\Phi_1, \ldots, \mathbf{r}\,\Phi_n)$, and it follows from (4) that $\mathsf{MA} \vdash \mathbf{r}\,A(\mathbf{r}\,\Phi_1, \ldots, \mathbf{r}\,\Phi_n)$. Thus we have to prove the statement for the proper axioms 1–11 of Varpakhovskii's calculus. We prove a stronger fact: if $A(p_1, \ldots, p_n)$ is a variant of one of the axioms 1–11 and $\Phi_1, \ldots, \Phi_n$ are closed formulas in the extended language of arithmetic, then $\mathsf{MA} \vdash \mathbf{r}\,A(\Phi_1, \ldots, \Phi_n)$. Obviously, this implies the statement of the theorem if we take $\mathbf{r}\,\Phi_1, \ldots, \mathbf{r}\,\Phi_n$ as $\Phi_1, \ldots, \Phi_n$. On the other hand, for this stronger statement of the theorem the rule of substitution obviously holds, thus we can consider the metavariables for formulas in the schemes 1–11 as propositional variables.

In proving deducibility of formulas in MA we shall use the following well-known tools:

- deduction theorem: if $\Gamma, \Phi \vdash \Psi$, then $\Gamma \vdash \Phi \supset \Psi$;
- the rule $(\vee \rightarrow)$: if $\Gamma, \Phi_1 \vdash \Psi$ and $\Gamma, \Phi_2 \vdash \Psi$, then $\Gamma, \Phi_1 \vee \Phi_2 \vdash \Psi$;
- the rule $(\& \rightarrow)$: if $\Gamma \vdash \Phi_1 \,\&\, \Phi_2$, then $\Gamma \vdash \Phi_1$ and $\Gamma \vdash \Phi_2$;
- the rule $(\rightarrow \&)$: if $\Gamma \vdash \Phi_1$ and $\Gamma \vdash \Phi_2$, then $\Gamma \vdash \Phi_1 \,\&\, \Phi_2$;
- the rule $(\exists \rightarrow)$: if $\Gamma, \Phi(x) \vdash \Psi$, where x is not free in Ψ and Γ, then $\Gamma, \exists x \Phi(x) \vdash \Psi$;
- the rule $(\rightarrow \exists)$: if $\Gamma \vdash \Phi(t)$ for a term t, then $\Gamma \vdash \exists x \Phi(x)$;
- the rule $(\forall \rightarrow)$: if $\Gamma \vdash \forall x \Phi(x)$, then $\Gamma \vdash \Phi(t)$ for any term t;
- the rule $(\rightarrow \forall)$: if $\Gamma \vdash \Phi(x)$, where x is not free in Γ, then $\Gamma \vdash \forall x \Phi(x)$.

Axiom 1.

$$\neg\neg S \supset ((S \Rightarrow A) \supset A)$$

Every closed arithmetical instance of this formula is of the form

$$\neg\neg \mathrm{S} \supset ((\mathrm{S} \Rightarrow \mathrm{A}) \supset \mathrm{A}),$$

where S and A are closed formulas in the extended language of arithmetic. Denote this formula by Φ. By Proposition 5.1, the formula $\mathbf{r}\,\Phi$ is equivalent in MA to the formula $\mathbf{r}\,\neg\neg\mathrm{S} \supset (\mathbf{r}\,(\mathrm{S} \Rightarrow \mathrm{A}) \supset \mathbf{r}\,\mathrm{A})$. By deduction theorem, it is enough to prove that in MA

$$\mathbf{r}\,\neg\neg\mathrm{S},\ \mathbf{r}\,(\mathrm{S} \Rightarrow \mathrm{A}) \vdash \mathbf{r}\,\mathrm{A},$$

i.e.,

$$\exists a\, a\,\mathbf{r}\,\neg\neg\mathrm{S},\ \exists b\, b\,\mathbf{r}\,(\mathrm{S} \Rightarrow \mathrm{A}) \vdash \exists c\, c\,\mathbf{r}\,\mathrm{A}.$$

By the rule $(\exists \rightarrow)$, it is enough to prove

$$a\,\mathbf{r}\,\neg\neg\mathrm{S},\ b\,\mathbf{r}\,(\mathrm{S} \Rightarrow \mathrm{A}) \vdash \exists c\, c\,\mathbf{r}\,\mathrm{A}.$$

Note that $a\,\mathbf{r}\,\neg\neg\mathrm{S}$ is the formula $\forall y\,\neg\forall x\,\neg\, x\,\mathbf{r}\,\mathrm{S}$ evidently equivalent in MA to the formula

$$\neg\,\neg\,\exists x\, x\,\mathbf{r}\,\mathrm{S}. \tag{5}$$

Futher, $b\,\mathbf{r}\,(\mathrm{S} \Rightarrow \mathrm{A})$ is the formula

$$\forall x\,((x\,\mathbf{r}\,\mathrm{S} \supset \exists y\,\{b\}(x) = y)\,\&\,\forall y\,(\{b\}(x) = y \supset y\,\mathbf{r}\,\mathrm{A}))$$

equivalent in MA to the conjunction of the following two formulas:

$$\forall x\,(x\,\mathbf{r}\,\mathrm{S} \supset \exists y\,\{b\}(x) = y); \tag{6}$$

$$\forall x\,\forall y\,(\{b\}(x) = y \supset y\,\mathbf{r}\,\mathrm{A}). \tag{7}$$

Obviously, the formula $\neg\,\neg\,\exists x\,\exists y\,\{b\}(x) = y$ is deducible from the formulas (5) and (6). As the formula $\exists x\,\exists y\,\{b\}(x) = y$ is almost negative, by Proposition 5.2, we obtain that it is deducible too. By the rule $(\exists \rightarrow)$, we can add the formula $\{b\}(x) = y$ to the hypothesises. It is easy to deduce the formula $y\,\mathbf{r}\,\mathrm{A}$ from the formulas $\{b\}(x) = y$ and (7). Therefore $\exists c\, c\,\mathbf{r}\,\mathrm{A}$ is deducible from the formulas $a\,\mathbf{r}\,\neg\neg\mathrm{S}$ and $b\,\mathbf{r}\,(\mathrm{S} \Rightarrow \mathrm{A})$. This completes the consideration of the axiom 1.

Axiom 2.

$$(S_1 \Rightarrow A_1)\,\&\,(S_2 \Rightarrow A_2) \supset ((S_1 \vee S_2) \Rightarrow (A_1 \vee A_2))$$

Every closed arithmetical instance of this formula is of the form

$$(\mathrm{S}_1 \Rightarrow \mathrm{A}_1)\,\&\,(\mathrm{S}_2 \Rightarrow \mathrm{A}_2) \supset ((\mathrm{S}_1 \vee \mathrm{S}_2) \Rightarrow (\mathrm{A}_1 \vee \mathrm{A}_2)),$$

where $\mathrm{S}_1, \mathrm{S}_2, \mathrm{A}_1, \mathrm{A}_2$ are closed formulas in the extended language of arithmetic. Denote this formula by Φ. By Proposition 5.1, the formula $\mathbf{r}\,\Phi$ is equivalent in MA to the formula

$$\exists a\, a\,\mathbf{r}\,(\mathrm{S}_1 \Rightarrow \mathrm{A}_1)\,\&\,\exists b\, b\,\mathbf{r}\,(\mathrm{S}_2 \Rightarrow \mathrm{A}_2) \supset \exists c\, c\,\mathbf{r}\,((\mathrm{S}_1 \vee \mathrm{S}_2) \Rightarrow (\mathrm{A}_1 \vee \mathrm{A}_2)).$$

By deduction theorem and the rules $(\& \rightarrow)$, $(\exists \rightarrow)$, and $(\rightarrow \exists)$, it is enough to prove that in MA

$$a\,\mathbf{r}\,(\mathrm{S}_1 \Rightarrow \mathrm{A}_1),\ b\,\mathbf{r}\,(\mathrm{S}_2 \Rightarrow \mathrm{A}_2) \vdash c\,\mathbf{r}\,((\mathrm{S}_1 \vee \mathrm{S}_2) \Rightarrow (\mathrm{A}_1 \vee \mathrm{A}_2)) \tag{8}$$

for an appropriate term c.

Note that the formula $a\,\mathbf{r}\,(\mathrm{S}_1 \Rightarrow \mathrm{A}_1)$ is of the form

$$\forall x\,((x\,\mathbf{r}\,\mathrm{S}_1 \supset \exists y\,\{a\}(x) = y)\,\&\,\forall y\,(\{a\}(x) = y \supset y\,\mathbf{r}\,\mathrm{A}_1))$$

and is equivalent in MA to the conjunction of the following two formulas:

$$\forall x\,(x\,\mathbf{r}\,\mathrm{S}_1 \supset \exists y\,\{a\}(x) = y); \tag{9}$$

$$\forall x\,\forall y\,(\{a\}(x) = y \supset y\,\mathbf{r}\,\mathrm{A}_1). \tag{10}$$

The formula $b\,\mathbf{r}\,(\mathrm{S}_2 \Rightarrow \mathrm{A}_2)$ is of the form

$$\forall x\,((x\,\mathbf{r}\,\mathrm{S}_2 \supset \exists y\,\{b\}(x) = y)\,\&\,\forall y\,(\{b\}(x) = y \supset y\,\mathbf{r}\,\mathrm{A}_2))$$

and is equivalent in MA to the conjunction of the following two formulas:

$$\forall x\,(x\,\mathbf{r}\,\mathrm{S}_2 \supset \exists y\,\{b\}(x) = y); \tag{11}$$

$$\forall x\,\forall y\,(\{b\}(x) = y \supset y\,\mathbf{r}\,\mathrm{A}_2). \tag{12}$$

Consider the Σ-formula

$$\begin{gathered}\mathsf{l}(x) = 0\,\&\,\exists z\,(\{a\}(\mathsf{r}(x)) = z\,\&\,y = \mathsf{j}(0,z))\vee\\ \vee\mathsf{l}(x) \neq 0\,\&\,\exists z\,(\{b\}(\mathsf{r}(x)) = z\,\&\,y = \mathsf{j}(1,z)).\end{gathered} \tag{13}$$

It is easy to prove that this formula is uniformized relative to y. By Proposition 3.1, we can find a term $t(a,b)$ such that the formula

$$\begin{aligned}\{t(a,b)\}(x) = y \equiv \mathsf{l}(x) = 0\,\&\,\exists z\,(\{a\}(\mathsf{r}(x)) = z\,\&\,y = \mathsf{j}(0,z))\vee\\ \vee\mathsf{l}(x) \neq 0\,\&\,\exists z\,(\{b\}(\mathsf{r}(x)) = z\,\&\,y = \mathsf{j}(1,z))\end{aligned} \tag{14}$$

is deducible in HA. We prove that (8) holds for $c = t(a,b)$.

Note that $c\,\mathbf{r}\,((\mathrm{S}_1 \vee \mathrm{S}_2) \Rightarrow (\mathrm{A}_1 \vee \mathrm{A}_2))$ is the formula

$$\forall x\,((x\,\mathbf{r}\,(\mathrm{S}_1 \vee \mathrm{S}_2) \supset \exists y\,\{c\}(x) = y)\,\&\,\forall y\,(\{c\}(x) = y \supset y\,\mathbf{r}\,(\mathrm{A}_1 \vee \mathrm{A}_2))$$

equivalent in MA to the conjunction of the following two formulas:

$$\forall x\,(x\,\mathbf{r}\,(\mathrm{S}_1 \vee \mathrm{S}_2) \supset \exists y\,\{c\}(x) = y); \tag{15}$$

$$\forall x\,\forall y\,(\{c\}(x) = y \supset y\,\mathbf{r}\,(\mathrm{A}_1 \vee \mathrm{A}_2)). \tag{16}$$

Thus we have to deduce both formulas (15) and (16) from the set Γ consisting of the formulas (9), (10), (11), and (12).

By deduction theorem and the rule $(\forall \rightarrow)$, in order to prove deducibility of the formula (15) from Γ, it is enough to deduce $\exists y\,\{c\}(x) = y$ from Γ and the formula $x\,\mathbf{r}\,(\mathrm{S}_1 \vee \mathrm{S}_2)$. Note that the last formula is equivalent in HA to the formula

$$(\mathsf{l}(x) = 0 \supset \mathsf{r}(x)\,\mathbf{r}\,\mathrm{S}_1)\,\&\,(\mathsf{l}(x) \neq 0 \supset \mathsf{r}(x)\,\mathbf{r}\,\mathrm{S}_2) \tag{17}$$

and $\exists y\,\{c\}(x) = y$ is equivalent to the disjunction of the formulas

$$\mathsf{l}(x) = 0\,\&\,\exists z\,\exists y\,(\{a\}(\mathsf{r}(x)) = z\,\&\,y = \mathsf{j}(0,z)) \tag{18}$$

and

$$\mathsf{l}(x) \neq 0\,\&\,\exists z\,\exists y\,(\{b\}(\mathsf{r}(x)) = z\,\&\,y = \mathsf{j}(1,z)). \tag{19}$$

As the formula $\mathsf{l}(x) = 0 \vee \mathsf{l}(x) \neq 0$ is deducible, we can consider two cases. In the first case, we assume that the hypothesis $\mathsf{l}(x) = 0$ is given. Then the formula

$\mathsf{r}(x)\,\mathbf{r}\,\mathrm{S}_1$ is derived from (17). It follows that the formula $\exists z\,\{a\}(\mathsf{r}(x)) = z$ is deducible from the formula (9). Thus

$$\Gamma,\ x\,\mathbf{r}\,(\mathrm{S}_1 \vee \mathrm{S}_2),\ \mathsf{l}(x) = 0 \vdash \exists z\,(\{a\}(\mathsf{r}(x)) = z. \tag{20}$$

Obviously,

$$\{a\}(\mathsf{r}(x)) = z \vdash \{a\}(\mathsf{r}(x)) = z \,\&\, \mathsf{j}(0, z) = \mathsf{j}(0, z).$$

Therefore, by the rule $(\rightarrow \exists)$,

$$\{a\}(\mathsf{r}(x)) = z \vdash \exists y\,(\{a\}(\mathsf{r}(x)) = z \,\&\, y = \mathsf{j}(0, z))$$

and

$$\{a\}(\mathsf{r}(x)) = z \vdash \exists z\,\exists y\,(\{a\}(\mathsf{r}(x)) = z \,\&\, y = \mathsf{j}(0, z)).$$

Hence, by the rule $(\exists \rightarrow)$,

$$\exists z\,\{a\}(\mathsf{r}(x)) = z \vdash \exists z\,\exists y\,(\{a\}(\mathsf{r}(x)) = z \,\&\, y = \mathsf{j}(0, z)).$$

Now it follows from (20), that

$$\Gamma,\ x\,\mathbf{r}\,(\mathrm{S}_1 \vee \mathrm{S}_2),\ \mathsf{l}(x) = 0 \vdash \mathsf{l}(x) = 0 \,\&\, \exists z\,\exists y\,(\{a\}(\mathsf{r}(x)) = z \,\&\, y = \mathsf{j}(0, z)).$$

This means that the formula (18) is deducible from Γ and the formula $\mathsf{l}(x) = 0$. This implies that

$$\Gamma,\ x\,\mathbf{r}\,(\mathrm{S}_1 \vee \mathrm{S}_2),\ \mathsf{l}(x) = 0 \vdash \exists y\,\{c\}(x) = y. \tag{21}$$

If the hypothesis $\mathsf{l}(x) \neq 0$ is given, we prove in the same way, using (11), that the formula (19) is deducible from Γ and $\mathsf{l}(x) \neq 0$. This yields that

$$\Gamma,\ x\,\mathbf{r}\,(\mathrm{S}_1 \vee \mathrm{S}_2),\ \mathsf{l}(x) \neq 0 \vdash \exists y\,\{c\}(x) = y. \tag{22}$$

Combining (21) and (22), we get $\Gamma \vdash x\,\mathbf{r}\,(\mathrm{S}_1 \vee \mathrm{S}_2) \supset \exists y\,\{c\}(x) = y$. By the rule $(\rightarrow \forall)$, the formula (15) is deducible from Γ, as was to be proved.

By the rule $(\rightarrow \forall)$ and deduction theorem, in order to prove deducibility of (16) from Γ, it is enough to deduce the formula $y\,\mathbf{r}\,(\mathrm{A}_1 \vee \mathrm{A}_2)$ from Γ and the formula $\{c\}(x) = y$. Note that $y\,\mathbf{r}\,(\mathrm{A}_1 \vee \mathrm{A}_2)$ is the disjunction of the formulas

$$\mathsf{l}(y) = 0 \,\&\, \mathsf{r}(y)\,\mathbf{r}\,\mathrm{A}_1 \tag{23}$$

and

$$\mathsf{l}(y) \neq 0 \,\&\, \mathsf{r}(y)\,\mathbf{r}\,\mathrm{A}_2 \tag{24}$$

and $\{c\}(x) = y$ is equivalent to the formula (13). By the rule $(\vee \rightarrow)$, we can divide the hypothesis (13) into two cases. In the first case, let the hypothesis

$$\mathsf{l}(x) = 0 \,\&\, \exists z\,(\{a\}(\mathsf{r}(x)) = z \,\&\, y = \mathsf{j}(0, z)) \tag{25}$$

be given. Let Δ be obtained from Γ by adding the formulas $\mathsf{l}(x) = 0$ and $\{a\}(\mathsf{r}(x)) = z \,\&\, y = \mathsf{j}(0, z)$. Obviously, the formula $\{a\}(\mathsf{r}(x)) = z \supset z\,\mathbf{r}\,\mathrm{A}_1$ is deducible from the hypothesis (10). Therefore the formulas $z\,\mathbf{r}\,\mathrm{A}_1$ and $y = \mathsf{j}(0, z)$ are deducible from Δ. It follows that the formulas $\mathsf{l}(y) = 0$, $\mathsf{r}(y) = z$, and $\mathsf{r}(y)\,\mathbf{r}\,\mathrm{A}_1$ are also deducible from Δ. This means that the formula (23) is deducible from Δ. By the rule $(\exists \rightarrow)$, we have that (23) is deducible from Γ and (25). Thus the formula $y\,\mathbf{r}\,(\mathrm{A}_1 \vee \mathrm{A}_2)$ is also deducible from Γ and (25). If the second disjunct of (13)

$$\mathsf{l}(x) \neq 0 \,\&\, \exists z\,(\{b\}(\mathsf{r}(x)) = z \,\&\, y = \mathsf{j}(1, z)) \tag{26}$$

is considered as a hypothesis, it can be proved, using (12), that the formula (24) is deducible from Γ and (26). Thus the formula $y\,\mathbf{r}\,(\mathrm{A}_1 \vee \mathrm{A}_2)$ is also deducible from Γ and (26). This means that the formula $y\,\mathbf{r}\,(\mathrm{A}_1 \vee \mathrm{A}_2)$ is deducible from Γ and the formula $\{c\}(x) = y$. Therefore the formula (16) is deducible from Γ, as was to be proved. This completes the consideration of the axiom 2.

Axiom 3.

$$(S \Rightarrow A_1)\,\&\,(A_1 \supset A_2) \supset (S \Rightarrow A_2)$$

Every closed arithmetical instance of this formula is of the form

$$(\mathrm{S} \Rightarrow \mathrm{A}_1)\,\&\,(\mathrm{A}_1 \supset \mathrm{A}_2) \supset (\mathrm{S} \Rightarrow \mathrm{A}_2),$$

where $\mathrm{S}, \mathrm{A}_1, \mathrm{A}_2$ are closed formulas in the extended language of arithmetic. Denote this formula by Φ. By Proposition 5.1, the formula $\mathbf{r}\,\Phi$ is equivalent in MA to the formula

$$\exists a\, a\,\mathbf{r}\,(\mathrm{S} \Rightarrow \mathrm{A}_1)\,\&\,\exists b\, b\,\mathbf{r}\,(\mathrm{A}_1 \supset \mathrm{A}_2) \supset \exists c\, c\,\mathbf{r}\,(\mathrm{S} \Rightarrow \mathrm{A}_2).$$

By deduction theorem and the rules $(\& \to)$, $(\exists \to)$, and $(\to \exists)$, it is enough to prove that in MA

$$a\,\mathbf{r}\,(\mathrm{S} \Rightarrow \mathrm{A}_1),\; b\,\mathbf{r}\,(\mathrm{A}_1 \supset \mathrm{A}_2) \vdash c\,\mathbf{r}\,(\mathrm{S} \Rightarrow \mathrm{A}_2) \tag{27}$$

for an appropriate term c.

Note that the formula $a\,\mathbf{r}\,(\mathrm{S} \Rightarrow \mathrm{A}_1)$ is of the form

$$\forall x\,((x\,\mathbf{r}\,\mathrm{S} \supset \exists y\,\{a\}(x) = y)\,\&\,\forall y\,(\{a\}(x) = y \supset y\,\mathbf{r}\,\mathrm{A}_1))$$

and is equivalent in MA to the conjunction of the following two formulas:

$$\forall x\,(x\,\mathbf{r}\,\mathrm{S} \supset \exists y\,\{a\}(x) = y); \tag{28}$$

$$\forall x\,\forall y\,(\{a\}(x) = y \supset y\,\mathbf{r}\,\mathrm{A}_1). \tag{29}$$

The formula $b\,\mathbf{r}\,(\mathrm{A}_1 \supset \mathrm{A}_2)$ is of the form

$$\forall x\,(x\,\mathbf{r}\,\mathrm{A}_1 \supset \exists y\,(\{b\}(x) = y\,\&\,y\,\mathbf{r}\,\mathrm{A}_2)). \tag{30}$$

The formula $c\,\mathbf{r}\,(\mathrm{S} \Rightarrow \mathrm{A}_2)$ is of the form

$$\forall x\,((x\,\mathbf{r}\,\mathrm{S} \supset \exists y\,\{c\}(x) = y)\,\&\,\forall y\,(\{c\}(x) = y \supset y\,\mathbf{r}\,\mathrm{A}_2))$$

and is equivalent in MA to the conjunction of the following two formulas:

$$\forall x\,(x\,\mathbf{r}\,\mathrm{S} \supset \exists y\,\{c\}(x) = y); \tag{31}$$

$$\forall x\,\forall y\,(\{c\}(x) = y \supset y\,\mathbf{r}\,\mathrm{A}_2). \tag{32}$$

Consider the Σ-formula

$$\exists z\,(\{a\}(x) = z\,\&\,\{b\}(z) = y). \tag{33}$$

It is easy to prove that this formula is uniformized relative to y. By Proposition 3.1, we can find a term $t(a, b)$ such that the formula

$$\{t(a, b)\}(x) = y \equiv \exists z\,(\{a\}(x) = z\,\&\,\{b\}(z) = y) \tag{34}$$

is deducible in HA. We prove that (27) holds for $c = t(a, b)$.

Obviously, we have to prove that the formulas (31) and (32) are deducible from the set Γ consisting of the formulas (28), (29), and (30). By the rule $(\to \forall)$ and deduction theorem, in order to prove deducibility of the formula (31), it is enough to prove that $\Gamma,\, x\,\mathbf{r}\,\mathrm{S} \vdash \exists y\,\{c\}(x) = y$. Note that the formula $\exists y\,\{c\}(x) = y$ is equivalent to the formula

$$\exists z\,(\{a\}(x) = z \,\&\, \exists y\,\{b\}(z) = y). \tag{35}$$

Using (28), we obtain that $\Gamma,\, x\,\mathbf{r}\,\mathrm{S} \vdash \exists z\,\{a\}(x) = z$. Thus, by the rule $(\exists \to)$, it is enough to prove that (35) is deducible from Γ and $\{a\}(x) = z$. But this is evident, because using (29) we have $\Gamma,\, \{a\}(x) = z \vdash z\,\mathbf{r}\,\mathrm{A}_1$ and using (30) we obtain $\Gamma,\, \{a\}(x) = z \vdash \exists y\,\{b\}(z) = y$ and then

$$\Gamma,\, \{a\}(x) = z \vdash \{a\}(x) = z \,\&\, \exists y\,\{b\}(z) = y.$$

By the rule $(\to \exists)$ we have

$$\Gamma,\, \{a\}(x) = z \vdash \exists z\,(\{a\}(x) = z \,\&\, \exists y\,\{b\}(z) = y)$$

as was to be proved.

By the rule $(\to \forall)$ and deduction theorem, in order to prove deducibility of the formula (32), it is enough to prove that $\Gamma,\, \{c\}(x) = y \vdash y\,\mathbf{r}\,\mathrm{A}_2$. Thus we have to prove deducibility of the formula $y\,\mathbf{r}\,\mathrm{A}_2$ from Γ and the formula (33). By the rules $(\exists \to)$ and $(\& \to)$, it is enough to prove that

$$\Gamma,\, \{a\}(x) = z,\, \{b\}(z) = y \vdash y\,\mathbf{r}\,\mathrm{A}_2.$$

But it is rather evident. Indeed, the formula $z\,\mathbf{r}\,\mathrm{A}_1$ is deducible from the formulas (29) and $\{a\}(x) = z$, and then the formula $y\,\mathbf{r}\,\mathrm{A}_2$ is deducible from the formulas $z\,\mathbf{r}\,\mathrm{A}_1$, $\{b\}(z) = y$, and (30). This completes the consideration of the axiom 3.

Axiom 4.

$$(S \Rightarrow A_1) \,\&\, (S \Rightarrow A_2) \supset (S \Rightarrow (A_1 \,\&\, A_2))$$

Every closed arithmetical instance of this formula is of the form

$$(\mathrm{S} \Rightarrow \mathrm{A}_1) \,\&\, (\mathrm{S} \Rightarrow \mathrm{A}_2) \supset (\mathrm{S} \Rightarrow (\mathrm{A}_1 \,\&\, \mathrm{A}_2)),$$

where $\mathrm{S}, \mathrm{A}_1, \mathrm{A}_2$ are closed formulas in the extended language of arithmetic. Denote this formula by Φ. By Proposition 5.1, the formula $\mathbf{r}\,\Phi$ is equivalent in MA to the formula

$$\exists a\, a\,\mathbf{r}\,(\mathrm{S} \Rightarrow \mathrm{A}_1) \,\&\, \exists b\, b\,\mathbf{r}\,(\mathrm{S} \Rightarrow \mathrm{A}_2) \supset \exists c\, c\,\mathbf{r}\,(\mathrm{S} \Rightarrow (\mathrm{A}_1 \,\&\, \mathrm{A}_2)).$$

By deduction theorem and the rules $(\& \to)$, $(\exists \to)$, and $(\to \exists)$, it is enough to prove that in MA

$$a\,\mathbf{r}\,(\mathrm{S} \Rightarrow \mathrm{A}_1),\, b\,\mathbf{r}\,(\mathrm{S} \Rightarrow \mathrm{A}_2) \vdash c\,\mathbf{r}\,(\mathrm{S} \Rightarrow (\mathrm{A}_1 \,\&\, \mathrm{A}_2)) \tag{36}$$

for an appropriate term c.

The formula $a\,\mathbf{r}\,(\mathrm{S} \Rightarrow \mathrm{A}_1)$ is equivalent in MA to the conjunction of the formulas (28) and (29). The formula $b\,\mathbf{r}\,(\mathrm{S} \Rightarrow \mathrm{A}_2)$ is equivalent in MA to the conjunction of the following two formulas:

$$\forall x\,(x\,\mathbf{r}\,\mathrm{S} \supset \exists y\,\{b\}(x) = y); \tag{37}$$

$$\forall x\,\forall y\,(\{b\}(x) = y \supset y\,\mathbf{r}\,\mathrm{A}_2). \tag{38}$$

The formula $c\,\mathbf{r}\,(\mathrm{S} \Rightarrow (\mathrm{A}_1 \,\&\, \mathrm{A}_2))$ is equivalent in MA to the conjunction of the following two formulas:

$$\forall x\,(x\,\mathbf{r}\,\mathrm{S} \supset \exists y\,\{c\}(x) = y); \tag{39}$$

$$\forall x\,\forall y\,(\{c\}(x) = y \supset y\,\mathbf{r}\,(\mathrm{A}_1 \,\&\, \mathrm{A}_2)). \tag{40}$$

Consider the Σ-formula

$$\exists u\,\exists v\,(\{a\}(x) = u \,\&\, \{b\}(x) = v \,\&\, y = \mathsf{j}(u, v)). \tag{41}$$

It is easy to prove that this formula is uniformized relative to y. By Proposition 3.1, we can find a term $t(a, b)$ such that the formula

$$\{t(a, b)\}(x) = y \equiv \exists u\,\exists v\,(\{a\}(x) = u \,\&\, \{b\}(x) = v \,\&\, y = \mathsf{j}(u, v)) \tag{42}$$

is deducible in HA. We prove that (36) holds for $c = t(a, b)$.

Obviously, we have to prove that the formulas (39) and (40) are deducible from the set Γ consisting of the formulas (28), (29), (37), and (38). By the rule $(\rightarrow \forall)$ and deduction theorem, in order to prove deducibility of the formula (39), it is enough to prove that $\Gamma,\, x\,\mathbf{r}\,\mathrm{S} \vdash \exists y\,\{c\}(x) = y$. Note that the formula $\exists y\,\{c\}(x) = y$ is equivalent to the formula

$$\exists u\,\exists v\,(\{a\}(x) = u \,\&\, \{b\}(x) = v \,\&\, \exists y\, y = \mathsf{j}(u, v)). \tag{43}$$

Using (28) and (37), we obtain that

$$\Gamma,\, x\,\mathbf{r}\,\mathrm{S} \vdash \exists u\,\{a\}(x) = u;\ \Gamma,\, x\,\mathbf{r}\,\mathrm{S} \vdash \exists v\,\{b\}(x) = v.$$

Thus, by the rule $(\exists \rightarrow)$, it is enough to prove that (43) is deducible from the set $\Gamma' = \Gamma \cup \{x\,\mathbf{r}\,\mathrm{S},\, \{a\}(x) = u,\, \{b\}(x) = v\}$. But this is evident, because $\Gamma' \vdash \exists y\, y = \mathsf{j}(u, v)$ and then

$$\Gamma' \vdash \{a\}(x) = u \,\&\, \{b\}(x) = v \,\&\, \exists y\, y = \mathsf{j}(u, v).$$

By the rule $(\rightarrow \exists)$ we have

$$\Gamma' \vdash \exists u\,\exists v\,(\{a\}(x) = u \,\&\, \{b\}(x) = v \,\&\, \exists y\, y = \mathsf{j}(u, v))$$

as was to be proved.

By the rule $(\rightarrow \forall)$ and deduction theorem, in order to prove deducibility of (40), it is enough to prove that $\Gamma,\, \{c\}(x) = y \vdash y\,\mathbf{r}\,(\mathrm{A}_1 \,\&\, A_2)$. Thus we have to prove deducibility of the formula $y\,\mathbf{r}\,(\mathrm{A}_1 \,\&\, \mathrm{A}_2)$ from Γ and the formula (41). By the rules $(\exists \rightarrow)$ and $(\& \rightarrow)$, it is enough to prove that $\Gamma' \vdash y\,\mathbf{r}\,(\mathrm{A}_1 \,\&\, \mathrm{A}_2)$, where $\Gamma' = \Gamma \cup \{\{a\}(x) = u,\, \{b\}(x) = v,\, y = \mathsf{j}(u, v)\}$. But this is rather evident. Indeed, the formula $u\,\mathbf{r}\,\mathrm{A}_1$ is deducible from the formulas (29) and $\{a\}(x) = u$. The formula $v\,\mathbf{r}\,\mathrm{A}_2$ is deducible from the formulas (38) and $\{b\}(x) = v$. But the formulas $\mathsf{l}(y) = u$ and $\mathsf{r}(y) = v$ are deducible from the formula $y = \mathsf{j}(u, v)$. Thus the formula

$$\mathsf{l}(y)\,\mathbf{r}\,\mathrm{A}_1 \,\&\, \mathsf{r}(y)\,\mathbf{r}\,\mathrm{A}_2 \tag{44}$$

is deducible from Γ'. Note that (44) is just the formula $y\,\mathbf{r}\,(\mathrm{A}_1 \,\&\, A_2)$. This completes the consideration of the axiom 4.

Axiom 5.

$$(A \supset B) \equiv (A(AB))$$

Every closed arithmetical instance of this formula is of the form

$$(\mathrm{A} \supset \mathrm{B}) \equiv (\mathrm{A}(\mathrm{AB})),$$

where A and B are formulas in the extended language of arithmetic. Denote this formula by Φ. By Proposition 5.1, the formula $\mathbf{r}\,\Phi$ is equivalent in MA to the conjunction of the formulas

$$\exists a\, a\,\mathbf{r}\,(\mathrm{A} \supset \mathrm{B}) \supset \exists b\, b\,\mathbf{r}\,(\mathrm{A}(\mathrm{AB})) \tag{45}$$

and

$$\exists b\, b\,\mathbf{r}\,(\mathrm{A}(\mathrm{AB})) \supset \exists a\, a\,\mathbf{r}\,(\mathrm{A} \supset \mathrm{B}). \tag{46}$$

Thus we have to prove deducibility in MA of the formulas (45) and (46).

Let us prove that the formula (45) is deducible. By deduction theorem and the rules $(\exists \to)$ and $(\to \exists)$, it is enough to prove that in MA

$$a\,\mathbf{r}\,(\mathrm{A} \supset \mathrm{B}) \vdash b\,\mathbf{r}\,(\mathrm{A}(\mathrm{AB})) \tag{47}$$

for an appropriate term b. Note that $a\,\mathbf{r}\,(\mathrm{A} \supset \mathrm{B})$ is the formula

$$\forall x\,(x\,\mathbf{r}\,\mathrm{A} \supset \exists y\,(\{a\}(x) = y \,\&\, y\,\mathbf{r}\,\mathrm{B})) \tag{48}$$

and $b\,\mathbf{r}\,(\mathrm{A}(\mathrm{AB}))$ is the formula

$$\forall x\,(x\,\mathbf{r}\,\mathrm{A} \supset \exists y\,(\{\mathsf{l}(b)\}(x) = y \,\&\, y = 1) \,\&\, \exists v\,\{\mathsf{r}(b)\}(x) = v) \,\&$$
$$\&\, \forall x\,\forall v\,(x\,\mathbf{r}\,\mathrm{A} \,\&\, \{\mathsf{r}(b)\}(x) = v \supset v\,\mathbf{r}\,\mathrm{B}).$$

Obviously, this formula is equivalent in MA to the conjunction of the following two formulas:

$$\forall x\,(x\,\mathbf{r}\,\mathrm{A} \supset \{\mathsf{l}(b)\}(x) = 1 \,\&\, \exists v\,\{\mathsf{r}(b)\}(x) = v); \tag{49}$$

$$\forall x\,\forall v\,(x\,\mathbf{r}\,\mathrm{A} \,\&\, \{\mathsf{r}(b)\}(x) = v \supset v\,\mathbf{r}\,\mathrm{B}). \tag{50}$$

Consider the Σ-formula $y = 1$. It is uniformized relative to y. By Proposition 3.1, there exists a natural number d_1 such that $\{d_1\}(x) = y \equiv y = 1$ is deducible in HA. Trivially, $\{d_1\}(x) = 1$ is deducible in HA. We prove that (47) holds for $b = \mathsf{j}(d_1, a)$. Evidently, we have to prove deducibility of the formulas (49) and (50) from the hypothesis (48). By the rules $(\to \forall)$ and $(\forall \to)$, deduction theorem, and the rule $(\to \&)$, in order to prove deducibility of the formula (49) from the hypothesis (48), it is enough to prove that

$$x\,\mathbf{r}\,\mathrm{A} \supset \exists y\,(\{a\}(x) = y \,\&\, y\,\mathbf{r}\,\mathrm{B}),\, x\,\mathbf{r}\,\mathrm{A} \vdash \{\mathsf{l}(b)\}(x) = 1 \tag{51}$$

and

$$x\,\mathbf{r}\,\mathrm{A} \supset \exists y\,(\{a\}(x) = y \,\&\, y\,\mathbf{r}\,\mathrm{B}),\, x\,\mathbf{r}\,\mathrm{A} \vdash \exists v\,\{\mathsf{r}(b)\}(x) = v). \tag{52}$$

The statement (51) follows from the fact that the formula $\mathsf{l}(b) = d_1$ is deducible in HA and the properties of the number d_1. The statement (52) follows easily from the fact that the formula $\mathsf{r}(b) = a$ is deducible in HA. By the rules $(\to \forall)$ and $(\forall \to)$ and deduction theorem, in order to prove deducibility of the formula (50) from the hypothesis (48), it is enough to prove that $x\,\mathbf{r}\,\mathrm{A} \supset \exists y\,(\{a\}(x) = y \,\&\, y\,\mathbf{r}\,\mathrm{B})$, $x\,\mathbf{r}\,\mathrm{A}$, $\{\mathsf{r}(b)\}(x) = v \vdash v\,\mathbf{r}\,\mathrm{B}$, but this follows easily from deducibility of the formula $\mathsf{r}(b) = a$ and the fact that the formula $\{a\}(x) = y$ is uniformized relative to y.

Now we prove that the formula (46) is deducible in MA. By deduction theorem and the rules $(\exists \to)$ and $(\to \exists)$, it is enough to prove that in MA

$$b\,\mathbf{r}\,(\mathrm{A}(\mathrm{AB})) \vdash a\,\mathbf{r}\,(\mathrm{A} \supset \mathrm{B}) \tag{53}$$

for an appropriate term a. Let a be the term $\mathsf{r}(b)$. We prove that the formula (48) is deducible in MA from the formulas (49) and (50). By the rules $(\to \forall)$, $(\forall \to)$, and deduction theorem, it is enough to prove that $\exists y\,(\{\mathsf{r}(b)\}(x) = y \,\&\, y\,\mathbf{r}\,\mathrm{B})$ is deducible from the hypothesises

$$x\,\mathbf{r}\,\mathrm{A} \supset \{\mathsf{l}(b)\}(x) = 1 \,\&\, \exists v\,\{\mathsf{r}(b)\}(x) = v,\ x\,\mathbf{r}\,\mathrm{A} \,\&\, \{\mathsf{r}(b)\}(x) = v \supset v\,\mathbf{r}\,\mathrm{B},\ x\,\mathbf{r}\,\mathrm{A},$$

but this is rather evident. This completes the consideration of the axiom 5.

Axiom 6.

$$(\theta(\theta_1 B_1 \nabla \ldots \nabla \theta_i B_i \nabla \theta_{i+1} B_{i+1} \nabla \ldots \nabla \theta_n B_n)) \equiv$$
$$\equiv (\theta(\theta_1 B_1 \nabla \ldots \nabla \theta_{i+1} B_{i+1} \nabla \theta_i B_i \nabla \ldots \nabla \theta_n B_n))$$

Here θ is a list of propositional formulas $A_1, \ldots, A_m$ and θ_i $(i = 1, \ldots, n)$ is a sublist $A_{(i,1)}, \ldots, A_{(i,m_i)}$ of θ; $B_1, B_2, \ldots, B_n$ are propositional formulas. Every closed arithmetical instance of this formula is of the form

$$(\Theta(\Theta_1 \mathrm{B}_1 \nabla \ldots \nabla \Theta_i \mathrm{B}_i \nabla \Theta_{i+1} \mathrm{B}_{i+1} \nabla \ldots \nabla \Theta_n \mathrm{B}_n)) \equiv$$
$$\equiv (\Theta(\Theta_1 \mathrm{B}_1 \nabla \ldots \nabla \Theta_{i+1} \mathrm{B}_{i+1} \nabla \Theta_i \mathrm{B}_i \nabla \ldots \nabla \Theta_n \mathrm{B}_n)),$$

where Θ is a finite sequence of arithmetical sentences $\mathrm{A}_1, \ldots, \mathrm{A}_m$ in the extended language; Θ_i $(i = 1, \ldots, n)$ is a subsequence $\mathrm{A}_{(i,1)}, \ldots, \mathrm{A}_{(i,m_i)}$ of Θ; $\mathrm{B}_1, \mathrm{B}_2, \ldots, \mathrm{B}_n$ are arithmetical sentences in the extended language. Let us denote this formula by Φ. By Proposition 5.1, it follows that the formula $\mathsf{r}\,\Phi$ is equivalent in MA to the conjunction of the following two formulas:

$$\begin{aligned}&\exists a\, a\,\mathsf{r}\,(\Theta(\Theta_1 \mathrm{B}_1 \nabla \ldots \nabla \Theta_i \mathrm{B}_i \nabla \Theta_{i+1} \mathrm{B}_{i+1} \nabla \ldots \nabla \Theta_n \mathrm{B}_n)) \supset\\ &\supset \exists b\, b\,\mathsf{r}\,(\Theta(\Theta_1 \mathrm{B}_1 \nabla \ldots \nabla \Theta_{i+1} \mathrm{B}_{i+1} \nabla \Theta_i \mathrm{B}_i \nabla \ldots \nabla \Theta_n \mathrm{B}_n));\end{aligned} \tag{54}$$

$$\begin{aligned}&\exists b\, b\,\mathsf{r}\,(\Theta(\Theta_1 \mathrm{B}_1 \nabla \ldots \nabla \Theta_{i+1} \mathrm{B}_{i+1} \nabla \Theta_i \mathrm{B}_i \nabla \ldots \nabla \Theta_n \mathrm{B}_n)) \supset\\ &\supset \exists a\, a\,\mathsf{r}\,(\Theta(\Theta_1 \mathrm{B}_1 \nabla \ldots \nabla \Theta_i \mathrm{B}_i \nabla \Theta_{i+1} \mathrm{B}_{i+1} \nabla \ldots \nabla \Theta_n \mathrm{B}_n)).\end{aligned} \tag{55}$$

It follows from the deduction theorem and the rules $(\exists \to)$ and $(\to \exists)$ that for proving deducibility of (54) in MA it is enough to prove that in MA

$$\begin{aligned}&a\,\mathsf{r}\,(\Theta(\Theta_1 \mathrm{B}_1 \nabla \ldots \nabla \Theta_i \mathrm{B}_i \nabla \Theta_{i+1} \mathrm{B}_{i+1} \nabla \ldots \nabla \Theta_n \mathrm{B}_n)) \vdash\\ &\vdash b\,\mathsf{r}\,(\Theta(\Theta_1 \mathrm{B}_1 \nabla \ldots \nabla \Theta_{i+1} \mathrm{B}_{i+1} \nabla \Theta_i \mathrm{B}_i \nabla \ldots \nabla \Theta_n \mathrm{B}_n))\end{aligned} \tag{56}$$

for an appropriate term b.

Note that $a\,\mathsf{r}\,(\Theta(\Theta_1 \mathrm{B}_1 \nabla \ldots \nabla \Theta_i \mathrm{B}_i \nabla \Theta_{i+1} \mathrm{B}_{i+1} \nabla \ldots \nabla \Theta_n \mathrm{B}_n))$ is the conjunction of the formulas

$$\begin{aligned}&\forall \mathbf{x}\,(\mathbf{x}\,\mathsf{r}\,\Theta \supset \exists v\,(\{\delta_0^n(a)\}(\mathbf{x}) = v \,\&\, \bigvee_{j=1}^{n} v = j \,\&\\ &\&\, \bigwedge_{j=1}^{n} (v = j \supset \exists u\,\{\delta_j^n(a)\}(\mathbf{x}_j) = u)))\end{aligned} \tag{57}$$

and

$$\forall \mathbf{x}\, \forall v\, (\bigwedge_{j=1}^{n} (\mathbf{x}_j \,\mathsf{r}\, \Theta_j \,\&\, \{\delta_j^n(a)\}(\mathbf{x}_j) = v \supset v \,\mathsf{r}\, \mathrm{B}_j)) \tag{58}$$

and $b\,\mathsf{r}\,(\Theta(\Theta_1\mathrm{B}_1\nabla\ldots\nabla\Theta_{i+1}\mathrm{B}_{i+1}\nabla\Theta_i\mathrm{B}_i\nabla\ldots\nabla\Theta_n\mathrm{B}_n))$ is the conjunction of the formulas

$$\begin{gathered}\forall \mathbf{x}\, (\mathbf{x} \,\mathsf{r}\, \Theta \supset \exists w\, (\{\delta_0^n(b)\}(\mathbf{x}) = w \,\&\, \bigvee_{j=1}^{n} w = j \,\&\\ \&\, \bigwedge_{j=1}^{n} (w = j \supset \exists u\, \{\delta_j^n(b)\}(\mathbf{x}_j) = u)))\end{gathered} \tag{59}$$

and

$$\forall \mathbf{x}\, \forall w\, (\bigwedge_{j=1}^{n} (\mathbf{x}'_j \,\mathsf{r}\, \Theta'_j \,\&\, \{\delta_j^n(b)\}(\mathbf{x}'_j) = w \supset w \,\mathsf{r}\, \mathrm{C}_j)), \tag{60}$$

where Θ'_j is Θ_j if $j \neq i, i+1$; Θ'_i is Θ_{i+1}; Θ'_{i+1} is Θ_i; $\mathbf{x}'_j$ is a list of variables of the same length as the list of formulas Θ'_j; C_j is B_j if $j \neq i, i+1$; C_i is B_{i+1}; C_{i+1} is B_i.

Consider the Σ-formula

$$\begin{gathered}\exists v\, (\{\delta_0^n(a)\}(\mathbf{x}) = v \,\&\\ \&\, (v = i \,\&\, y = i+1 \vee v = i+1 \,\&\, y = i \vee v \neq i \,\&\, v \neq i+1 \,\&\, y = v)).\end{gathered}$$

Clearly, it is uniformized relative to y. By Proposition 3.1, there exists a term $d(a)$ such that the formula

$$\begin{gathered}\{d(a)\}(\mathbf{x}) = y \equiv \exists v\, (\{\delta_0^n(a)\}(\mathbf{x}) = v \,\&\\ (v = i \,\&\, y = i+1 \vee v = i+1 \,\&\, y = i \vee v \neq i \,\&\, v \neq i+1 \,\&\, y = v))\end{gathered} \tag{61}$$

is deducible in HA. Now let b be the term

$$\langle d(a), \delta_1^n(a), \ldots, \delta_{i-1}^n(a), \delta_{i+1}^n(a), \delta_i^n(a), \delta_{i+2}^n(a), \ldots, \delta_n^n(a)\rangle.$$

We prove that (56) holds. Evidently, we have to prove deducibility of the formulas (59) and (60) from the hypothesises (57) and (58). By the rules $(\rightarrow \forall)$, $(\forall \rightarrow)$ and deduction theorem, in order to prove deducibility of the formula (59) from the hypothesises (57) and (58), it is enough to prove that the formula

$$\exists w\, (\{\delta_0^n(b)\}(\mathbf{x}) = w \,\&\, \bigvee_{j=1}^{n} w = j \,\&\, \bigwedge_{j=1}^{n} (w = j \supset \exists u\, \{\delta_j^n(b)\}(\mathbf{x}_j) = u))$$

is deducible from the hypothesises $\mathbf{x} \,\mathsf{r}\, \Theta$ and

$$\begin{gathered}\mathbf{x} \,\mathsf{r}\, \Theta \supset\\ \supset \exists v\, (\{\delta_0^n(a)\}(\mathbf{x}) = v \,\&\, \bigvee_{j=1}^{n} v = j \,\&\, \bigwedge_{j=1}^{n} (v = j \supset \exists u\, \{\delta_j^n(a)\}(\mathbf{x}_j) = u)).\end{gathered}$$

Note that the formula $\delta_0^n(b) = d(a)$ is deducible in HA. Therefore we have to prove deducibility of the formula

$$\exists w\, (\{d(a)\}(\mathbf{x}) = w \,\&\, \bigvee_{j=1}^{n} w = j \,\&\, \bigwedge_{j=1}^{n} (w = j \supset \exists u\, \{\delta_j^n(b)\}(\mathbf{x}_j) = u)) \tag{62}$$

from the hypothesises under consideration. As the formula

$$\exists v\,(\{\delta_0^n(a)\}(\mathbf{x}) = v \,\&\, \bigvee_{j=1}^{n} v = j \,\&\, \bigwedge_{j=1}^{n} (v = j \supset \exists u\,\{\delta_j^n(a)\}(\mathbf{x}_j) = u))$$

is deducible from these hypothesises, by the rule $(\exists \to)$, it is enough to prove deducibility of the formula (62) from the hypothesises

$$\{\delta_0^n(a)\}(\mathbf{x}) = v,\ \bigvee_{j=1}^{n} v = j,\ \bigwedge_{j=1}^{n} (v = j \supset \exists u\,\{\delta_j^n(a)\}(\mathbf{x}_j) = u). \tag{63}$$

First we show that the formula $\exists w\,\{d(a)\}(\mathbf{x}) = w$ is deducible from the hypothesises (63). Note that this formula is equivalent in MA to the formula

$$\exists v\,(\{\delta_0^n(a)\}(\mathbf{x}) = v \,\&$$
$$\&\,\exists w\,(v = i \,\&\, w = i+1 \vee v = i+1 \,\&\, w = i \vee v \neq i \,\&\, v \neq i+1 \,\&\, w = v)).$$

By the rule $(\to \exists)$, it is enough to prove deducibility of the formula

$$\{\delta_0^n(a)\}(\mathbf{x}) = v \,\&$$
$$\&\ \exists w\,(v = i \,\&\, w = i+1 \vee v = i+1 \,\&\, w = i \vee v \neq i \,\&\, v \neq i+1 \,\&\, w = v)$$

from the hypothesises (63). The first conjunct $\{\delta_0^n(a)\}(\mathbf{x}) = v$ is obviously deducible. Deducibility of the second conjunct

$$\exists w\,(v = i \,\&\, w = i+1 \vee v = i+1 \,\&\, w = i \vee v \neq i \,\&\, v \neq i+1 \,\&\, w = v)$$

is proved by using the hypothesis $\bigvee_{j=1}^{n} v = j$ and considering the cases $v = j$ for $j = 1, \ldots, n$. In any case, by the rule $(\to \exists)$, it is enough to prove the formula $v = i \,\&\, t = i+1 \vee v = i+1 \,\&\, t = i \vee v \neq i \,\&\, v \neq i+1 \,\&\, t = v$ for an appropriate term t. Obviously, we can let t be $i+1$ if $j = i$, i if $j = i+1$, and j in other cases. Now we can use an additional hypothesis $\exists w\,\{d(a)\}(\mathbf{x}) = w$ in proving deducibility of the formula (62). By the rules $(\exists \to)$ and $(\to \exists)$, it follows that it is enough to prove deducibility of the formula

$$\{d(a)\}(\mathbf{x}) = w \,\&\, \bigvee_{j=1}^{n} w = j \,\&\, \bigwedge_{j=1}^{n} (w = j \supset \exists u\,\{\delta_j^n(b)\}(\mathbf{x}_j) = u) \tag{64}$$

from the hypothesises (63) and $\{d(a)\}(\mathbf{x}) = w$. The first conjunct in (64) is obviously deducible. Taking into account the structure of the formula $\{d(a)\}(\mathbf{x}) = w$ and the rule $(\exists \to)$, we can use the hypothesises $\{\delta_0^n(a)\}(\mathbf{x}) = v$ and

$$v = i \,\&\, w = i+1 \vee v = i+1 \,\&\, w = i \vee v \neq i \,\&\, v \neq i+1 \,\&\, w = v \tag{65}$$

in proving deducibility of the second and the third conjuncts in (64). The formula $\bigvee_{j=1}^{n} w = j$ is deduced by using hypothesises $\bigvee_{j=1}^{n} v = j$ and (65) and considering the cases $v = j$ for $j = 1, \ldots, n$. If $j = i$, then the formula $w = i+1$ is deducible from (65); if $j = i+1$, then the formula $w = i$ is deducible; in other cases, the formula $w = v$ is deducible, therefore $w = j$ is deducible. In any case, the formula $\bigvee_{j=1}^{n} w = j$ is deducible from the hypothesises under consideration. In order to prove deducibility of the third conjunct in (64), it is enough to deduce $\exists u\,\{\delta_j^n(b)\}(\mathbf{x}_j) = u$ $(j = 1, \ldots, n)$ from the hypothesis $w = j$ and other hypothesises. Let j be fixed. If $j = i + 1$, then the formula $\delta_j^n(b) = \delta_i^n(a)$ is deducible in HA and $v = i$ is deducible from (65). Thus we have to deduce the formula $\exists u\,\{\delta_i^n(a)\}(\mathbf{x}_j) = u$, but this is evident because the formula

$$v = i \supset \exists u\,\{\delta_i^n(a)\}(\mathbf{x}_i) = u$$

is deducible from the hypothesises (63). Other values of j are considered in a similar way. Thus we have proved that the formula (62) is deducible from the hypothesises (63).

Now we prove that (60) is deducible from (57) and (58). By the rule $(\to \forall)$ and deduction theorem, it is enough to prove deducibility of the formulas $w\,\mathsf{r}\,\mathrm{C}_j$ from the hypothesises (57), (58), $\mathbf{x}'_j\,\mathsf{r}\,\Theta'_j$, and $\{\delta_j^n(b)\}(\mathbf{x}'_j) = w$ for $j = 1, \ldots, n$. Let $j = i$. Then the formula $\delta_j^n(b) = \delta_{i+1}^n(a)$ is obviously deducible, Θ'_j is Θ_{i+1}, $\mathbf{x}'_j$ is the list of variables of the same length as the list of formulas Θ_{i+1}, and C_j is the formula B_{i+1}. Thus we have to deduce the formula $w\,\mathsf{r}\,\mathrm{B}_{i+1}$ from the hypothesises (57), (58), $\mathbf{x}'_j\,\mathsf{r}\,\Theta_{i+1}$, and $\{\delta_{i+1}^n(a)\}(\mathbf{x}'_j) = w$, but this is evident because $\mathbf{x}'_j\,\mathsf{r}\,\Theta_{i+1}\,\&\,\{\delta_{i+1}^n(a)\}(\mathbf{x}'_j) = w \supset w\,\mathsf{r}\,\mathrm{B}_{i+1}$ is deducible from the hypothesis (58). Other values of j are considered in a similar way.

We have proved deducibility of the formula (55) from the formula (54). As the situation is absolutely symmetric, deducibility of the formula (54) from the formula (55) is proved too. This completes the consideration of the axiom 6.

Axiom 7.

$$(\theta(\theta_1(B' \vee B'')\nabla\theta_2 B_2\nabla\ldots\nabla\theta_n B_n)) \supset (\theta(\theta_1 B'\nabla\theta_1 B''\nabla\theta_2 B_2\nabla\ldots\nabla\theta_n B_n))$$

Here θ is a list of propositional formulas $A_1, \ldots, A_m$ and θ_i $(i = 1, \ldots, n)$ is a sublist $A_{(i,1)}, \ldots, A_{(i,m_i)}$ of θ; $B', B'', B_2, \ldots, B_n$ are propositional formulas. A closed arithmetical instance of the axiom 7 is of the form

$$(\Theta(\Theta_1(\mathrm{B}' \vee \mathrm{B}'')\nabla\Theta_2\mathrm{B}_2\nabla\ldots\nabla\Theta_n\mathrm{B}_n)) \supset$$

$$\supset (\Theta(\Theta_1\mathrm{B}'\nabla\Theta_1\mathrm{B}''\nabla\Theta_2\mathrm{B}_2\nabla\ldots\nabla\Theta_n\mathrm{B}_n)).$$

Let us denote this formula by Φ. Here Θ is a finite sequence of arithmetical sentences $\mathrm{A}_1, \ldots, \mathrm{A}_m$ in the extended language; Θ_i $(i = 1, \ldots, n)$ is a subsequence $\mathrm{A}_{(i,1)}, \ldots, \mathrm{A}_{(i,m_i)}$ of Θ; $\mathrm{B}', \mathrm{B}'', \mathrm{B}_2, \ldots, \mathrm{B}_n$ are arithmetical sentences in the extended language. By Proposition 5.1, it follows that the formula $\exists e\, e\,\mathsf{r}\,\Phi$ is equivalent in MA to the formula

$$\begin{aligned} &\exists a\, a\,\mathsf{r}\,(\Theta(\Theta_1(\mathrm{B}' \vee \mathrm{B}'')\nabla\Theta_2\mathrm{B}_2\nabla\ldots\nabla\Theta_n\mathrm{B}_n)) \supset \\ &\supset \exists b\, b\,\mathsf{r}\,(\Theta(\Theta_1\mathrm{B}'\nabla\Theta_1\mathrm{B}''\nabla\Theta_2\mathrm{B}_2\nabla\ldots\nabla\Theta_n\mathrm{B}_n)). \end{aligned} \tag{66}$$

By deduction theorem and the rules $(\exists \to)$ and $(\to \exists)$, for proving deducibility of (66) in MA it is enough to prove that in MA

$$\begin{aligned} &a\,\mathsf{r}\,(\Theta(\Theta_1(\mathrm{B}' \vee \mathrm{B}'')\nabla\Theta_2\mathrm{B}_2\nabla\ldots\nabla\Theta_n\mathrm{B}_n)) \vdash \\ &\vdash b\,\mathsf{r}\,(\Theta(\Theta_1\mathrm{B}'\nabla\Theta_1\mathrm{B}''\nabla\Theta_2\mathrm{B}_2\nabla\ldots\nabla\Theta_n\mathrm{B}_n)) \end{aligned} \tag{67}$$

for an appropriate term b.

Note that $a\,\mathsf{r}\,(\Theta(\Theta_1(\mathrm{B}'\vee\mathrm{B}'')\nabla\Theta_2\mathrm{B}_2\nabla\ldots\nabla\Theta_n\mathrm{B}_n))$ is the conjunction of the formulas

$$\begin{gathered}\forall\mathbf{x}\,(\mathbf{x}\,\mathsf{r}\,\Theta\supset\exists v\,(\{\delta_0^n(a)\}(\mathbf{x})=v\,\&\,\bigvee_{j=1}^{n}v=j\,\&\\ \&\,\bigwedge_{j=1}^{n}(v=j\supset\exists u\,\{\delta_j^n(a)\}(\mathbf{x}_j)=u)))\end{gathered}\tag{68}$$

and

$$\begin{gathered}\forall\mathbf{x}\,\forall v\,((\mathbf{x}_1\,\mathsf{r}\,\Theta_1\,\&\,\{\delta_1^n(a)\}(\mathbf{x}_1)=v\supset v\,\mathsf{r}\,(\mathrm{B}'\vee\mathrm{B}''))\,\&\\ \&\,\bigwedge_{j=2}^{n}(\mathbf{x}_j\,\mathsf{r}\,\Theta_j\,\&\,\{\delta_j^n(a)\}(\mathbf{x}_j)=v\supset v\,\mathsf{r}\,\mathrm{B}_j))\end{gathered}\tag{69}$$

and $b\,\mathsf{r}\,(\Theta(\Theta_1\mathrm{B}'\nabla\Theta_1\mathrm{B}''\nabla\Theta_2\mathrm{B}_2\nabla\ldots\nabla\Theta_n\mathrm{B}_n))$ is the conjunction of the formulas

$$\begin{gathered}\forall\mathbf{x}\,(\mathbf{x}\,\mathsf{r}\,\Theta\supset\exists w\,(\{\delta_0^{n+1}(b)\}(\mathbf{x})=w\,\&\,\bigvee_{j=1}^{n+1}w=j\,\&\\ \&\,\bigwedge_{j=1}^{n+1}(w=j\supset\exists u\,\{\delta_j^{n+1}(b)\}(\mathbf{x}'_j)=u)))\end{gathered}\tag{70}$$

and

$$\forall\mathbf{x}\,\forall w\,(\bigwedge_{j=1}^{n+1}(\mathbf{x}'_j\,\mathsf{r}\,\Theta'_j\,\&\,\{\delta_j^{n+1}(b)\}(\mathbf{x}'_j)=w\supset w\,\mathsf{r}\,\mathrm{C}_j)),\tag{71}$$

where Θ'_1 and Θ'_2 are Θ_1; Θ'_j is Θ_{j-1} for $j=3,\ldots,n+1$; $\mathbf{x}'_1$ and $\mathbf{x}'_2$ are $\mathbf{x}_1$; $\mathbf{x}'_j$ is $\mathbf{x}_{j-1}$ for $j=3,\ldots,n+1$; C_1 is B'; C_2 is B''; C_j is B_{j-1} for $j=3,\ldots,n+1$.

Consider the Σ-formula

$$\begin{gathered}\exists v\,(\{\delta_0^n(a)\}(\mathbf{x})=v\,\&\,(v=1\,\&\,\exists u(\{\delta_1^n(a)\}(\mathbf{x}_1)=u\,\&\,(\mathsf{l}(u)=0\,\&\,y=1\vee\\ \vee\mathsf{l}(u)\neq0\,\&\,y=2)\vee v\neq1\,\&\,y=v+1)).\end{gathered}$$

Clearly, it is uniformized relative to y. By Proposition 3.1, there exists a term $d(a)$ such that the formula

$$\begin{gathered}\{d(a)\}(\mathbf{x})=y\equiv\exists v\,(\{\delta_0^n(a)\}(\mathbf{x})=v\,\&\,(v=1\,\&\,\exists u(\{\delta_1^n(a)\}(\mathbf{x}_1)=u\,\&\\ \&\,(\mathsf{l}(u)=0\,\&\,y=1\vee\mathsf{l}(u)\neq0\,\&\,y=2)\vee v\neq1\,\&\,y=v+1))\end{gathered}\tag{72}$$

is deducible in HA.

Consider the Σ-formula $\exists v\,(\{\delta_1^n(a)\}(\mathbf{x}_1)=v\,\&\,\mathsf{l}(v)=0\,\&\,y=\mathsf{r}(v))$. Clearly, it is uniformized relative to y. By Proposition 3.1, there exists a term $d_1(a)$ such that the formula

$$\{d_1(a)\}(\mathbf{x}_1)=y\equiv\exists v\,(\{\delta_1^n(a)\}(\mathbf{x}_1)=v\,\&\,\mathsf{l}(v)=0\,\&\,y=\mathsf{r}(v))\tag{73}$$

is deducible in HA.

Consider the Σ-formula $\exists v\,(\{\delta_1^n(a)\}(\mathbf{x}_1)=v\,\&\,\mathsf{l}(v)\neq0\,\&\,y=\mathsf{r}(v))$. Clearly, it is uniformized relative to y. By Proposition 3.1, there exists a term $d_2(a)$ such that the formula

$$\{d_2(a)\}(\mathbf{x}_1)=y\equiv\exists v\,(\{\delta_1^n(a)\}(\mathbf{x}_1)=v\,\&\,\mathsf{l}(v)\neq0\,\&\,y=\mathsf{r}(v))\tag{74}$$

is deducible in HA.

Now let b be the term $\langle d(a), d_1(a), d_2(a), \delta_2^n(a), \dots, \delta_n^n(a)\rangle$. We prove that (67) holds. Evidently, we have to prove deducibility of the formulas (70) and (71) from the hypothesises (68) and (69).

By the rules $(\to\forall)$, $(\forall\to)$ and deduction theorem, in order to prove deducibility of the formula (70) from the hypothesises (68) and (69), it is enough to prove that the formula

$$\exists w\,(\{\delta_0^{n+1}(b)\}(\mathbf{x}) = w \,\&\, \bigvee_{j=1}^{n+1} w = j \,\&\, \bigwedge_{j=1}^{n+1} (w = j \supset \exists u\,\{\delta_j^{n+1}(b)\}(\mathbf{x}'_j) = u))$$

is deducible from the hypothesises $\mathbf{x}\,\mathsf{r}\,\Theta$ and

$$\mathbf{x}\,\mathsf{r}\,\Theta \supset$$

$$\supset \exists v\,(\{\delta_0^n(a)\}(\mathbf{x}) = v \,\&\, \bigvee_{j=1}^{n} v = j \,\&\, \bigwedge_{j=1}^{n} (v = j \supset \exists u\,\{\delta_j^n(a)\}(\mathbf{x}_j) = u)).$$

Note that the formula $\delta_0^{n+1}(b) = d(a)$ is deducible in HA. Therefore we have to prove deducibility of the formula

$$\exists w\,(\{d(a)\}(\mathbf{x}) = w \,\&\, \bigvee_{j=1}^{n+1} w = j \,\&\, \bigwedge_{j=1}^{n+1} (w = j \supset \exists u\,\{\delta_j^{n+1}(b)\}(\mathbf{x}'_j) = u)) \tag{75}$$

from the hypothesises under consideration. As the formula

$$\exists v\,(\{\delta_0^n(a)\}(\mathbf{x}) = v \,\&\, \bigvee_{j=1}^{n} v = j \,\&\, \bigwedge_{j=1}^{n} (v = j \supset \exists u\,\{\delta_j^n(a)\}(\mathbf{x}_j) = u))$$

is deducible from these hypothesises, by the rule $(\exists\to)$, it is enough to prove deducibility of the formula (75) from the hypothesises

$$\{\delta_0^n(a)\}(\mathbf{x}) = v,\ \bigvee_{j=1}^{n} v = j,\ \bigwedge_{j=1}^{n} (v = j \supset \exists u\,\{\delta_j^n(a)\}(\mathbf{x}_j) = u). \tag{76}$$

First we show that the formula $\exists w\,\{d(a)\}(\mathbf{x}) = w$ is deducible from the hypothesises (76). Note that this formula is equivalent in MA to the formula

$$\exists v\,(\{\delta_0^n(a)\}(\mathbf{x}) = v \,\&\, \exists w\,(v = 1 \,\&\, \exists u(\{\delta_1^n(a)\}(\mathbf{x}_1) = u \,\&$$

$$\&\, (\mathsf{l}(u) = 0 \,\&\, w = 1 \vee \mathsf{l}(u) \neq 0 \,\&\, w = 2)) \vee v \neq 1 \,\&\, w = v + 1)).$$

By the rule $(\to\exists)$, it is enough to prove deducibility of the formula

$$\{\delta_0^n(a)\}(\mathbf{x}) = v \,\&\, \exists w\,(v = 1 \,\&\, \exists u(\{\delta_1^n(a)\}(\mathbf{x}_1) = u \,\&$$

$$\&\, (\mathsf{l}(u) = 0 \,\&\, w = 1 \vee \mathsf{l}(u) \neq 0 \,\&\, w = 2)) \vee v \neq 1 \,\&\, w = v + 1)$$

from the hypothesises (76). The first conjunct $\{\delta_0^n(a)\}(\mathbf{x}) = v$ is obviously deducible. Deducibility of the second conjunct

$$\begin{gathered}\exists w\,(v = 1 \,\&\, \exists u(\{\delta_1^n(a)\}(\mathbf{x}_1) = u \,\&\\ \&\, (\mathsf{l}(u) = 0 \,\&\, w = 1 \vee \mathsf{l}(u) \neq 0 \,\&\, w = 2)) \vee v \neq 1 \,\&\, w = v + 1)\end{gathered} \tag{77}$$

is proved by using the hypothesis $\bigvee_{j=1}^{n} v = j$ and considering the cases $v = j$ for $j = 1, \dots, n$.

If $j = 1$, we note that the formula $\exists u\,\{\delta_1^n(a)\}(\mathbf{x}_1') = u$ is deducible from the hypothesises (76) because $\mathbf{x}_1'$ is $\mathbf{x}_1$. Thus by the rule $(\exists\to)$, we can use an additional hypothesis $\{\delta_1^n(a)\}(\mathbf{x}_1') = u$. As $\mathsf{l}(u) = 0 \vee \mathsf{l}(u) \neq 0$ is deducible in HA, consider two cases. First let the hypothesis $\mathsf{l}(u) = 0$ be given. Then the formula

$$\{\delta_1^n(a)\}(\mathbf{x}_1') = u \,\&\, (\mathsf{l}(u) = 0 \,\&\, 1 = 1 \vee \mathsf{l}(u) \neq 0 \,\&\, 1 = 2)$$

is deducible. It follows that the formula

$$v = 1 \,\&\, \exists u(\{\delta_1^n(a)\}(\mathbf{x}_1') = u \,\&\, \\ \&\, (\mathsf{l}(u) = 0 \,\&\, 1 = 1 \vee \mathsf{l}(u) \neq 0 \,\&\, 1 = 2)) \vee v \neq 1 \,\&\, 1 = v + 1))$$

and then (77) are also deducible. In the case of the hypothesis $\mathsf{l}(u) \neq 0$, the formula

$$\{\delta_1^n(a)\}(\mathbf{x}_1') = u \,\&\, (\mathsf{l}(u) = 0 \,\&\, 2 = 1 \vee \mathsf{l}(u) \neq 0 \,\&\, 2 = 2)$$

and then (77) are deducible.

If $j = 2, \ldots, n$, it is evident that the formula $v \neq 1 \,\&\, j+1 = v+1$ and then (77) are deducible.

Thus we have proved deducibility of the formula $\exists w\,\{d(a)\}(\mathbf{x}) = w$ from the hypothesises under consideration. Now we can use this formula as an additional hypothesis in proving deducibility of the formula (75). By the rules $(\exists\to)$ and $(\to\exists)$, it follows that it is enough to prove deducibility of the formula

$$\{d(a)\}(\mathbf{x}) = w \,\&\, \bigvee_{j=1}^{n+1} w = j \,\&\, \bigwedge_{j=1}^{n+1} (w = j \supset \exists u\,\{\delta_j^{n+1}(b)\}(\mathbf{x}_j') = u) \tag{78}$$

from the hypothesises (76) and $\{d(a)\}(\mathbf{x}) = w$. The first conjunct in (78) is obviously deducible. Taking into account the structure of the formula $\{d(a)\}(\mathbf{x}) = w$ and the rule $(\exists\to)$, we can use the hypothesises $\{\delta_0^n(a)\}(\mathbf{x}) = v$ and

$$v = 1 \,\&\, \exists u(\{\delta_1^n(a)\}(\mathbf{x}_1) = u \,\&\, (\mathsf{l}(u) = 0 \,\&\, w = 1 \vee \mathsf{l}(u) \neq 0 \,\&\, w = 2))\vee \\ \vee v \neq 1 \,\&\, w = v + 1 \tag{79}$$

in proving deducibility of the second and the third conjuncts in (78). The formula $\bigvee_{j=1}^{n+1} w = j$ is deduced by using the hypothesises $\bigvee_{j=1}^{n} v = j$ and (79) and considering the cases $v = j$ for $j = 1, \ldots, n$. If $j = 1$, then the formula $w = 1 \vee w = 2$ is deducible from (79) by considerations like those used above in proving deducibility of the formula $\exists w\,\{d(a)\}(\mathbf{x}) = w$; if $j = 2, \ldots, n$, then the formula $w = j + 1$ is deducible. Thus the formula $\bigvee_{j=1}^{n+1} w = j$ is deducible from the hypothesises under consideration.

In order to prove deducibility of the third conjunct in (78), it is enough to deduce $\exists u\,\{\delta_j^{n+1}(b)\}(\mathbf{x}_j') = u$ $(j = 1, \ldots, n+1)$ from the hypothesis $w = j$ and other hypothesises. Let j be fixed. If $j = 1$, then the formula

$$v = 1 \,\&\, \exists u(\{\delta_1^n(a)\}(\mathbf{x}_1) = u \,\&\, (\mathsf{l}(u) = 0 \,\&\, 1 = 1 \vee \mathsf{l}(u) \neq 0 \,\&\, 1 = 2))\vee \\ \vee v \neq 1 \,\&\, 1 = v + 1 \tag{80}$$

is deducible from the hypothesis (79). This gives us the formula

$$\exists u(\{\delta_1^n(a)\}(\mathbf{x}_1) = u \,\&\, \mathsf{l}(u) = 0). \tag{81}$$

Note that the formula $\delta_1^{n+1}(b) = d_1(a)$ is deducible in HA. Thus we have to deduce the formula $\exists u\,\{d_1(a)\}(\mathbf{x}_1') = u$. Taking into account (73) and the fact that $\mathbf{x}_1'$ is $\mathbf{x}_1$, we see that it is enough to deduce the formula

$$\exists u\,\exists v\,(\{\delta_1^n(a)\}(\mathbf{x}_1) = v\,\&\,\mathsf{l}(v) = 0\,\&\,u = \mathsf{r}(v)),$$

but this is rather evident because the formula (81) is deducible.

If $j = 2$, then the formula

$$\exists u(\{\delta_1^n(a)\}(\mathbf{x}_1) = u\,\&\,\mathsf{l}(u) \neq 0) \tag{82}$$

is deducible from the hypothesis (79). Note that $\delta_2^{n+1}(b) = d_2(a)$ is deducible in HA. Thus we have to deduce the formula $\exists u\,\{d_2(a)\}(\mathbf{x}_2') = u$. Taking into account (74) and the fact that $\mathbf{x}_2'$ is $\mathbf{x}_1$, we see that it is enough to deduce the formula

$$\exists u\,\exists v\,(\{\delta_1^n(a)\}(\mathbf{x}_1) = v\,\&\,\mathsf{l}(v) \neq 0\,\&\,\exists u\,u = \mathsf{r}(v)),$$

but this is rather evident because the formula (82) is deducible.

If $j = 3, \ldots, n+1$, then the formula $\delta_j^{n+1}(b) = \delta_{j-1}^n(a)$ is deducible in HA and the formula $v = j-1$ is deducible from the hypothesis (79). Taking into account the fact that $\mathbf{x}_j'$ is $\mathbf{x}_{j-1}$, we see that it is enough to deduce the formula $\exists u\,\{\delta_{j-1}^n(a)\}(\mathbf{x}_{j-1}) = u$, but this evident because this formula is deducible from the third hypothesis in (76). Thus we have proved that the formula (70) is deducible from the hypothesises (68) and (69).

In order to prove deducibility of the formula (71) from the hypothesises (68) and (69), it is enough for any $j = 1, 2, \ldots, n+1$ to prove that the formula $\mathbf{x}_j'\,\mathsf{r}\,\Theta_j'\,\&\,\{\delta_j^{n+1}(b)\}(\mathbf{x}_j') = w \supset w\,\mathsf{r}\,\mathrm{C}_j$ is deducible from these hypothesises.

If $j = 1$, then C_j is B', $\mathbf{x}_j'$ is $\mathbf{x}_1$, Θ_j' is Θ_1, and $\delta_j^{n+1}(b) = d_1(a)$ is deducible in HA. Thus we have to prove deducibility of the formula

$$\mathbf{x}_1\,\mathsf{r}\,\Theta_1\,\&\,\{d_1(a)\}(\mathbf{x}_1) = w \supset w\,\mathsf{r}\,\mathrm{B}'.$$

By the deduction theorem and the rule ($\& \to$), it is enough to prove that $w\,\mathsf{r}\,\mathrm{B}'$ is deducible from the hypothesises (68), (69) and the formulas $\mathbf{x}_1\,\mathsf{r}\,\Theta_1$, $\{d_1(a)\}(\mathbf{x}_1) = w$. Note that $\{d_1(a)\}(\mathbf{x}_1) = w$ is the formula

$$\exists v\,(\{\delta_1^n(a)\}(\mathbf{x}_1) = v\,\&\,\mathsf{l}(v) = 0\,\&\,y = \mathsf{r}(v)).$$

By the rule ($\exists \to$), we can use the formula

$$\{\delta_1^n(a)\}(\mathbf{x}_1) = v\,\&\,\mathsf{l}(v) = 0\,\&\,w = \mathsf{r}(v) \tag{83}$$

as an additional hypothesis. Note also that the formula

$$\mathbf{x}_1\,\mathsf{r}\,\Theta_1\,\&\,\{\delta_1^n(a)\}(\mathbf{x}_1) = v \supset v\,\mathsf{r}\,(\mathrm{B}' \vee \mathrm{B}'') \tag{84}$$

is deducible from the hypothesis (69). Obviously, the formulas $\mathsf{l}(v) = 0$, $w = \mathsf{r}(v)$, and $v\,\mathsf{r}\,(\mathrm{B}' \vee \mathrm{B}'')$ are deducible from the formulas (83), (84) and the hypothesis $\mathbf{x}_1\,\mathsf{r}\,\Theta_1$. Recall that $v\,\mathsf{r}\,(\mathrm{B}' \vee \mathrm{B}'')$ is the formula

$$(\mathsf{l}(v) = 0\,\&\,\mathsf{r}(v)\mathbf{r}\,\mathrm{B}') \vee (\mathsf{l}(v) \neq 0\,\&\,\mathsf{r}(v)\mathbf{r}\,\mathrm{B}'').$$

Now it is evident that the formula $w\,\mathsf{r}\,\mathrm{B}'$ is deducible from the hypothesises under consideration.

If $j = 2$, then C_j is B'', $\mathbf{x}'_j$ is $\mathbf{x}_1$, Θ'_j is Θ_1, and $\delta_j^{n+1}(b) = d_2(a)$ is deducible in HA. Thus it is enough to prove that $w \,\mathsf{r}\, \mathrm{B}''$ is deducible from the hypothesises (68), (69) and the formulas $\mathbf{x}_1 \,\mathsf{r}\, \Theta_1$, $\{d_2(a)\}(\mathbf{x}_1) = w$. As in the previous case, we prove that the formulas $\mathsf{l}(v) \neq 0$, $w = \mathsf{r}(v)$, $v \,\mathsf{r}\, (\mathrm{B}' \vee \mathrm{B}'')$ and then $w \,\mathsf{r}\, \mathrm{B}''$ are deducible from the hypothesises under consideration.

If $j = 3, \ldots, n+1$, then C_j is B_{j-1}, $\mathbf{x}'_j$ is $\mathbf{x}_{j-1}$, Θ'_j is Θ_{j-1}, and the formula $\delta_j^{n+1}(b) = \delta_{j-1}^n(a)$ is deducible in HA. Thus we have to prove deducibility of the formula $\mathbf{x}_{j-1} \,\mathsf{r}\, \Theta_{j-1} \,\&\, \{\delta_{j-1}^n(a)\}(\mathbf{x}_{j-1}) = w \supset w \,\mathsf{r}\, \mathrm{B}_{j-1}$ from the hypothesises (68), (69), but this formula is evidently deducible from (69). This completes the consideration of the axiom 7.

Axiom 8.

$$(\theta(\theta_1(A \supset B)\nabla\theta_2 B_2 \nabla \ldots \nabla \theta_n B_n)) \supset (\theta^*(\theta_1^* B \nabla \theta_2 B_2 \nabla \ldots \nabla \theta_n B_n))$$

Here θ is a list of propositional formulas $A_1, \ldots, A_m$ and θ_i $(i = 1, \ldots, n)$ is a sublist $A_{(i,1)}, \ldots, A_{(i,m_i)}$ of θ; $A, B, B_2, \ldots, B_n$ are propositional formulas; θ^* is the sequence $A_1, \ldots, A_m, A$; θ_1^* is the sequence $A_{(1,1)}, \ldots, A_{(1,m_1)}, A$. A closed arithmetical instance of the axiom 8 is of the form

$$(\Theta(\Theta_1(\mathrm{A} \supset \mathrm{B})\nabla\Theta_2 \mathrm{B}_2 \nabla \ldots \nabla \Theta_n \mathrm{B}_n)) \supset (\Theta^*(\Theta_1^* \mathrm{B} \nabla \Theta_2 \mathrm{B}_2 \nabla \ldots \nabla \Theta_n \mathrm{B}_n)).$$

Let us denote this formula by Φ. Here Θ is a finite sequence of arithmetical sentences $\mathrm{A}_1, \ldots, \mathrm{A}_m$ in the extended language; Θ_i $(i = 1, \ldots, n)$ is a subsequence $\mathrm{A}_{(i,1)}, \ldots, \mathrm{A}_{(i,m_i)}$ of Θ; $\mathrm{A}, \mathrm{B}, \mathrm{B}_2, \ldots, \mathrm{B}_n$ are arithmetical sentences in the extended language; Θ^* is the sequence $\mathrm{A}_1, \ldots, \mathrm{A}_m, \mathrm{A}$; Θ_1^* is the sequence $\mathrm{A}_{(1,1)}, \ldots, \mathrm{A}_{(1,m_1)}, \mathrm{A}$. By Proposition 5.1, it follows that the formula $\exists e\, e \,\mathsf{r}\, \Phi$ is equivalent in MA to the formula

$$\begin{aligned} &\exists a\, a \,\mathsf{r}\, (\Theta(\Theta_1(\mathrm{A} \supset \mathrm{B})\nabla\Theta_2 \mathrm{B}_2 \nabla \ldots \nabla \Theta_n \mathrm{B}_n)) \supset \\ &\supset \exists b\, b \,\mathsf{r}\, (\Theta^*(\Theta_1^* \mathrm{B} \nabla \Theta_2 \mathrm{B}_2 \nabla \ldots \nabla \Theta_n \mathrm{B}_n)). \end{aligned} \tag{85}$$

It follows from the deduction theorem and the rules $(\exists \to)$ and $(\to \exists)$ that for proving deducibility of the formula (85) in MA it is enough to prove that in MA

$$\begin{aligned} &a \,\mathsf{r}\, (\Theta(\Theta_1(\mathrm{A} \supset \mathrm{B})\nabla\Theta_2 \mathrm{B}_2 \nabla \ldots \nabla \Theta_n \mathrm{B}_n)) \vdash \\ &\vdash b \,\mathsf{r}\, (\Theta^*(\Theta_1^* \mathrm{B} \nabla \Theta_2 \mathrm{B}_2 \nabla \ldots \nabla \Theta_n \mathrm{B}_n)) \end{aligned} \tag{86}$$

for an appropriate term b.

Note that $a \,\mathsf{r}\, (\Theta(\Theta_1(\mathrm{A} \supset \mathrm{B})\nabla\Theta_2 \mathrm{B}_2 \nabla \ldots \nabla \Theta_n \mathrm{B}_n))$ is the conjunction of the formulas

$$\begin{aligned} &\forall \mathbf{x}\, (\mathbf{x} \,\mathsf{r}\, \Theta \supset \exists v\, (\{\delta_0^n(a)\}(\mathbf{x}) = v \,\&\, \bigvee_{j=1}^{n} v = j \,\& \\ &\& \bigwedge_{j=1}^{n} (v = j \supset \exists u\, \{\delta_j^n(a)\}(\mathbf{x}_j) = u))) \end{aligned} \tag{87}$$

and

$$\begin{aligned} &\forall \mathbf{x}\, \forall v\, ((\mathbf{x}_1 \,\mathsf{r}\, \Theta_1 \,\&\, \{\delta_1^n(a)\}(\mathbf{x}_1) = v \supset v \,\mathsf{r}\, (\mathrm{A} \supset \mathrm{B})) \,\& \\ &\& \bigwedge_{j=2}^{n} (\mathbf{x}_i \,\mathsf{r}\, \Theta_j \,\&\, \{\delta_j^n(a)\}(\mathbf{x}_j) = v \supset v \,\mathsf{r}\, \mathrm{B}_j)) \end{aligned} \tag{88}$$

and $b\,\mathsf{r}\,(\Theta^*(\Theta_1^*\mathrm{B}\nabla\Theta_2\mathrm{B}_2\nabla\dots\nabla\Theta_n\mathrm{B}_n))$ is the conjunction of the formulas

$$\begin{aligned}\forall\mathbf{x},x_{m+1}\,(\mathbf{x}\,\mathsf{r}\,\Theta\,\&\,x_{m+1}\,\mathsf{r}\,\mathrm{A}\supset\exists w\,(\{\delta_0^n(b)\}(\mathbf{x},x_{m+1})=w\,\&\,\bigvee_{j=1}^{n}w=j\,\&\\ \&\,(w=1\supset\exists u\,\{\delta_1^n(b)\}(\mathbf{x}_1,x_{m+1})=u)\,\&\\ \&\,\bigwedge_{j=2}^{n}(w=j\supset\exists u\,\{\delta_j^n(b)\}(\mathbf{x}_j)=u)))\end{aligned}\tag{89}$$

and

$$\begin{aligned}\forall\mathbf{x},x_{m+1}\,\forall w\,((\mathbf{x}_1\,\mathsf{r}\,\Theta_1\,\&\,x_{m+1}\,\mathsf{r}\,\mathrm{A}\,\&\,\{\delta_1^n(b)\}(\mathbf{x}_1,x_{m+1})=w\supset w\,\mathsf{r}\,\mathrm{B})\,\&\\ \&\,\bigwedge_{j=2}^{n}(\mathbf{x}_j\,\mathsf{r}\,\Theta_j\,\&\,\{\delta_j^n(b)\}(\mathbf{x}_j)=w\supset w\,\mathsf{r}\,\mathrm{B}_j)).\end{aligned}\tag{90}$$

Consider the Σ-formula $\{\delta_0^n(a)\}(\mathbf{x})=y$. Clearly, it is uniformized relative to y. By Proposition 3.1, there exists a term $d_0(a)$ such that

$$\{d_0(a)\}(\mathbf{x},x_{m+1})=y\equiv\{\delta_0^n(a)\}(\mathbf{x})=y\tag{91}$$

is deducible in HA.

Consider the Σ-formula $\exists z\,(\{\delta_1^n(a)\}(\mathbf{x}_1)=z\,\&\,\{z\}(x_{m+1})=y)$. Clearly, it is uniformized relative to y. By Proposition 3.1, there exists a term $d_1(a)$ such that the formula

$$\{d_1(a)\}(\mathbf{x}_1,x_{m+1})=y\equiv\exists z\,(\{\delta_1^n(a)\}(\mathbf{x}_1)=z\,\&\,\{z\}(x_{m+1})=y)\tag{92}$$

is deducible in HA.

Let $b=\langle d_0(a),d_1(a),\delta_2^n(a),\dots,\delta_n^n(a)\rangle$. We prove that (86) holds. Evidently, we have to prove deducibility of (89) and (90) from (87) and (88).

By the rules $(\to\forall)$, $(\forall\to)$ and deduction theorem, in order to prove deducibility of the formula (89) from the hypothesises (87) and (88), it is enough to prove that the formula

$$\begin{aligned}\exists w\,(\{\delta_0^n(b)\}(\mathbf{x},x_{m+1})=w\,\&\,\bigvee_{j=1}^{n}w=j\,\&\\ \&\,(w=1\supset\exists u\,\{\delta_1^n(b)\}(\mathbf{x}_1,x_{m+1})=u)\,\&\\ \&\,\bigwedge_{j=2}^{n}(w=j\supset\exists u\,\{\delta_j^n(b)\}(\mathbf{x}_i)=u))\end{aligned}$$

is deducible from the hypothesis (88) and the formulas $\mathbf{x}\,\mathsf{r}\,\Theta$, $x_{m+1}\,\mathsf{r}\,\mathrm{A}$, and

$$\begin{aligned}\mathbf{x}\,\mathsf{r}\,\Theta\supset\exists v\,(\{\delta_0^n(a)\}(\mathbf{x})=v\,\&\,\bigvee_{j=1}^{n}v=j\,\&\\ \&\,\bigwedge_{j=1}^{n}(v=j\supset\exists u\,\{\delta_j^n(a)\}(\mathbf{x}_j)=u)).\end{aligned}$$

Note that the formula $\delta_0^n(b)=d_0(a)$ is deducible in HA. Therefore we have to prove deducibility of the formula

$$\exists w\,(\{d_0(a)\}(\mathbf{x},x_{m+1})=w\,\&\,\bigvee_{j=1}^{n}w=j\,\&\,\bigwedge_{j=1}^{n}(w=j\supset\exists u\,\{\delta_j^n(b)\}(\mathbf{x}_j)=u))$$

from the hypothesises under consideration. As the formula

$$\exists v\,(\{\delta_0^n(a)\}(\mathbf{x}) = v \,\&\, \bigvee_{j=1}^{n} v = j \,\&\, \bigwedge_{j=1}^{n} (v = j \supset \exists u\,\{\delta_j^n(a)\}(\mathbf{x}_j) = u))$$

is deducible from these hypothesises, by the rule $(\exists \to)$, we can add the formulas

$$\{\delta_0^n(a)\}(\mathbf{x}) = v,\ \bigvee_{j=1}^{n} v = j,\ \bigwedge_{j=1}^{n} (v = j \supset \exists u\,\{\delta_j^n(a)\}(\mathbf{x}_j) = u) \tag{93}$$

to the hypothesises.

First we show that $\exists w\,\{d_0(a)\}(\mathbf{x}, x_{m+1}) = w$ is deducible from (93). Note that this formula is equivalent in MA to the formula $\exists w\,\{\delta_0^n(a)\}(\mathbf{x}) = w$. Now its deducibility from the hypothesises (93) is evident. Moreover, we see that it is enough to prove deducibility of the formulas $(v = 1 \supset \exists u\,\{\delta_1^n(b)\}(\mathbf{x}_1, x_{m+1}) = u)$, $\bigvee_{j=1}^{n} v = j$, and $\bigwedge_{j=2}^{n} (v = j \supset \exists u\,\{\delta_j^n(b)\}(\mathbf{x}_j) = u)$ from the hypothesises (93). Deducibility of the second of them is evident. Note that $\delta_j^n(b) = \delta_j^n(a)$ is deducible in HA for $j = 2, \dots, n$, therefore deducibility of the third formula is evident. Consider the first formula. It is enough to deduce $\exists u\,\{\delta_1^n(b)\}(\mathbf{x}_1, x_{m+1}) = u$ from the hypothesis $v = 1$ and other hypothesises. Obviously, the formula $\exists z\{\delta_1^n(a)\}(\mathbf{x}_1) = z$ is deducible from the hypothesises (93) and $v = 1$. By the rule $(\exists \to)$, we can add the formula

$$\{\delta_1^n(a)\}(\mathbf{x}_1) = z \tag{94}$$

to the hypothesises. Note that the formula $\delta_1^n(b) = d_1(a)$ is deducible in HA. Thus we have to deduce the formula $\exists u\,\{d_1(a)\}(\mathbf{x}_1) = u$. Taking into account (92), by the rule $(\to \exists)$, it is enough to deduce the formula $\exists u\,\{z\}(x_{m+1}) = u$. Now we use the hypothesises $\mathbf{x}\,\mathsf{r}\,\Theta$, $x_{m+1}\,\mathsf{r}\,\mathrm{A}$, and (88). Evidently, the formula $\mathbf{x}_1\,\mathsf{r}\,\Theta_1 \,\&\, \{\delta_1^n(a)\}(\mathbf{x}_1) = z \supset z\,\mathsf{r}\,(\mathrm{A} \supset \mathrm{B})$ is deducible from the last one. As the formula $\mathbf{x}_1\,\mathsf{r}\,\Theta_1$ is deducible from $\mathbf{x}\,\mathsf{r}\,\Theta$, taking into account the hypothesis (94), we obtain deducibility of the formula $z\,\mathsf{r}\,(\mathrm{A} \supset \mathrm{B})$. Now deducibility of the formula $\exists u\,\{z\}(x_{m+1}) = u$ immediately follows from the hypothesis $x_{m+1}\,\mathsf{r}\,\mathrm{A}$ and the definition of the formula $z\,\mathsf{r}\,(\mathrm{A} \supset \mathrm{B})$. We have proved that (89) is deducible from (87) and (88).

In order to prove deducibility of the formula (90) from the hypothesises (87) and (88), it is enough to prove that the formulas

$$\mathbf{x}_1\,\mathsf{r}\,\Theta_1 \,\&\, x_{m+1}\,\mathsf{r}\,\mathrm{A} \,\&\, \{\delta_1^n(b)\}(\mathbf{x}_1, x_{m+1}) = w \supset w\,\mathsf{r}\,\mathrm{B}$$

and

$$\mathbf{x}_j\,\mathsf{r}\,\Theta_j \,\&\, \{\delta_j^n(b)\}(\mathbf{x}_j) = w \supset w\,\mathsf{r}\,\mathrm{B}_j$$

for $j = 1, 2, \dots, n$ are deducible from these hypothesises. Consider the first of these formulas. Note that the formula $\delta_1^n(b) = d_1(a)$ is deducible in HA. Thus, by deduction theorem, it is enough to deduce the formula $w\,\mathsf{r}\,\mathrm{B}$ from the hypothesises

$$\mathbf{x}_1\,\mathsf{r}\,\Theta_1,\ x_{m+1}\,\mathsf{r}\,\mathrm{A},\ \{d_1(a)\}(\mathbf{x}_1, x_{m+1}) = w.$$

Using (92), we get that $\exists z\,(\{\delta_1^n(a)\}(\mathbf{x}_1) = z \,\&\, \{z\}(x_{m+1}) = w)$ is deducible from the hypothesises under consideration. By the rule $(\exists\to)$, we can use the formula $\{\delta_1^n(a)\}(\mathbf{x}_1) = z \,\&\, \{z\}(x_{m+1}) = w$ as an additional hypothesis. Note that $\mathbf{x}_1 \,\mathsf{r}\, \Theta_1 \,\&\, \{\delta_1^n(a)\}(\mathbf{x}_1) = z \supset z \,\mathsf{r}\, (\mathrm{A} \supset \mathrm{B})$ is deducible from the hypothesis (88). Thus we have deduced the formulas $x_{m+1} \,\mathsf{r}\, \mathrm{A}$, $\{z\}(x_{m+1}) = w$, and $z \,\mathsf{r}\, (\mathrm{A} \supset \mathrm{B})$. Now deducibility of the formula $w \,\mathsf{r}\, \mathrm{B}$ immediately follows from the definition of the formula $z \,\mathsf{r}\, (\mathrm{A} \supset \mathrm{B})$. We have proved that (90) is deducible from (87) and (88). This completes the consideration of the axiom 8.

Axiom 9.

$$(A \supset A_p) \,\&\, (\theta(\nabla_{i=1}^n \theta_i B_i)) \supset (\theta'(\nabla_{i=1}^n \theta_i' B_i))$$

Here A is a propositional formula; θ is a finite sequence of propositional formulas $A_1, \ldots, A_m$; $1 \le p \le m$; θ_i $(i = 1, \ldots, n)$ is a subsequence $A_{(i,1)}, \ldots, A_{(i,m_i)}$ of θ; θ' and θ_i' are the sequences obtained by replacing in θ and θ' the formula A_p by A; $B_1, \ldots, B_n$ are propositional formulas. A closed arithmetical instance of the axiom 9 is of the form

$$(\mathrm{A} \supset \mathrm{A}_p) \,\&\, (\Theta(\nabla_{i=1}^n \Theta_i \mathrm{B}_i)) \supset (\Theta'(\nabla_{i=1}^n \Theta_i' \mathrm{B}_i)).$$

Let us denote this formula by Φ. Here A is an arithmetical sentence in the extended language; Θ is a finite sequence of arithmetical sentences $\mathrm{A}_1, \ldots, \mathrm{A}_m$ in the extended language; Θ_i $(i = 1, \ldots, n)$ is a subsequence $\mathrm{A}_{(i,1)}, \ldots, \mathrm{A}_{(i,m_i)}$ of Θ; Θ' and Θ_i' are the sequences obtained by replacing in Θ and Θ' the formula A_p by A; $\mathrm{B}_1, \ldots, \mathrm{B}_n$ are arithmetical sentences in the extended language. By Proposition 5.1, it follows that the formula $\exists e\, e \,\mathsf{r}\, \Phi$ is equivalent in MA to the formula

$$\exists a\, a \,\mathsf{r}\, (\mathrm{A} \supset \mathrm{A}_p) \,\&\, \exists b\, b \,\mathsf{r}\, (\Theta(\nabla_{i=1}^n \Theta_i \mathrm{B}_i)) \supset \exists c\, c \,\mathsf{r}\, (\Theta'(\nabla_{i=1}^n \Theta_i' \mathrm{B}_i)). \tag{95}$$

It follows from the deduction theorem and the rules $(\&\to)$, $(\exists\to)$, and $(\to\exists)$ that for proving deducibility of (95) in MA it is enough to prove that in MA

$$a \,\mathsf{r}\, (\mathrm{A} \supset \mathrm{A}_p),\; b \,\mathsf{r}\, (\Theta(\nabla_{i=1}^n \Theta_i \mathrm{B}_i)) \vdash c \,\mathsf{r}\, (\Theta'(\nabla_{i=1}^n \Theta_i' \mathrm{B}_i)) \tag{96}$$

for an appropriate term c.

Note that $a \,\mathsf{r}\, (\mathrm{A} \supset \mathrm{A}_p)$ is the formula

$$\forall x\,(x \,\mathbf{r}\, \mathrm{A} \supset \exists y\,(\{a\}(x) = y \,\&\, y \,\mathbf{r}\, \mathrm{A}_p)), \tag{97}$$

$b \,\mathsf{r}\, (\Theta(\nabla_{i=1}^n \Theta_i \mathrm{B}_i))$ is the conjunction of the formulas

$$\begin{aligned} \forall \mathbf{x}\,(\bigwedge_{i=1}^m x_i \,\mathsf{r}\, \mathrm{A}_i \supset \exists v\,(\{\delta_0^n(b)\}(\mathbf{x}) = v \,\&\, \bigvee_{j=1}^n v = j \,\&\, \\ \&\, \bigwedge_{j=1}^n (v = j \supset \exists u\,\{\delta_j^n(b)\}(\mathbf{x}_j) = u))) \end{aligned} \tag{98}$$

and

$$\forall \mathbf{x}\,\forall v\,(\bigwedge_{j=1}^n (\mathbf{x}_j \,\mathsf{r}\, \Theta_j \,\&\, \{\delta_j^n(b)\}(\mathbf{x}_j) = v \supset v \,\mathsf{r}\, \mathrm{B}_j)), \tag{99}$$

and $c\,\mathsf{r}\,(\Theta'(\nabla_{i=1}^{n}\Theta_i'\mathrm{B}_i))$ is the conjunction of the formulas

$$\forall\mathbf{x}\,((\bigwedge_{\substack{i=1\\ i\neq p}}^{n} x_i\,\mathsf{r}\,\mathrm{A}_i)\,\&\,x_p\,\mathsf{r}\,\mathrm{A}\supset\exists w\,(\{\delta_0^n(c)\}(\mathbf{x})=w\,\&\,\bigvee_{j=1}^{n} w=j\,\&\,\&\bigwedge_{j=1}^{n}(w=j\supset\exists u\,\{\delta_j^n(c)\}(\mathbf{x}_j)=u))) \tag{100}$$

and

$$\forall\mathbf{x}\,\forall w\,(\bigwedge_{j=1}^{n}(\mathbf{x}_j\,\mathsf{r}\,\Theta_j'\,\&\,\{\delta_j^n(c)\}(\mathbf{x}_j)=w\supset w\,\mathsf{r}\,\mathrm{B}_j)). \tag{101}$$

Consider the Σ-formula

$$\exists z\,(\{a\}(x_p)=z\,\&\,\{\delta_n^0(b)\}(x_1,\dots,x_{p-1},z,x_{p+1},\dots,x_m)=y).$$

Clearly, it is uniformized relative to y. By Proposition 3.1, there exists a term $c_0(a,b)$ such that

$$\{c_0(a)\}(\mathbf{x})=y\equiv\equiv\exists z\,(\{a\}(x_p)=z\,\&\,\{\delta_n^0(b)\}(x_1,\dots,x_{p-1},z,x_{p+1},\dots,x_m)=y) \tag{102}$$

is deducible in HA.

For $j=1,\dots,n$ let $c_j(a,b)$ be $\delta_n^j(b)$ if $p\notin\{(j,1),\dots,(j,m_j)\}$. In the converse case, if $p=(j,i)$, consider the Σ-formula

$$\exists z\,(\{a\}(x_p)=z\,\&\,\{\delta_n^j(b)\}(x_{(j,1)},\dots,x_{(j,i-1)},z,x_{(j,i+1)},\dots,x_{(j,m_j)})=y).$$

Clearly, it is uniformized relative to y. By Proposition 3.1, there exists a term $c_j(a,b)$ such that

$$\{c_j(a,b)\}(\mathbf{x}_j)=y\equiv\exists z\,(\{a\}(x_p)=z\,\&\,\&\,\{\delta_n^j(b)\}(x_{(j,1)},\dots,x_{(j,i-1)},z,x_{(j,i+1)},\dots,x_{(j,m_j)})=y) \tag{103}$$

is deducible in HA.

Let $c=\langle c_0(a,b),c_1(a,b),\dots,c_n(a,b)\rangle$. We prove that (96) holds. Thus we have to prove deducibility of (100) and (101) from (97), (98), and (99).

By the rules $(\to\forall)$, $(\forall\to)$ and deduction theorem, in order to prove deducibility of the formula (100) from the hypothesises (97), (98), and (99), it is enough to prove that the formula

$$\exists w\,(\{\delta_0^n(c)\}(\mathbf{x})=w\,\&\,\bigvee_{j=1}^{n} w=j\,\&\,\bigwedge_{j=1}^{n}(w=j\supset\exists u\,\{\delta_j^n(c)\}(\mathbf{x}_j)=u))$$

is deducible from (97), (99) and the formulas $\bigwedge_{\substack{j=1\\ j\neq p}}^{n} x_j\,\mathsf{r}\,\mathrm{A}_j$, $x_p\,\mathsf{r}\,\mathrm{A}$, and

$$\bigwedge_{i=1}^{m} x_i\,\mathsf{r}\,\mathrm{A}_i\supset\supset\exists v\,(\{\delta_0^n(b)\}(\mathbf{x})=v\,\&\,\bigvee_{j=1}^{n} v=j\,\&\,\bigwedge_{j=1}^{n}(v=j\supset\exists u\,\{\delta_j^n(b)\}(\mathbf{x}_j)=u)). \tag{104}$$

Note that the formula $\delta_0^n(c) = c_0(a,b)$ is deducible in HA. Therefore we have to prove deducibility of the formula

$$\exists w\,(\{c_0(a,b)\}(\mathbf{x}) = w \,\&\, \bigvee_{j=1}^{n} w = j \,\&\, \bigwedge_{j=1}^{n}(w = j \supset \exists u\,\{\delta_j^n(c)\}(\mathbf{x}_j) = u)) \tag{105}$$

from the hypothesises under consideration. Obviously,

$$\exists z\,(\{a\}(x_p) = z \,\&\, z\,\mathbf{r}\,\mathrm{A}_p)$$

is deducible from the hypothesises (97) and $x_p\,\mathsf{r}\,\mathrm{A}$. By the rule $(\exists \to)$, we can add the formulas

$$\{a\}(x_p) = z,\ z\,\mathbf{r}\,\mathrm{A}_p \tag{106}$$

to the hypothesises. The formula

$$\exists v\,(\{\delta_0^n(b)\}(x_1,\ldots,x_{p-1},z,x_{p+1},\ldots,x_m) = v \,\&\, \bigvee_{j=1}^{n} v = j \,\&$$

$$\&\, \bigwedge_{j=1}^{n}(v = j \supset \exists u\,\{\delta_j^n(b)\}(\mathbf{x}'_j) = u)),$$

where $\mathbf{x}'_j$ is $x_{(j,1)},\ldots,x_{(j,k-1)},z,x_{(j,k+1)},\ldots,x_{(j,m_j)}$ if $p = (j,k)$ and $\mathbf{x}_j$ else, is deducible from (106) and (104). By the rule $(\exists \to)$, we can add the formulas

$$\begin{gathered}\{\delta_0^n(b)\}(x_1,\ldots,x_{p-1},z,x_{p+1},\ldots,x_m) = v,\ \bigvee_{j=1}^{n} v = j,\\ \bigwedge_{j=1}^{n}(v = j \supset \exists u\,\{\delta_j^n(b)\}(\mathbf{x}'_j) = u)\end{gathered} \tag{107}$$

to the hypothesises.

Now we prove deducibility of the formula (105) from the hypothesises under consideration. First we show that the formula $\exists w\,\{c_0(a,b)\}(\mathbf{x}) = w$ is deducible from the hypothesises. Note that this formula is equivalent in MA to the formula

$$\exists w\,\exists z\,(\{a\}(x_p) = z \,\&\, \{\delta_n^0(b)\}(x_1,\ldots,x_{p-1},z,x_{p+1},\ldots,x_m) = w).$$

Now its deducibility from the hypothesises (106) and (107) is evident. Moreover, we see that it is enough to prove deducibility of the formulas $\bigvee_{j=1}^{n} v = j$ and $\bigwedge_{j=1}^{n}(v = j \supset \exists u\,\{\delta_j^n(c)\}(\mathbf{x}_j) = u)$ from the hypothesises. Deducibility of the first of them is evident. In order to prove deducibility of the second formula, note that the formula $\delta_j^n(c) = c_j(a,b)$ is deducible in HA for $j = 1,\ldots,n$. Thus it is enough for $j = 1,2,\ldots,n$ to deduce the formula $\exists u\,\{c_j(a,b)\}(\mathbf{x}'_j) = u$ from the hypothesis $v = j$ and other hypothesises. Let j be fixed. If $p \notin \{(j,1),\ldots,(j,m_j)\}$, then $c_j(a,b) = \delta_j^n(b)$, $\mathbf{x}'_j$ is $\mathbf{x}_j$, thus we have to prove deducibility of the formula $\exists u\,\{\delta_j^n(b)\}(\mathbf{x}'_j) = u$ from the hypothesises $v = j$ and (107), but this is evident. If $p = (j,k)$ for some $k \in \{(j,1),\ldots,(j,m_j)\}$, then $\exists u\,\{c_j(a,b)\}(\mathbf{x}_j) = u$ is the formula

$$\exists u \exists z(\{a\}(x_p) = z \,\&\, \{\delta_n^j(b)\}(x_{(j,1)},\ldots,x_{(j,i-1)},z,x_{(j,i+1)},\ldots,x_{(j,m_j)}) = u).$$

This formula is evidently deducible from the hypothesises (106) and (107) because $\mathbf{x}'_j$ is $x_{(j,1)}, \ldots, x_{(j,i-1)}, z, x_{(j,i+1)}, \ldots, x_{(j,m_j)}$. We have proved that (100) is deducible from the hypothesises (97), (98), and (99).

In order to prove deducibility of the formula (101) from (97), (98), and (99), by the rule ($\to \forall$), deduction theorem, and the rule ($\& \to$), it is enough to prove that the formula $w \,\mathsf{r}\, \mathrm{B}_j$ for $j = 1, 2, \ldots, n$ is deducible from the hypothesises (97), (98), (99), $\mathbf{x}_j \,\mathsf{r}\, \Theta'_j$, and $\{\delta^n_j(c)\}(\mathbf{x}_j) = w$. Let j be fixed. Note that $\delta^n_j(c) = c_j(a, b)$ is deducible in HA. If $p \notin \{(j,1), \ldots, (j, m_j)\}$, then $c_j(a,b) = \delta^n_j(b)$, $\mathbf{x}'_j$ is $\mathbf{x}_j$, Θ'_j is Θ_j. Thus we have to prove deducibility of the formula $w \,\mathsf{r}\, \mathrm{B}_j$ from the formulas $\mathbf{x}_j \,\mathsf{r}\, \Theta_j$, $\{\delta^n_j(b)\}(\mathbf{x}_j) = w$ and other hypothesis, but this is evident in view of the hypothesis (99). If $p = (j,k)$ for some $k \in \{(j,1), \ldots, (j,m_j)\}$, then Θ'_j is $\mathrm{A}_{(j,1)}, \ldots, \mathrm{A}_{(j,i-1)}, \mathrm{A}, \mathrm{A}_{(j,i+1)}, \ldots, \mathrm{A}_{(j,m_j)}$, $\mathbf{x}_j \,\mathsf{r}\, \Theta'_j$ is equivalent to the formula

$$(108) \qquad \Big(\bigwedge_{\substack{i=1 \\ i \neq p}}^{n} x_i \,\mathsf{r}\, \mathrm{A}_i \Big) \,\&\, x_p \,\mathsf{r}\, \mathrm{A},$$

$\mathbf{x}'_j$ is $x_{(j,1)}, \ldots, x_{(j,k-1)}, z, x_{(j,k+1)}, \ldots, x_{(j,m_j)}$, and $\{\delta^n_j(c)\}(\mathbf{x}_j) = w$ is the formula

$$(109) \qquad \exists z\, (\{a\}(x_p) = z \,\&\, \{\delta^j_n(b)\}(x'_j) = w).$$

Thus we have to prove deducibility of the formula $w \,\mathsf{r}\, \mathrm{B}_j$ from the formulas (97), (98), (99), (108), and (109). By the rule ($\exists \to$), we can replace the hypothesis (109) by the formulas

$$(110) \qquad \{a\}(x_p) = z, \ \{\delta^j_n(b)\}(x'_j) = w.$$

Obviously, the formula $z \,\mathbf{r}\, \mathrm{A}_p$ is deducible from the hypothesises (108), (110), and (97). It follows that the formula $\mathbf{x}'_j \,\mathsf{r}\, \Theta_j$ is deducible from (108). On the other hand, the formula

$$\mathbf{x}'_j \,\mathsf{r}\, \Theta_j \,\&\, \{\delta^n_j(b)\}(\mathbf{x}'_j) = v \supset v \,\mathsf{r}\, \mathrm{B}_j$$

is deducible from the hypothesis (99). Thus we have deducibility of the formula $\{\delta^n_j(b)\}(\mathbf{x}'_j) = v \supset v \,\mathsf{r}\, \mathrm{B}_j$. Using (110), we obtain deducibility of the formula $w \,\mathsf{r}\, \mathrm{B}_j$. We have proved that (101) is deducible from (97), (98), and (99). This completes the consideration of the axiom 9.

Axiom 10.

$$(B_1 \supset B) \,\&\, (\theta(\theta_1 B_1 \nabla \theta_2 B_2 \nabla \ldots \nabla \theta_n B_n)) \supset (\theta(\theta_1 B \nabla \theta_2 B_2 \nabla \ldots \nabla \theta_n B_n))$$

Here θ is a list of propositional formulas $A_1, \ldots, A_m$ and θ_i $(i = 1, \ldots, n)$ is a sublist $A_{(i,1)}, \ldots, A_{(i,m_i)}$ of θ; $B, B_1, \ldots, B_n$ are propositional formulas. A closed arithmetical instance of the axiom 10 is of the form

$$(\mathrm{B}_1 \supset \mathrm{B}) \,\&\, (\Theta(\Theta_1 \mathrm{B}_1 \nabla \Theta_2 \mathrm{B}_2 \nabla \ldots \nabla \Theta_n \mathrm{B}_n)) \supset$$
$$\supset (\Theta(\Theta_1 \mathrm{B} \nabla \Theta_2 \mathrm{B}_2 \nabla \ldots \nabla \Theta_n \mathrm{B}_n)).$$

Let us denote this formula by Φ. Here Θ is a finite sequence of arithmetical sentences $\mathrm{A}_1, \ldots, \mathrm{A}_m$ in the extended language; Θ_i $(i = 1, \ldots, n)$ is a subsequence $\mathrm{A}_{(i,1)}, \ldots, \mathrm{A}_{(i,m_i)}$ of Θ; $\mathrm{B}, \mathrm{B}_1, \ldots, \mathrm{B}_n$ are arithmetical sentences in the extended language. By Proposition 5.1, it follows that the formula $\exists e\, e\, \mathsf{r}\, \Phi$ is equivalent in MA to the formula

$$\exists a\, a\, \mathsf{r}\, (\mathrm{B}_1 \supset \mathrm{B}) \,\&\, \exists b\, b\, \mathsf{r}\, (\Theta(\Theta_1 \mathrm{B}_1 \nabla \Theta_2 \mathrm{B}_2 \nabla \ldots \nabla \Theta_n \mathrm{B}_n)) \supset \supset \exists c\, c\, \mathsf{r}\, (\Theta(\Theta_1 \mathrm{B} \nabla \Theta_2 \mathrm{B}_2 \nabla \ldots \nabla \Theta_n \mathrm{B}_n)). \tag{111}$$

It follows from the deduction theorem and the rules $(\& \rightarrow)$, $(\exists \rightarrow)$, and $(\rightarrow \exists)$ that for proving deducibility of (111) in MA it is enough to prove that in MA

$$a\, \mathsf{r}\, (\mathrm{B}_1 \supset \mathrm{B}),\ b\, \mathsf{r}\, (\Theta(\Theta_1 \mathrm{B}_1 \nabla \Theta_2 \mathrm{B}_2 \nabla \ldots \nabla \Theta_n \mathrm{B}_n)) \vdash \vdash c\, \mathsf{r}\, (\Theta(\Theta_1 \mathrm{B} \nabla \Theta_2 \mathrm{B}_2 \nabla \ldots \nabla \Theta_n \mathrm{B}_n)) \tag{112}$$

for an appropriate term c.

Note that $a\, \mathsf{r}\, (\mathrm{B}_1 \supset \mathrm{B})$ is the formula

$$\forall x\, (x\, \mathbf{r}\, \mathrm{B}_1 \supset \exists y\, (\{a\}(x) = y \,\&\, y\, \mathbf{r}\, \mathrm{B})), \tag{113}$$

$b\, \mathsf{r}\, (\Theta(\Theta_1 \mathrm{B}_1 \nabla \Theta_2 \mathrm{B}_2 \nabla \ldots \nabla \Theta_n \mathrm{B}_n))$ is the conjunction of the formulas

$$\forall \mathbf{x}\, (\mathbf{x}\, \mathsf{r}\, \Theta \supset \exists v\, (\{\delta_0^n(b)\}(\mathbf{x}) = v \,\&\, \bigvee_{j=1}^{n} v = j \,\&\, \& \bigwedge_{j=1}^{n} (v = j \supset \exists u\, \{\delta_j^n(b)\}(\mathbf{x}_j) = u))) \tag{114}$$

and

$$\forall \mathbf{x}\, \forall v\, (\bigwedge_{j=1}^{n} (\mathbf{x}_j\, \mathsf{r}\, \Theta_j \,\&\, \{\delta_j^n(b)\}(\mathbf{x}_j) = v \supset v\, \mathsf{r}\, \mathrm{B}_j)), \tag{115}$$

and $c\, \mathsf{r}\, (\Theta(\Theta_1 \mathrm{B} \nabla \Theta_2 \mathrm{B}_2 \nabla \ldots \nabla \Theta_n \mathrm{B}_n))$ is the conjunction of the formulas

$$\forall \mathbf{x}\, ((\mathbf{x}\, \mathsf{r}\, \Theta \supset \exists w\, (\{\delta_0^n(c)\}(\mathbf{x}) = w \,\&\, \bigvee_{j=1}^{n} w = j \,\&\, \& \bigwedge_{j=1}^{n} (w = j \supset \exists u\, \{\delta_j^n(c)\}(\mathbf{x}_j) = u))) \tag{116}$$

and

$$\forall \mathbf{x}\, \forall w\, (\{\delta_1^n(c)\}(\mathbf{x}_1) = w \supset w\, \mathsf{r}\, \mathrm{B} \,\&\, \& \bigwedge_{j=2}^{n} (\mathbf{x}_j\, \mathsf{r}\, \Theta_j \,\&\, \{\delta_j^n(c)\}(\mathbf{x}_j) = w \supset w\, \mathsf{r}\, \mathrm{B}_j)). \tag{117}$$

Consider the Σ-formula

$$\exists z\, (\{\delta_1^n(b)\}(\mathbf{x}_1) = z \,\&\, \{a\}(z) = y).$$

Clearly, it is uniformized relative to y. By Proposition 3.1, there exists a term $c_1(a, b)$ such that

$$\{c_1(a, b)\}(\mathbf{x}) = y \equiv \exists z\, (\{\delta_1^n(b)\}(\mathbf{x}_1) = z \,\&\, \{a\}(z) = y) \tag{118}$$

is deducible in HA.

Let $c = \langle \delta_0^n(b), c_1(a, b), \delta_2^n(b), \ldots, \delta_n^n(b) \rangle$. We prove that (111) holds. Thus we have to prove deducibility of (116) and (117) from (113), (114), and (115).

By the rules ($\to\forall$), ($\forall\to$) and deduction theorem, in order to prove deducibility of the formula (116) from the hypothesises (113), (114), and (115), it is enough to prove that the formula

$$\exists w\,(\{\delta_0^n(c)\}(\mathbf{x}) = w \,\&\, \bigvee_{j=1}^{n} w = j \,\&\, \bigwedge_{j=1}^{n} (w = j \supset \exists u\,\{\delta_j^n(c)\}(\mathbf{x}_j) = u)))$$

is deducible from (113), (115) and the formulas $\mathbf{x}\,\mathsf{r}\,\Theta$ and

$$\mathbf{x}\,\mathsf{r}\,\Theta \supset \exists v\,(\{\delta_0^n(b)\}(\mathbf{x}) = v \,\&\, \bigvee_{j=1}^{n} v = j \,\&\, \bigwedge_{j=1}^{n} (v = j \supset \exists u\,\{\delta_j^n(b)\}(\mathbf{x}_j) = u)). \tag{119}$$

Note that the formula $\delta_0^n(c) = \delta_0^n(b)$ is deducible in HA. Therefore we have to prove deducibility of the formula

$$\exists w\,(\{\delta_0^n(b)\}(\mathbf{x}) = w \,\&\, \bigvee_{j=1}^{n} w = j \,\&\, \bigwedge_{j=1}^{n} (w = j \supset \exists u\,\{\delta_j^n(c)\}(\mathbf{x}_j) = u)) \tag{120}$$

from the hypothesises under consideration. Obviously, the formula

$$\exists v\,(\{\delta_0^n(b)\}(\mathbf{x}) = v \,\&\, \bigvee_{j=1}^{n} v = j \,\&\, \bigwedge_{j=1}^{n} (v = j \supset \exists u\,\{\delta_j^n(b)\}(\mathbf{x}_j) = u))$$

is deducible from these hypothesises. By the rule ($\exists\to$), we can add the formulas

$$\{\delta_0^n(b)\}(\mathbf{x}) = v,\ \bigvee_{j=1}^{n} v = j,\ \bigwedge_{j=1}^{n} (v = j \supset \exists u\,\{\delta_j^n(b)\}(\mathbf{x}_j) = u) \tag{121}$$

to the hypothesises. Obviously, deducibility of the formula (120) from the hypothesises will follow from deducibility of the formulas $\{\delta_0^n(b)\}(\mathbf{x}) = v$, $\bigvee_{j=1}^{n} v = j$, and $\bigwedge_{j=1}^{n} (v = j \supset \exists u\,\{\delta_j^n(c)\}(\mathbf{x}_j) = u)$. The first two formulas are obviously deducible from (121). In order to prove deducibility of the third one, it is enough to prove deducibility of the formulas

$$v = j \supset \exists u\,\{\delta_j^n(c)\}(\mathbf{x}_j) = u$$

for $j = 1, \ldots, n$. This is evident if $j \neq 0$ because $\delta_j^n(c) = \delta_j^n(b)$ is deducible in HA and $v = j \supset \exists u\,\{\delta_j^n(b)\}(\mathbf{x}_j) = u$ is a conjunct in the third formula in (121). Consider the case $j = 1$. Note that the formula $\delta_1^n(c) = c_1(a, b)$ is deducible in HA. By deduction theorem, it is enough to deduce the formula $\exists u\,\{c_1(a, b)\}(\mathbf{x}_1) = u$ from $v = 1$ and other hypothesises. By (118), we have to prove deducibility of the formula

$$\exists u\,\exists z\,(\{\delta_1^n(b)\}(\mathbf{x}_1) = z \,\&\, \{a\}(z) = u). \tag{122}$$

Obviously, the formula $\exists z\,\{\delta_1^n(b)\}(\mathbf{x}_1) = z$ is deducible from (121) and the hypothesis $v = 1$. By the rule ($\exists\to$), we can add the formula

$$\{\delta_1^n(b)\}(\mathbf{x}_1) = z \tag{123}$$

to the hypothesises. As the formula $\mathbf{x}_1 \,\mathsf{r}\, \Theta_1$ is deducible from the hypothesis $\mathbf{x} \,\mathsf{r}\, \Theta$, using the hypothesis (115), we obtain deducibility of the formula $z \,\mathsf{r}\, \mathrm{B}_1$. Now, using the hypothesis (113), we have deducibility of the formula $\exists u\,(\{a\}(z) = u \,\&\, u \,\mathbf{r}\, \mathrm{B})$. By the rule $(\exists \to)$, we can add the formula

$$\{a\}(z) = u \,\&\, u \,\mathbf{r}\, \mathrm{B} \tag{124}$$

to the hypothesises. Deducibility of (122) from (123) and (124) is evident. We have proved that (116) is deducible from the hypothesises (113), (114), and (115).

In order to prove deducibility of the formula (117) from the hypothesises (113), (114), and (115), by the rule $(\to \forall)$, deduction theorem, and the rule $(\& \to)$, it is enough to prove that

1) the formula $w \,\mathsf{r}\, \mathrm{B}$ is deducible from the hypothesises (113), (114), (115), $\mathbf{x}_1 \,\mathsf{r}\, \Theta_1$, $\{\delta_1^n(c)\}(\mathbf{x}_1) = w$;

2) the formula $w \,\mathsf{r}\, \mathrm{B}_j$ for $j = 1, 2, \ldots, n$ is deducible from (113), (114), (115), $\mathbf{x}_j \,\mathsf{r}\, \Theta_j$, $\{\delta_j^n(c)\}(\mathbf{x}_j) = w$.

Let us prove 1). As the formula $\delta_1^n(c) = c_1(a, b)$ is deducible in HA, by (118), we can replace the hypothesis $\{\delta_1^n(c)\}(\mathbf{x}_1) = w$ by

$$\exists z\,(\{\delta_1^n(b)\}(\mathbf{x}_1) = z \,\&\, \{a\}(z) = w)$$

and then, by the rule $(\exists \to)$, by the formula

$$\{\delta_1^n(b)\}(\mathbf{x}_1) = z \,\&\, \{a\}(z) = w. \tag{125}$$

The formula $z \,\mathsf{r}\, \mathrm{B}_1$ is obviously deducible from (115), $\mathbf{x}_1 \,\mathsf{r}\, \Theta_1$, and (125). Now we have deducibility of $w \,\mathsf{r}\, \mathrm{B}$ using the hypothesises (113) and (125).

The statement 2) is evident, because if $j = 1, 2, \ldots, n$, then $\delta_j^n(c) = \delta_j^n(b)$ is deducible in HA.

We have proved that (117) is deducible from (113), (114), and (115). This completes the consideration of the axiom 10.

Before considering the next axiom we introduce a new notion. A propositional (an arithmetical) formula is called ST-formula if it is of the form

$$\neg S \supset \bigvee_{t=1}^{l} \neg\neg T_t \tag{126}$$

or

$$\neg T_1 \supset S, \tag{127}$$

where $S, T_1, \ldots, T_l$ are propositional (arithmetical) formulas.

Axiom 11.

$$\theta((\nabla_{i=1}^{r} \theta_i B_i) \nabla (\nabla_{i=r+1}^{n} \theta_i(\neg S)) \supset (S \Rightarrow \bigvee_{i=1}^{r} (\neg\neg T^i \supset B_i))$$

Here θ is a sequence of propositional formulas $A_1, \ldots, A_m$ of the form (126) or (127) with a fixed propositional formula S; θ_i $(i = 1, \ldots, n)$ is a subsequence $A_{(i,1)}, \ldots, A_{(i,m_i)}$ of θ; $B_1, \ldots, B_n$ are propositional formulas; T^i $(i = 1, \ldots, r)$

is a conjunction of formulas T_t chosen in one subformula from every element of the list θ_i. A closed arithmetical instance of the axiom 11 is of the form

$$\Theta((\nabla_{i=1}^r \Theta_i \mathrm{B}_i)\nabla(\nabla_{i=r+1}^n \theta_i(\neg\mathrm{S})) \supset (\mathrm{S} \Rightarrow \bigvee_{i=1}^r (\neg\neg\mathrm{T}^i \supset \mathrm{B}_i)).$$

Let us denote this formula by Φ. Here Θ is a list $\mathrm{A}_1, \dots, \mathrm{A}_m$ of arithmetical sentences of the form

$$\neg\mathrm{S} \supset \bigvee_{t=1}^l \neg\neg\mathrm{T}_t \tag{128}$$

or

$$\neg\mathrm{T}_1 \supset \mathrm{S} \tag{129}$$

with a fixed sentence S; Θ_i $(i = 1, \dots, n)$ is a sublist $\mathrm{A}_{(i,1)}, \dots, \mathrm{A}_{(i,m_i)}$ of Θ; $\mathrm{B}_1, \dots, \mathrm{B}_n$ are arithmetical sentences; T^i $(i = 1, \dots, r)$ is an arithmetical instance of the propositional formula T^i.

By Proposition 5.1, it follows that the formula $\exists e\, e \,\mathsf{r}\, \Phi$ is equivalent in MA to the formula

$$\exists a\, a \,\mathsf{r}\, \Theta((\nabla_{i=0}^r \Theta_i \mathrm{B}_i)\nabla(\nabla_{i=r+1}^n \theta_i(\neg\mathrm{S})) \supset \exists b\, b \,\mathsf{r}\, (\mathrm{S} \Rightarrow \bigvee_{i=0}^r (\neg\neg\mathrm{T}^i \supset \mathrm{B}_i)). \tag{130}$$

It follows from the deduction theorem and the rules $(\exists \rightarrow)$ and $(\rightarrow \exists)$ that for proving deducibility of (130) in MA it is enough to prove that in MA

$$a \,\mathsf{r}\, \Theta((\nabla_{i=0}^r \Theta_i \mathrm{B}_i)\nabla(\nabla_{i=r+1}^n \theta_i(\neg\mathrm{S})) \vdash b \,\mathsf{r}\, (\mathrm{S} \Rightarrow \bigvee_{i=1}^r (\neg\neg\mathrm{T}^i \supset \mathrm{B}_i)) \tag{131}$$

for an appropriate term b.

Note that $a \,\mathsf{r}\, \Theta((\nabla_{i=1}^r \Theta_i \mathrm{B}_i)\nabla(\nabla_{i=r+1}^n \theta_i(\neg\mathrm{S}))$ is equivalent in MA to the conjunction of the formulas

$$\begin{aligned} &\forall \mathbf{x}\, (\mathbf{x} \,\mathsf{r}\, \Theta \supset \exists v\, (\{\delta_0^n(a)\}(\mathbf{x}) = v \,\&\, \bigvee_{j=1}^n v = j \,\& \\ &\& \bigwedge_{j=1}^n (v = j \supset \exists u\, \{\delta_j^n(a)\}(\mathbf{x}_j) = u))) \end{aligned} \tag{132}$$

and

$$\begin{aligned} &\forall \mathbf{x} \forall v\, (\bigwedge_{j=1}^r (\mathbf{x}_j \,\mathsf{r}\, \Theta_j \,\&\, \{\delta_j^n(a)\}(\mathbf{x}_j) = v \supset v \,\mathsf{r}\, \mathrm{B}_i) \,\& \\ &\& \bigwedge_{j=r+1}^n (\mathbf{x}_j \,\mathsf{r}\, \Theta_j \,\&\, \{\delta_j^n(a)\}(\mathbf{x}_j) = v \supset v \,\mathsf{r}\, \neg\mathrm{S})), \end{aligned} \tag{133}$$

and

$$b \,\mathsf{r}\, (\mathrm{S} \Rightarrow \bigvee_{i=1}^r (\neg\neg\mathrm{T}^i \supset \mathrm{B}_i))$$

is a conjunction of the formulas

$$\forall s\, (s \,\mathsf{r}\, \mathrm{S} \supset \exists u\, \{b\}(s) = u) \tag{134}$$

and

$$\forall s\,\forall u\,(\{b\}(s) = u \supset u\,\mathsf{r}\,\bigvee_{i=1}^{r}(\neg\neg \mathrm{T}^i \supset \mathrm{B}_i)). \tag{135}$$

For every natural $k \geq 1$ and every $j \in \{1, \ldots, k\}$ we define a term $f_{k,j}(z)$ inductively: $f_{1,1}(z)$ is z; $f_{i+1,j}(z)$ is $\langle 0, f_{i,j}(z)\rangle$ if $1 \leq j \leq i$ and $\langle 1, z\rangle$ if $j = i+1$. Note that for a given k we have a concrete sequence of terms $f_{k,1}(z), \ldots, f_{k,k}(z)$. If a number z is fixed, then the values $f_{k,i}(z)$ have the following property: for every $i \in \{1, \ldots, k\}$, if z realizes an arithmetical formula C_i, then $f_{k,i}(z)$ realizes the formula $\bigvee_{j=1}^{k} C_j$. Moreover, the formula

$$z\,\mathsf{r}\,C_i \supset f_{k,i}(z)\,\mathsf{r}\,\bigvee_{j=1}^{k} C_j$$

is deducible in HA.

Consider the Σ-formula

$$\bigvee_{i=1}^{r}(x = i \,\&\, y = f_{r,i}(z)).$$

Clearly, it is uniformized relative to y. By Proposition 3.1, there exists a number f such that

$$\{f\}(x, z) = y \equiv \bigvee_{i=1}^{r}(x = i \,\&\, y = f_{r,i}(z)) \tag{136}$$

is deducible in HA.

For every $j = 1, \ldots, m$, $i = 1, \ldots, n$ let l_{ij} be the term $f_{l,k}(0)$ if the formula A_j is of the form (128) and is contained in Θ_i, and T_k $(1 \leq k \leq l)$ is a conjunct in T^i; otherwise let $l_{ij} = 0$. Note that for every fixed arithmetical instance of Axiom 11 l_{ij} are fixed terms. The terms l_{ij} have the following property: if the formula A_j is of the form (128) and is contained in Θ_i, then in MA

$$0\,\mathsf{r}\,\neg\neg\mathrm{T}_k \vdash l_{ij}\,\mathsf{r}\,\bigvee_{t=1}^{l}\neg\neg\mathrm{T}_t. \tag{137}$$

Consider the Σ-formula $y = s$. Clearly, it is uniformized relative to y. By Proposition 3.1, there exists a term $[s]$ such that

$$\{[s]\}(x) = y \equiv y = s \tag{138}$$

is deducible in HA.

Let $j \in \{1, \ldots, m\}$ be such that A_j is of the form (126). Consider the Σ-formula

$$\exists v\,(\{u\}(u) = v \,\&\, \bigvee_{i=1}^{n}(v = i \,\&\, y = l_{ij})).$$

Clearly, it is uniformized relative to y. By Proposition 3.1, there exists a term $g_j(u,s)$ such that

$$\{g_j(u,s)\}(x) = y \equiv \exists v\,(\{u\}(u) = v \,\&\, \bigvee_{i=1}^{n} (v = i \,\&\, y = l_{ij})) \tag{139}$$

is deducible in HA. If j is such that A_j is of the form (127), consider the Σ-formula $y = [s]$. Clearly, it is uniformized relative to y. By Proposition 3.1, there exists a number $g_j(u,s)$ such that

$$\{g_j(u,s)\}(x) = y \equiv y = [s] \tag{140}$$

is deducible in HA. Thus for every $j = 1,\dots,m$ a term $g_j(u,s)$ is defined. Let $\mathbf{g}(u,s) = \mathbf{v}$, where $\mathbf{v}$ is the list $v_1,\dots,v_m$, denote the formula

$$\bigwedge_{j=1}^{m} g_j(u,s) = v_j.$$

Note that for every $j = 1,\dots,m$ and every term t

$$s \,\mathsf{r}\, \mathrm{S} \vdash g_j(t,s) \,\mathsf{r}\, \mathrm{A}_j \tag{141}$$

holds. Indeed, if j is such that A_j is of the form $\neg\mathrm{S} \supset \bigvee_{t=1}^{l} \neg\neg\mathrm{T}_t$, then $\forall v\, v \,\mathsf{r}\, \mathrm{A}_j$ is evidently deducible from $s \,\mathsf{r}\, \mathrm{S}$. It follows that $g_j(t,s) \,\mathsf{r}\, \mathrm{A}_j$ is also deducible. If A_j is of the form $\neg\mathrm{T}_1 \supset \mathrm{S}$, then $g_j(t,s)$ is the term $[s]$, and $g_j(t,s) \,\mathsf{r}\, \mathrm{A}_j$ is the formula

$$\forall y\,(y \,\mathsf{r}\, \neg\mathrm{T}_1 \supset \exists v\,(\{[s]\}(y) = v \,\&\, s \,\mathsf{r}\, \mathrm{S})).$$

Deducibility of this formula from the hypothesis $s \,\mathsf{r}\, \mathrm{S}$ is rather evident because by (138), the formula $\{[s]\}(y) = s$ is deducible in HA.

Consider the Σ-formula

$$\{\delta_0^n(a)\}(\mathbf{g}(u,s)) = y.$$

Clearly, it is uniformized relative to y. By Proposition 3.1, there exists a term $e(a,s)$ such that

$$\{e(a,s)\}(u) = y \equiv \{\delta_0^n(a)\}(\mathbf{g}(u,s)) = y) \tag{142}$$

is deducible in HA.

Consider the Σ-formula $\{e(a,s)\}(e(a,s)) = y \,\&\, \bigvee_{i=1}^{r} y = i$. Clearly, it is uniformized relative to y. By Proposition 3.1, there exists a term $h(a)$ such that

$$\{h(a)\}(s) = y \equiv \{e(a,s)\}(e(a,s)) = y \,\&\, \bigvee_{i=1}^{r} y = i \tag{143}$$

is deducible in HA.

Consider the Σ-formula

$$\exists z(\{h(a)\}(s) = z \,\&\, \bigvee_{i=1}^{r} (z = i \,\&$$

$$\&\, \exists w\,(\{\delta_i^n\}(\mathbf{g}_i(e(a,s),s)) = w \,\&\, \{f\}(i,[w]) = y))).$$

Clearly, it is uniformized relative to y. By Proposition 3.1, there exists a term $b(a)$ such that

$$\{b(a)\}(s) = y \equiv \exists z\,(\{h(a)\}(s) = z \,\&\, \bigvee_{i=1}^{r} (z = i \,\& \\ \&\, \exists w\,(\{\delta_i^n\}(\mathbf{g}_i(e(a,s),s)) = w \,\&\, \{f\}(i,[w]) = y))) \tag{144}$$

is deducible in HA.

We shall prove that (131) holds if b is the term $b(a)$. That is we prove deducibility of the formulas (134) and (135) from the hypothesises (132) and (133). Obviously, for proving deducibility of (134) it is enough to prove that the formula $\exists u\,\{b\}(s) = u$ is deducible in MA from the formulas (132), (133), and $s\,\mathsf{r}\,\mathrm{S}$.

It follows from (144) that $\exists u\,\{b\}(s) = u$ is equivalent to

$$\exists z\,(\{h(a)\}(s) = z \,\& \\ \&\, \bigvee_{i=1}^{r} (z = i \,\&\, \exists w\,(\{\delta_i^n\}(\mathbf{g}_i(e(a,s),s)) = w \,\&\, \exists u\,\{f\}(i,[w]) = u))). \tag{145}$$

First we prove deducibility of the formula $\exists z\,\{h(a)\}(s) = z$. By (143), this formula is equivalent to $\exists z\,(\{e(a,s)\}(e(a,s)) = z \,\&\, \bigvee_{i=1}^{r} z = i)$. Let us prove deducibility of $\exists z\,\{e(a,s)\}(e(a,s)) = z$. By (142), this formula is equivalent to $\exists z\,\{\delta_0^n(a)\}(\mathbf{g}(e(a,s),s)) = z$. By (141) with $e(a,s)$ as t, for every $j = 1,\dots,m$ the formula $g_j(e(a,s),s)\,\mathsf{r}\,\mathrm{A}_j$ is deducible from the hypothesises under consideration. This means that the formulas $\mathbf{g}(e(a,s),s)\,\mathsf{r}\,\Theta$ and $\mathbf{g}_i(e(a,s),s)\,\mathsf{r}\,\Theta_i$ $(i = 1,\dots,n)$ are also deducible. Therefore the formula

$$\exists z\,(\{\delta_0^n(a)\}(\mathbf{g}(e(a,s),s)) = z \,\&\, \bigvee_{j=1}^{n} z = j \,\& \\ \&\, \bigwedge_{j=1}^{n} (z = j \supset \exists u\,\{\delta_j^n(a)\}(\mathbf{g}_j(e(a,s),s)) = u)) \tag{146}$$

is deducible from (132). By the rule $(\exists \rightarrow)$, we can add the formulas

$$\{\delta_0^n(a)\}(\mathbf{g}(e(a,s),s)) = z,\ \bigvee_{j=1}^{n} z = j, \\ \bigwedge_{j=1}^{n} (z = j \supset \exists u\,\{\delta_j^n(a)\}(\mathbf{g}_j(e(a,s),s)) = u) \tag{147}$$

to the hypothesises. Obviously, the formula $\exists z\,\{e(a,s)\}(e(a,s)) = z$ is deducible from (147). Now for proving deducibility of the formula $\exists z\,\{h(a)\}(s) = z$ it is enough to deduce $\bigvee_{i=1}^{r} z = i$. By the rule $(\vee \rightarrow)$, using (147), we can consider the cases $z = j$ for $j = 1,\dots,n$. If $j = r+1,\dots,n$, then the formula $\exists u\,\{\delta_j^n(a)\}(\mathbf{g}_j(e(a,s),s)) = u$ is evidently deducible from (147). On the other hand,

$$\{\delta_j^n(a)\}(\mathbf{g}_j(e(a,s),s)) = u \supset u\,\mathsf{r}\,\neg\mathrm{S}$$

is deducible from (133). It easily follows that the formula $\exists u\, u\,\mathsf{r}\,\neg\mathrm{S}$ is deducible, but this contradicts to the hypothesis $s\,\mathsf{r}\,\mathrm{S}$. Thus the hypothesis $z = j$ for $j = r+1,\dots,n$ leads to a contradiction, and we have that the formulas $\bigvee_{i=1}^{r} z = i$ and

$\{h(a)\}(s) = z$ are deducible. Now for proving deducibility of the formula (134) it is enough to prove that the formula

$$\bigvee_{i=j}^{r}(z = j \,\&\, \exists w\,(\{\delta_j^n\}(\mathbf{g}_j(e(a,s),s)) = w \,\&\, \exists u\,\{f\}(j,[w]) = u))$$

is deducible from the hypothesises under consideration. By the rule $(\vee \rightarrow)$, using deducibility of the formula $\bigvee\limits_{i=1}^{r} z = i$, we can consider the cases $z = j$ for $j = 1, \ldots, r$. Let j be fixed. We have ro prove deducibility of the formula

$$\exists w\,(\{\delta_j^n\}(\mathbf{g}_j(e(a,s),s)) = w \,\&\, \exists u\,\{f\}(j,[w]) = u)).$$

The formula $\exists w\,\{\delta_j^n\}(\mathbf{g}_j(e(a,s),s)) = w$ is evidently deducible from (147). By the rule $(\exists \rightarrow)$, we can add the formula $\{\delta_j^n\}(\mathbf{g}_j(e(a,s),s)) = w$ to the hypothesises. Obviously, it is enough to prove deducibility of the formula $\exists u\,\{f\}(j,[w]) = u$. By (136), this formula is equivalent to the formula $\exists u \bigvee\limits_{i=1}^{r}(j = i \,\&\, u = f_{r,i}([w]))$. Thus it is enough to deduce $\exists u\, u = f_{r,j}([w])$, but this is evident because $f_{r,j}([w])$ is a term. Thus we have proved deducibility of the formula (134) from the hypothesises (132) and (133).

In order to prove deducibility of the formula (135) from the hypothesises (132) and (133), it is enough to prove that $u\,\mathsf{r} \bigvee\limits_{i=1}^{r}(\neg\neg \mathrm{T}^i \supset \mathrm{B}_i)$ is deducible in MA from these hypothesises and the formula $\{b\}(s) = u$. By (144), the hypothesis $\{b\}(s) = u$ is equivalent to

$$\exists z\,(\{h(a)\}(s) = z \,\&\, \bigvee_{i=1}^{r}(z = i \,\&$$
$$\&\, \exists w\,(\{\delta_i^n\}(\mathbf{g}_i(e(a,s),s)) = w \,\&\, \{f\}(i,[w]) = u))).$$

By the rule $(\exists \rightarrow)$, we can add the formulas $\{h(a)\}(s) = z$ and

$$\bigvee_{i=1}^{r}(z = i \,\&\, \exists w\,(\{\delta_i^n\}(\mathbf{g}_i(e(a,s),s)) = w \,\&\, \{f\}(i,[w]) = u)) \tag{148}$$

to the hypothesises. By (143), the formula $\{h(a)\}(s) = z$ is equivalent to $\{e(a,s)\}(e(a,s)) = z \,\&\, \bigvee\limits_{i=1}^{r} z = i$. Thus we can use additional hypothesises (148), $\{e(a,s)\}(e(a,s)) = z$, and $\bigvee\limits_{i=1}^{r} z = i$. Using the last one, by the rule $(\vee \rightarrow)$, we can consider the cases $z = i$ for $i = 1, \ldots, r$. Let i be fixed. The formula $\exists w\,(\{\delta_i^n\}(\mathbf{g}_i(e(a,s),s)) = w \,\&\, \{f\}(i,[w]) = u)$ is evidently deducible from (148). By the rule $(\exists \rightarrow)$, we can add the formulas $\{\delta_i^n\}(\mathbf{g}_i(e(a,s),s)) = w$ and $\{f\}(i,[w]) = u$ to the hypothesises. We prove deducibility of the formula $[w]\,\mathsf{r}\,(\neg\neg \mathrm{T}^i \supset \mathrm{B}_i)$. Obviously, it is enough to prove that $\exists y\,(\{[w]\}(x) = y \,\&\, y\,\mathsf{r}\,\mathrm{B}_i)$ is deducible from the formula $x\,\mathsf{r}\,\neg\neg \mathrm{T}^i$ and other hypothesises. Recall that T^i is a conjunction $\mathrm{T}_{i_1} \,\&\, \ldots \,\&\, \mathrm{T}_{i_m}$, where T_{i_j} is a subformula in A_j $(j = 1, \ldots, m)$. It is easy to prove that the formula

$$0\,\mathsf{r}\,\neg\neg \mathrm{T}_{i_j} \tag{149}$$

is deducible from $x \,\mathsf{r}\, \neg\neg\mathrm{T}^i$. Then for every j such that A_j is contained in Θ_i, the formula $g_j(e(a,s),s) \,\mathsf{r}\, \mathrm{A}_j$ is deducible. Indeed, if A_j is of the form $\neg\mathrm{S} \supset \bigvee_{t=1}^{l} \neg\neg\mathrm{T}_t$, then the formula $g_j(e(a,s),s) \,\mathsf{r}\, \mathrm{A}_j$ is equivalent to

$$\forall x\,(x \,\mathsf{r}\, \neg\mathrm{S} \supset \exists y\,(\{g_j(e(a,s),s)\}(x) = y \,\&\, y \,\mathsf{r} \bigvee_{t=1}^{l} \neg\neg\mathrm{T}_t)).$$

By (139), the formula $\{g_j(e(a,s),s)\}(x) = y$ is equivalent to

$$\exists z\,(\{e(a,s)\}(e(a,s)) = z \,\&\, \bigvee_{i=1}^{n}(z = i \,\&\, y = l_{ij})).$$

Note that $\{e(a,s)\}(e(a,s)) = i \,\&\, \bigvee_{k=1}^{r} i = k$ is deducible from the hypothesises under consideration. Thus we have only to prove deducibility of the formula $l_{ij} \,\mathsf{r} \bigvee_{t=1}^{l} \neg\neg\mathrm{T}_t$ but this is evident by (137) and deducibility of the formula (149). If A_j is of the form (129) and is contained in Θ_i, then A_j is the formula $\neg\mathrm{T}_{i_j} \supset \mathrm{S}$. In this case, the formula $g_j(e(a,s),s) \,\mathsf{r}\, \mathrm{A}_j$ is obviously deducible from (149). Thus we have proved that the formula $g_j(e(a,s),s) \,\mathsf{r}\, \mathrm{A}_j$ is deducible if A_j is contained in Θ_i. This means that the formula $\mathbf{g}_i(e(a,s),s)) \,\mathsf{r}\, \Theta_i$ is deducible from the hypothesises under consideration. Now the formula $w \,\mathsf{r}\, \mathrm{B}_i$ is obviously deducible from (133). It is evident that

$$[w] \,\mathsf{r}\, (\neg\neg\mathrm{T}^i \supset \mathrm{B}_i) \tag{150}$$

is also deducible from the hypothesises. As the formula $\{f\}(i,[w]) = u$ is a hypothesis, it follows from deducibility of (136) that the formula $u = f_{r,i}([w])$ is deducible from the hypothesises under consideration. Now it follows from deducibility of the formula (150) and the property of the term $f_{r,i}([w])$ that the formula $u \,\mathsf{r} \bigvee_{i=1}^{r}(\neg\neg\mathrm{T}^i \supset \mathrm{B}_i)$ is deducible from the hypothesises $\{b\}(s) = u$, (132) and (133). This completes the consideration of the axiom 11. ⊣

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