Intuitionistic Logic of Proofs

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1 Introduction

The logic of proofs LP was introduced in [3] and thoroughly studied in [1]. LP is a natural extension of the propositional calculus in the language representing proofs as formal objects. Proof expressing terms are constructed using constants, variables, and symbols of natural operations on derivations. Then formula $t{:}F$ has the intended interpretation "t is a proof of F". LP is complete with respect to the interpretation of $t{:}F$ in the Peano arithmetic by an arithmetical formula "t is a PA-derivation of F".

The intuitionistic logic of proofs ILP from [2] is the fragment of LP that has intuitionistic propositional axioms instead of classical ones. However, ILP is not arithmetically complete with respect to HA and, thus, it is not a logic of proofs of HA, the finding of which could be interesting [4]. On the other hand, a properly defined intuitionistic logic of proofs can yield a natural representation for the admissible rules of HA. The latter are important in connection with the problem of the provability logic of HA [8].

In [4] the basic intuitionistic logic of proofs iBLP and the intuitionistic logic of proofs iLP are introduced. Unlike iLP, iBLP has no operations on proof terms. iBLP is proved to be correct and complete with respect to the arithmetical semantics introduced. iLP is conjectured to also be correct and complete, i. e. to be the logic of proofs of HA.

This paper is organized as follows. In Section 3 we study some basic properties of iBLP and iLP.

We suggest a Kripke-style semantics for iBLP and iLP in Section 4. It is obtained by combining the semantic characterization of the admissible rules of IPC by Iemhoff [7] and the Mkrtychev-Fitting approach [11], [5] to the semantics for the logic of proofs. The projective formula technique developed by Ghilardi [6] is also extensively used. We prove the completeness theorem for the both logics with respect to the semantics introduced. Then we establish some corollaries.

In Section 5 we define the arithmetical interpretation of iLP with respect to HA and prove the completeness theorem, thereby verifying the conjecture from [4]. With this in view, we modify the completeness proof for LP by Artemov [1] using some technical lemmas from [4] and Section 4 as well as de Jongh's theorem [12].

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2 Conventions

Consider the set of propositional variables $\mathrm{Var} \rightleftharpoons \{p_i \mid i \in \omega\}$, proof variables $\mathrm{PVar} \rightleftharpoons \{u_i \mid i \in \omega\}$, and proof constants $\mathrm{PCnst} \rightleftharpoons \{c_i \mid i \in \omega\}$.

The formulas of the propositional language $\mathcal{PC}$ are constructed in the usual way using atoms $\mathrm{At} = \mathrm{Var} \cup \{\bot\}$ and the propositional connectives. The language $\mathcal{BLP}$ consists of the formulas defined by the grammar

$$F ::= \bot \mid p_{2i+1} \mid t{:}F \mid (F \vee F) \mid (F \wedge F) \mid (F \rightarrow F) \text{ for all } i \in \omega,$$

where the set of proof terms $\mathrm{Tm}_{\mathcal{BLP}}$ is defined by the grammar

$$t ::= u_i \text{ for all } i \in \omega.$$

Similarly, we define the language $\mathcal{ILP} \supset \mathcal{BLP}$ using the following grammars

$$F ::= \bot \mid p_{2i+1} \mid t{:}F \mid (F \vee F) \mid (F \wedge F) \mid (F \rightarrow F) \text{ and}$$

$$t ::= c_i \mid u_i \mid {!}\,t \mid \mathfrak{f}_i\, t \mid (t \cdot t) \mid (t + t) \text{ for all } i \in \omega.$$

A *quasiatom* is a formula of the form $t{:}F$.

The language $\mathcal{A}$ of the first-order arithmetic is defined in the usual way, however we use the 0-ary predicate symbol $\bot$ instead of $\neg$ and add the symbols for all definitions of primitive recursive functions [13]. The set of all formulas from $\mathcal{A}$ containing no free variables is denoted by $\mathcal{A}_0$.

$(F \leftrightarrow G)$ is defined as $((F \rightarrow G) \wedge (G \rightarrow F))$, $\neg F$ as $(F \rightarrow \bot)$, and $\top$ as $\bot \rightarrow \bot$. The operators have the following precedences from highest to lowest: $!$, $\mathfrak{f}_i$, $\cdot$, $+$, $:$, $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$. We assume that $t_1 \cdot t_2 \cdot \ldots \cdot t_n$ means $((\ldots(t_1 \cdot t_2) \cdot \ldots) \cdot t_n)$ and $t_1 + t_2 + \ldots + t_n$ means $((\ldots(t_1 + t_2) + \ldots) + t_n)$.

Denote the set of proof atoms $\mathrm{PVar} \cup \mathrm{PCnst}$ by PAt. Let F be a formula and t a proof term. Define the set of subformulas $SFm(F)$ and subterms $STm(t)$ in the natural way. Denote $SFm(F) \cap \mathrm{Var}$ by $\mathrm{Var}(F)$ and define $\mathrm{PAt}(\cdot)$ for formulas and terms similarly. We also use this notation for finite sets of formulas, e. g. $\mathrm{Var}(\{F_1, \ldots, F_n\}) \rightleftharpoons \bigcup_i \mathrm{Var}(F_i)$

We write $e[a_1, \ldots, a_n]$ if the expression e contains no (free) variables other than $a_1, \ldots, a_n$.

3 Logic iLP

The intuitionistic propositional logic IPC is defined in $\mathcal{PC}$ in the usual way [9].

Let $a_1, \ldots, a_n, b$ be any formulas. A *rule* is an expression of the form $\dfrac{a_1, \ldots, a_n}{b}$. For an appropriate substitution σ a rule $\dfrac{\sigma(a_1), \ldots, \sigma(a_n)}{\sigma(b)}$ is called an *instance* of $\dfrac{a_1, \ldots, a_n}{b}$. Note that we distinguish rules from *inference rules*, where $a_1, \ldots, a_n, b$ are schemata. A substitution is called *propositional* if it maps Var to $\mathcal{PC}$.

Suppose A and B are propositional formulas. The rule $\dfrac{A}{B}$ is *admissible* in IPC if $\vdash_{\mathbf{IPC}} \sigma(A)$ implies $\vdash_{\mathbf{IPC}} \sigma(B)$ for any propositional substitution σ. Then we write $A \rhd_{\mathbf{IPC}} B$.

Choose a bijection $\nu\colon \{t{:}F \mid t{:}F \in \mathcal{ILP}\} \rightarrow \omega$. The *0-translation* is a map $\cdot^0\colon \mathcal{ILP} \rightarrow \mathcal{PC}$ such that the following conditions hold:

- if $F \in \mathrm{At}$, then $F^0 = F$.
- $\cdot^0$ commutes with the propositional connectives;
- $(t\colon F)^0 = p_{2\nu(t\colon F)}$.

Clearly, $\cdot^0$ is a bijection. The reverse map is denoted by $\cdot^{-0}$.

The basic intuitionistic logic of proofs iBLP in $\mathcal{BLP}$ has the following axiom schemata:

B1 Axiom schemata of IPC;

B2 $u\colon F \to F$; *(reflexion)*

B3 $u\colon F \vee \neg u\colon F$;

B4 $\left(\bigwedge_{i=1}^{n} v_i\colon F_i\right) \to G$ if $\bigwedge_{i=0}^{n} (F_i \wedge v_i\colon F_i)^0 \rhd_{\mathbf{IPC}} G^0$;

and inference rule

MP $\dfrac{A \to B,\ A}{B}$. *(modus ponens)*

Suppose the language contains the propositional connectives. For any $n \in \omega$ an instance of the rule

$$\frac{\bigwedge_{i=1}^{n} (r_i \to q_i) \to r_{n+1} \vee r_{n+2}}{\bigvee_{j=1}^{n+2} \left(\bigwedge_{i=1}^{n} (r_i \to q_i) \to r_j\right)}$$

is called a *Visser's* rule. The set of all Visser's rules for given n is denoted by V_n.

The intuitionistic logic of proofs iLP in $\mathcal{ILP}$ is defined by the following axiom schemata:

A1 Axiom schemata of IPC

A2 $t\colon F \to F$ *(reflexion)*

A3 $t\colon F \vee \neg t\colon F$

A4 $s\colon (F \to G) \to (t\colon F \to s \cdot t\colon G)$ *(application)*

A5 $t\colon F \to\ !t\colon t\colon F$ *(proof checker)*

A6 $s\colon F \to s + t\colon F$, $t\colon F \to s + t\colon F$ *(sum)*

A7$_\mathrm{n}$ $t\colon F \to \mathfrak{f}_n t\colon G$ if $\dfrac{F}{G} \in \mathrm{V}_n$

and inference rules:

MP $\dfrac{A \to B,\ A}{B}$ *(modus ponens)*

CS $\dfrac{A}{c\colon A}$ if A is an axiom of iLP and $c \in \mathrm{PCnst}$ *(axiom necessitation)*

3.1 Elementary properties

Lemma 3.1 (substitution lemma)**.**

1. *If* $\Gamma(u,p) \vdash_{\mathbf{iBLP}} A(u,p)$*, then* $\Gamma(u\,/\,v, p\,/\,F) \vdash_{\mathbf{iBLP}} A(u\,/\,v, p\,/\,F)$ *for any* $v \in \mathrm{PVar}$ *and* $F \in \mathcal{BLP}$*.*
2. *If* $\Gamma(u,p) \vdash_{\mathbf{iLP}} A(u,p)$*, then* $\Gamma(u\,/\,t, p\,/\,F) \vdash_{\mathbf{iLP}} A(u\,/\,t, p\,/\,F)$ *for any* $t \in \mathrm{Tm}_{\mathcal{ILP}}$ *and* $F \in \mathcal{ILP}$*.*

Proof. Replace all occurrences of u and p in the derivation of A from Γ by the respective terms and formulas. We obtain correct derivations. Let us consider the case of B4. Indeed, the composition of the considered substitution and the 0-translation is a propositional substitution. Therefore it keeps admissibility in IPC. □

Lemma 3.2 (deduction theorem)**.**

1. *If* $\Gamma, A \vdash_{\mathbf{iBLP}} B$*, then* $\Gamma \vdash_{\mathbf{iBLP}} A \to B$*.*
2. *If* $\Gamma, A \vdash_{\mathbf{iLP}} B$*, then* $\Gamma \vdash_{\mathbf{iLP}} A \to B$*.*

Proof. Induction on the derivation of B from Γ, A. The standard proof for IPC [9] is valid for the first claim. We must additionally consider the case of CS to prove the second one. It can be done as for LP [1]. □

Lemma 3.3. *Suppose* $F^0 \vdash_{\mathbf{IPC}} G^0$*. Then* $F \vdash_{\mathbf{iLP}} G$ *and if also* $\{F, G\} \subset \mathcal{BLP}$*, then* $F \vdash_{\mathbf{iBLP}} G$*.*

Proof. Replace all formulas in the derivation of G^0 from F^0 with their 0-preimages. By Axioms A1 and B1, we obtain a derivation of G from F in iLP. Suppose $\{F, G\} \subset \mathcal{BLP}$. Assume that the iLP derivation is not an iBLP one. Then the IPC derivation contains H such that $H^{-0} \notin \mathcal{BLP}$, i. e., it contains $p \in \mathrm{Var}(H)$ such that $p^{-0} \notin \mathcal{BLP}$. Since $p \notin \mathrm{Var}\big(\{F^0\} \cup \{G^0\}\big)$, we can substitute any $p_{2k+1} \notin \mathrm{Var}\big(\{F^0\} \cup \{G^0\}\big)$ for all occurrences of p in the derivation of G^0 from F^0. □

Lemma 3.4 (internalization of proofs)**.** *Suppose*

$$A_1, \dots, A_n \vdash_{\mathbf{iLP}} B;$$

then there exists a proof term $t[v_1, \dots, v_n]$ *such that*

$$v_1\colon A_1, \dots, v_n\colon A_n \vdash_{\mathbf{iLP}} t\colon B.$$

Proof. Induction on the derivation of B from $A_1, \dots, A_n$ as in the case of LP [1]. □

3.2 Internalization of admissible rules

We write $F \vdash_{\mathbf{L+R}} G$ if there is a derivation of G from F in a logic L extended by a set of rules R, i. e., there exist formulas $F_0 = F, F_1, \dots, F_n = G$ in the language of L such that for all $i \in \{1, \dots, n\}$ either $F_0, \dots, F_{i-1} \vdash_{\mathbf{L}} F_i$ or there is $j \in \{1, \dots, i\}$ such that $\dfrac{F_j}{F_i} \in R$. A set R of rules admissible in IPC is a *basis* for the admissible rules of IPC if $A \rhd_{\mathbf{IPC}} B$ implies $A \vdash_{\mathbf{IPC+R}} B$.

Theorem 3.5 ([7], [8])**.** *The Visser's rules are a basis for the admissible rules of* IPC*.*

Lemma 3.6. $\vdash_{\mathbf{iBLP}} A$ *implies* $\vdash_{\mathbf{iLP}} A$*.*

Proof. Let us check all axioms and inference rules of iBLP. Consider the non-trivial case of B4. Suppose $\bigwedge_{i=1}^{n}(v_i{:}F_i \wedge F_i)^0 \rhd_{\mathbf{IPC}} G^0$. We must derive $(\bigwedge_{i=1}^{n} v_i{:}F_i) \to G$ in iLP. By Theorem 3.5, $\bigwedge_{i=1}^{n}(v_i{:}F_i \wedge F_i)^0 \vdash_{\mathbf{IPC}+\bigcup\limits_{\mathbf{m}\in\omega}\mathbf{V_m}} G^0$. By definition, there exist propositional $D_0 \rightleftharpoons \bigwedge_{i=1}^{n}(v_i{:}F_i \wedge F_i)^0, \ldots, D_l \rightleftharpoons G^0$ such that if $j \in \{1,\ldots,l\}$ then either $D_0,\ldots,D_{j-1} \vdash_{\mathbf{IPC}} D_j$ or $\dfrac{D_\alpha}{D_j} \in \mathrm{V_m}$ for some $\alpha \in \{1,\ldots,j\}$ and $m \in \omega$. Put $C_j \rightleftharpoons D_j^{-0} \in \mathcal{BLP}$. From Lemma 3.3 it follows that for any $j \in \{1,\ldots,l\}$ either

1. $C_0,\ldots,C_{j-1} \vdash_{\mathbf{iLP}} C_j$ or
2. $\dfrac{C_\alpha}{C_j} \in \mathrm{V_m}$ for some $\alpha \in \{1,\ldots,j\}$ and $m \in \omega$.

We prove $\vdash_{\mathbf{iLP}} (\bigwedge_{i=1}^{n} v_i{:}F_i) \to C_j$ by induction on j up to l. Suppose $\dfrac{C_\alpha}{C_{k+1}} \in \mathrm{V_m}$ for some $\alpha \in \{1,\ldots,k\}$. Then for some ground terms g, h and for $s \rightleftharpoons h\cdot(g\cdot{!v_1}\cdot\ldots\cdot{!v_n})$ we get:

$$\vdash_{\mathbf{iLP}} h{:}((\bigwedge_{i=1}^{n} v_i{:}F_i) \to C_\alpha); \quad \text{(the induction hypoth. and Lemma 3.4)}$$

$$\vdash_{\mathbf{iLP}} (\bigwedge_{i=1}^{n} v_i{:}F_i) \to (!v_\beta{:}v_\beta{:}F_\beta) \text{ for all } \beta \in \{1,\ldots,n\}; \qquad \text{(A5)}$$

$$\vdash_{\mathbf{iLP}} v_1{:}F_1 \to (\ldots \to (v_n{:}F_n \to (\bigwedge_{i=1}^{n} v_i{:}F_i))\ldots); \qquad \text{(A1)}$$

$$\vdash_{\mathbf{iLP}} g{:}(v_1{:}F_1 \to (\ldots \to (v_n{:}F_n \to (\bigwedge_{i=1}^{n} v_i{:}F_i))\ldots); \qquad \text{(Lemma 3.4)}$$

$$\bigwedge_{i=1}^{n} v_i{:}F_i \vdash_{\mathbf{iLP}} g\cdot{!v_1}\cdot\ldots\cdot{!v_n}{:}(\bigwedge_{i=1}^{n} v_i{:}F_i); \qquad \text{(MP, A4 n-fold)}$$

$$\bigwedge_{i=1}^{n} v_i{:}F_i \vdash_{\mathbf{iLP}} h\cdot(g\cdot{!v_1}\cdot\ldots\cdot{!v_n}){:}C_\alpha; \qquad \text{(A4)}$$

$$\vdash_{\mathbf{iLP}} s{:}C_\alpha \to \mathfrak{f}_m s{:}C_{k+1}; \qquad \text{(A7}_\mathrm{m}\text{)}$$

$$\bigwedge_{i=1}^{n} v_i{:}F_i \vdash_{\mathbf{iLP}} C_{k+1}. \qquad \text{(A2)}$$

□

Theorem 3.7 ([4])**.** *If* $F \rhd_{\mathbf{IPC}} G$*, then there exists a proof term* $t[u]$ *such that* $\vdash_{\mathbf{iLP}} u{:}F^{-0} \to t{:}G^{-0}$*. If also* $\{F,G\} \subset \mathcal{PC} \cap \mathcal{ILP}$*, then* $\vdash_{\mathbf{iLP}} u{:}F \to t{:}G$

Proof. Suppose $F \rhd_{\mathbf{IPC}} G$. Then $F \wedge (u{:}F^{-0})^0 \rhd_{\mathbf{IPC}} G$. Hence $u{:}F^{-0} \to G^{-0}$ is an axiom of iBLP and, by Lemma 3.6, $\vdash_{\mathbf{iLP}} u{:}F^{-0} \to G^{-0}$. Lemma 3.4 yields that there is a ground term g such that $\vdash_{\mathbf{iLP}} g{:}\left(u{:}F^{-0} \to G^{-0}\right)$. By Axiom A4, $\vdash_{\mathbf{iLP}} {!u}{:}u{:}F^{-0} \to g\cdot{!u}{:}G^{-0}$. Using A5, we get $\vdash_{\mathbf{iLP}} u{:}F^{-0} \to g\cdot{!u}{:}G^{-0}$. If $H \in \mathcal{PC} \cap \mathcal{ILP}$, then $H^{-0} \rightleftharpoons H$. □

A substitution of formulas from $\mathcal{A}_0$ for propositional variables is called *arithmetical.* Let the language of a theory T contain $\mathcal{A}_0$ and A, B be propositional formulas. The rule $\dfrac{A}{B}$ is *admissible* in T if $\vdash_{\mathbf{T}} \tau(A)$ implies $\vdash_{\mathbf{T}} \tau(B)$ for any arithmetical substitution τ. Then we write $A \rhd_{\mathbf{T}} B$.

The intuitionistic arithmetic HA in $\mathcal{A}$ is defined in the usual way [9], however we add the axioms for all definitions of primitive recursive functions [13], [1]. It is clear that we obtain a definitional extension of the canonically defined HA.

Theorem 3.8 ([14])**.** *$F \rhd_{\mathbf{IPC}} G$ iff $F \rhd_{\mathbf{HA}} G$.*

The following theorem is now obvious.

Theorem 3.9. *Suppose $\{F, G\} \subset \mathcal{PC} \cap \mathcal{ILP}$. Then from $F \rhd_{\mathbf{HA}} G$ it follows that there exists a proof term $t[u]$ such that $\vdash_{\mathbf{iLP}} u{:}F \to t{:}G$.*

4 Kripke semantics

We consider Kripke models $\mathsf{K} = (W, \preccurlyeq, \Vdash)$ of the language $\mathcal{PC}$ such that the set of nodes W is non-empty, the accessibility relation $\preccurlyeq\ \subseteq W \times W$ is reflexive, transitive and antisymmetric, and the forcing relation $\Vdash\ \subseteq W \times \mathrm{Var}$ is monotone, i. e., $x \Vdash p$ and $x \preccurlyeq y$ imply $y \Vdash p$. Without loss of generality, all nodes of all the considered models can be assumed to be members of a certain set $\widetilde{W}$. A forcing relation is extended to $\mathcal{PC}$:

$$\begin{array}{rcl} x \Vdash F \wedge G & \rightleftharpoons & x \Vdash F \text{ and } x \Vdash G; \\ x \Vdash F \vee G & \rightleftharpoons & x \Vdash F \text{ or } x \Vdash G; \\ x \Vdash F \to G & \rightleftharpoons & \forall y \succcurlyeq x\, (y \Vdash F \Rightarrow y \Vdash G). \end{array}$$

It is clear that $x \Vdash \bot$ for no x and $\Vdash$ is still monotone.

A Kripke model is called *rooted* if its *frame* $(W, \preccurlyeq)$ is finite and has a least node (*root*).

Theorem 4.1 ([10])**.** *$\vdash_{\mathbf{IPC}} A$ iff $\mathsf{K} \Vdash A$ for all (rooted) models K.*

A *cone* K_x of a model K is a triple $(W_x, \preccurlyeq_x, \Vdash_x)$ such that $W_x = \{y \mid x \preccurlyeq y\}$, $\preccurlyeq_x\ =\ \preccurlyeq \cap\, (W_x \times W_x)$, and $\Vdash_x\ =\ \Vdash \cap\, (W_x \times \mathrm{Var})$. Clearly, a cone is a Kripke model. The set of all cones of K is denoted by $\mathit{Cone}(\mathsf{K})$. A Kripke model is called an Iemhoff model if the following conditions hold:

1. the frames of all its cones are finite;
2. every finite set U of its nodes has a *tight predecessor*, i. e., a node x_U such that $x_U \preccurlyeq x$ for all $x \in U$ and if $x_U \prec y$, then there exists $x' \in U$ such that $x' \preccurlyeq y$.

Consider two Kripke models such that their frames are isomorphic posets and the corresponding nodes force the same variables. Such models are said to be isomorphic and to be *variants* of each other if they are rooted and their forcing relations possibly disagree at the roots. Trivially, isomorphic models force the same formulas (at the corresponding nodes). Let $\{\mathsf{K}_1, \ldots, \mathsf{K}_n\}$ be a set of rooted models $\mathsf{K}_i = (W_i, \preccurlyeq_i, \Vdash_i)$. Put $\left(\sum \mathsf{K}_i\right)' = \big(W', \preccurlyeq', \Vdash'\big)$, where

$$W' = \bigcup_{i=1}^{n} W_i \cup \{w_0\} \ \text{ for some } w_0 \in \widetilde{W} \setminus \bigcup_{i=1}^{n} W_i,$$

$$\preccurlyeq' = \bigcup_{i=1}^{n} \preccurlyeq_i \cup \left\{ (w_0, w_0), (w_0, w) \mid w \in \bigcup_{i=1}^{n} W_i \right\} \text{ and } \Vdash' = \bigcup_{i=1}^{n} \Vdash_i .$$

Obviously, $(\sum \mathsf{K}_i)'$ is a rooted model defined uniquely up to isomorphism. A class $\mathcal{K}$ of rooted models is called *extensible* if for any finite $\{\mathsf{K}_i\} \subseteq \mathcal{K}$ there is a variant of $(\sum \mathsf{K}_i)'$ in $\mathcal{K}$. Without loss of generality, we consider extensible classes to be closed under isomorphism. By definition, put $\mathcal{K}(F) \rightleftharpoons \{\mathsf{K} \mid \mathsf{K} \Vdash F\}$ for $F \in \mathcal{PC}$.

Consider a rooted model $\mathsf{K} = (W, \preccurlyeq, \Vdash)$ and a map $\Xi \colon \widetilde{W} \to 2^{\mathrm{Var}}$. A *Smorynski closure* of K for Ξ is a triple

$$Sm(\mathsf{K}, \Xi) \rightleftharpoons \left(\bigcup_{n<\omega} W_n, \bigcup_{n<\omega} \preccurlyeq_n, \bigcup_{n<\omega} \Vdash_n \right),$$

where W_n, $\preccurlyeq_n$ and $\Vdash_n$ are defined recursively:

- $W_0 \rightleftharpoons W$, $W_{n+1} \rightleftharpoons W_n \cup \left\{ \nu_n(Q) \mid Q \in 2^{W_n} \right\}$;
- $\preccurlyeq_0 \rightleftharpoons \preccurlyeq$,
  $\preccurlyeq_{n+1} \rightleftharpoons \preccurlyeq_n \cup \left\{ (\nu_n(Q), \nu_n(Q)), (\nu_n(Q), w) \mid \exists v \in Q\, (v \preccurlyeq_n w), Q \in 2^{W_n} \right\}$;
- $\Vdash_0 \rightleftharpoons \Vdash$, $\Vdash_{n+1} \rightleftharpoons \Vdash_n \cup \left\{ (\nu_n(Q), p) \mid p \in \Xi(\nu_n(Q)), Q \in 2^{W_n} \right\}$;
- $\nu_n \colon 2^{W_n} \to \widetilde{W} \setminus W_n$ is an injection.

Lemma 4.2. *Suppose $\mathsf{K} = (W, \preccurlyeq, \Vdash)$ is a rooted model such that $Cone(\mathsf{K}) \subseteq \mathcal{K}$ for an extensible $\mathcal{K}$. Then there exists an Iemhoff model K^{I} such that for a certain map $\Xi \colon \widetilde{W} \to 2^{\mathrm{Var}}$ the following conditions hold:*

1. *$Sm(\mathsf{K}, \Xi)$ is a Kripke model defined uniquely up to isomorphism;*
2. *K^{I} is isomorphic to $Sm(\mathsf{K}, \Xi)$;*
3. *$\mathsf{K} \in Cone(\mathsf{K}^{\mathsf{I}}) \subseteq \mathcal{K}$.*

Proof. Put $W' = \bigcup_{n<\omega} W_n$, $\preccurlyeq' = \bigcup_{n<\omega} \preccurlyeq_n$, and $\Vdash' = \bigcup_{n<\omega} \Vdash_n$. Clearly, $\preccurlyeq' \subseteq W' \times W'$ is reflexive, transitive and antisymmetric. Let us show that there exists Ξ such that for all $x \in W'$ the triple $K'_x = \left(W'_x, \preccurlyeq'_x, \Vdash'_x\right)$ is a rooted model and $Cone(K'_x) \subseteq \mathcal{K}$. Induction on $d(x) = \min\{n \in \omega \mid x \in W_n\}$. If $d(x) = 0$, then $K'_x = \mathsf{K}$ and $Cone(\mathsf{K}) \subseteq \mathcal{K}$. Assume that our claim is proved for all x such that $d(x) < n$. Consider x such that $d(x) = n$. By the construction of $Sm(\mathsf{K}, \Xi)$, the node x is attached from $\preccurlyeq'$-below to a set $\{x_1, \ldots, x_k\} \subset W'$ such that $d(x_i) < n$ for all $i \in \{1, \ldots, k\}$. By the induction hypothesis, $K'_{x_1}, \ldots, K'_{x_k}$ are rooted models and all their cones belong to $\mathcal{K}$. Since $\mathcal{K}$ is extensible, we can define $\Xi(x)$ so that K'_x is a rooted model in $\mathcal{K}$. Let $\Xi(y)$ be empty at any $y \in \widetilde{W} \setminus \bigcup_{0<n<\omega} W_n$. Thus, we obtain Ξ such that $\Vdash'$ is monotone. Therefore $Sm(\mathsf{K}, \Xi)$ is, up to isomorphism, a Kripke model. Since $\mathcal{K}$ is closed under isomorphism, all the cones of this model belong to $\mathcal{K}$. Note that for any finite $Q \subset W'$ and for $m = \max\{d(x) \mid x \in Q\}$ the node $\nu_m(Q)$ is a tight predecessor of Q. Let us choose one of the (mutually isomorphic) models satisfying Sm definition as K^{I}. Then the second and the third claims will be proved. $\square$

4.1 Models of $\mathcal{ILP}$ and $\mathcal{BLP}$

A map $\mathcal{E}\colon \mathrm{Tm}_{\mathcal{ILP}} \to 2^{\mathcal{ILP}}$ is called an *evidence function* for $\mathcal{ILP}$ if the following conditions hold:

- if $F \to G \in \mathcal{E}(s)$ and $F \in \mathcal{E}(t)$, then $G \in \mathcal{E}(s \cdot t)$;
- if $F \in \mathcal{E}(t)$, then $t\colon F \in \mathcal{E}(!\,t)$;
- if $F \in \mathcal{E}(t)$, then $F \in \mathcal{E}(s+t) \cap \mathcal{E}(t+s)$;
- if $F \in \mathcal{E}(t)$ and $\dfrac{F}{G} \in \mathrm{V}_n$, then $G \in \mathcal{E}(\mathfrak{f}_n\, t)$;
- if c is a proof constant and B is an axiom of iLP, then $B \in \mathcal{E}(c)$.

By definition, a model of $\mathcal{ILP}$ (or $\mathcal{ILP}$-model) is a pair $\mathsf{K}^{\mathsf{I,ILP}} = \big(\mathsf{K}^{\mathsf{I}}, \mathcal{E}\big)$, where K^{I} is an Iemhoff model and $\mathcal{E}$ is an evidence function for $\mathcal{ILP}$.

Let us define the forcing of $\mathcal{ILP}$ formulas at the nodes of $\mathsf{K}^{\mathsf{I,ILP}}$. Extend the forcing relation of K^{I} to $\mathcal{ILP}$ the similar way as to $\mathcal{PC}$ using the following additional condition for quasiatoms:

$$x \Vdash t\colon F \rightleftharpoons \mathsf{K}^{\mathsf{I,ILP}} \Vdash F \text{ and } F \in \mathcal{E}(t)\,.$$

It is clear that $x \Vdash \bot$ for no x and $\Vdash$ is still monotone.

Similarly, a model of $\mathcal{BLP}$ (or $\mathcal{BLP}$-model) is a pair $\mathsf{K}^{\mathsf{I,BLP}} = \big(\mathsf{K}^{\mathsf{I}}, \mathcal{E}\big)$, where K^{I} is an Iemhoff model and the evidence function for $\mathcal{BLP}$ $\mathcal{E}\colon \mathrm{PVar} \to 2^{\mathcal{BLP}}$ is an arbitrary map. The forcing is defined as for models of $\mathcal{ILP}$.

4.2 Correctness

To each $\mathcal{ILP}$-model $\mathsf{K}^{\mathsf{I,ILP}} = ((W, \preccurlyeq, \Vdash), \mathcal{E})$ assign the triple $\mathsf{K}^{\mathsf{I,ILP},0} \rightleftharpoons \big(W, \preccurlyeq, \Vdash^0\big)$, where $\Vdash^0 \rightleftharpoons \Vdash \cup \{(x, (t\colon F)^0) \mid x \in W \text{ and } \mathsf{K}^{\mathsf{I,ILP}} \Vdash t\colon F\}$. Likewise, $\mathsf{K}^{\mathsf{I,BLP},0}$ corresponds to the $\mathcal{BLP}$-model $\mathsf{K}^{\mathsf{I,BLP}}$. It is obvious that $\mathsf{K}^{\mathsf{I,ILP},0}$ and $\mathsf{K}^{\mathsf{I,BLP},0}$ are Iemhoff models.

Lemma 4.3.

1. $\mathsf{K}^{\mathsf{I,ILP}}, x \Vdash A$ *and* $\mathsf{K}^{\mathsf{I,ILP},0}, x \Vdash A^0$ *are equivalent for any node* x *of any* $\mathcal{ILP}$*-model* $\mathsf{K}^{\mathsf{I,ILP}}$*.*
2. $\mathsf{K}^{\mathsf{I,BLP}}, x \Vdash A$ *and* $\mathsf{K}^{\mathsf{I,BLP},0}, x \Vdash A^0$ *are equivalent for any node* x *of any* $\mathcal{BLP}$*-model* $\mathsf{K}^{\mathsf{I,BLP}}$*.*

Proof. Induction on the construction of A. □

Theorem 4.4 ([7])**.** *$A \rhd_{\mathbf{IPC}} B$ iff $\mathsf{K}^{\mathsf{I}} \Vdash A$ implies $\mathsf{K}^{\mathsf{I}} \Vdash B$ for all Iemhoff models K^{I}.*

Remark 4.5. From this theorem it follows that IPC is complete with respect to Iemhoff models. Indeed, $\nvdash_{\mathbf{IPC}} A$ implies $\top \not\rhd_{\mathbf{IPC}} A$.

Theorem 4.6.

1. *If $\vdash_{\mathbf{iLP}} A$, then $\mathsf{K}^{\mathsf{I,ILP}} \Vdash A$ for all $\mathcal{ILP}$-models $\mathsf{K}^{\mathsf{I,ILP}}$.*

2. *If* $\vdash_{\mathbf{iBLP}} A$, *then* $\mathsf{K}^{\mathsf{I},\mathsf{BLP}} \Vdash A$ *for all* $\mathcal{BLP}$*-models* $\mathsf{K}^{\mathsf{I},\mathsf{BLP}}$.

Proof. Prove the first claim by induction on the derivation of A.

A1 Trivially, the 0-translation of any $\mathcal{ILP}$-instance of an IPC axiom schema is an axiom of IPC. By Theorem 4.1, every Kripke model forces all the axioms of IPC. So does $\mathsf{K}^{\mathsf{I},\mathsf{ILP},0}$. Now use Lemma 4.3.

A2 Suppose $x \Vdash u{:}F$. Then, by definition, $\mathsf{K}^{\mathsf{I},\mathsf{ILP}} \Vdash F$. Hence $x \Vdash F$.

A3 If $x \Vdash t{:}F$, then $x \Vdash t{:}F \vee \neg t{:}F$. If $x \nVdash t{:}F$, then either $\mathsf{K}^{\mathsf{I},\mathsf{ILP}} \nVdash F$ or $F \notin \mathcal{E}(t)$. In both cases $\mathsf{K}^{\mathsf{I},\mathsf{ILP}}, y \nVdash t{:}F$ for any y. Therefore $x \Vdash \neg t{:}F$.

A5 If $x \Vdash s{:}(F \to G)$ and $x \Vdash t{:}F$, then $F \to G \in \mathcal{E}(s)$, $F \in \mathcal{E}(t)$, $\mathsf{K}^{\mathsf{I},\mathsf{ILP}} \Vdash F \to G$, and $\mathsf{K}^{\mathsf{I},\mathsf{ILP}} \Vdash F$. By definitions, this yields that $G \in \mathcal{E}(s \cdot t)$ and $\mathsf{K}^{\mathsf{I},\mathsf{ILP}} \Vdash G$, i. e., $x \Vdash s \cdot t{:}G$. The cases of A5 and A6 are considered similarly.

$\mathrm{A7_n}$ Suppose $x \Vdash t{:}F$, i. e., $\mathsf{K}^{\mathsf{I},\mathsf{ILP}} \Vdash F$ and $F \in \mathcal{E}(t)$. By Lemma 4.3, $\mathsf{K}^{\mathsf{I},\mathsf{ILP},0} \Vdash F^0$. From Theorem 3.5 it follows that $F^0 \rhd_{\mathbf{IPC}} G^0$. Using Theorem 4.4, we obtain $\mathsf{K}^{\mathsf{I},\mathsf{ILP},0} \Vdash G^0$. Lemma 4.3 yields that $\mathsf{K}^{\mathsf{I},\mathsf{ILP}} \Vdash G$. By definition, $G \in \mathcal{E}(\mathfrak{f}_n\, t)$. Thus $x \Vdash \mathfrak{f}_n\, t{:}G$.

MP It is easy to see that if a model $\mathsf{K}^{\mathsf{I},\mathsf{ILP}}$ forces A and $A \to B$, then it also forces B.

CS By the above, all the axioms of iLP are forced by every model $\mathsf{K}^{\mathsf{I},\mathsf{ILP}}$. Using the definition of evidence function, we get $\mathsf{K}^{\mathsf{I},\mathsf{ILP}} \Vdash c{:}B$ for any axiom B of iLP.

The second claim is proved in the same way. So we only need to consider the case of B4.

B4 Suppose $x \Vdash (\bigwedge_{i=1}^{n} v_i{:}F_i)$ and $\bigwedge_{i=1}^{n} (F_i \wedge v_i{:}F_i)^0 \rhd_{\mathbf{IPC}} G^0$. By definition, $\mathsf{K}^{\mathsf{I},\mathsf{BLP}} \Vdash \bigwedge_{i=1}^{n} (F_i \wedge v_i{:}F_i)$. Lemma 4.3 yields that $\mathsf{K}^{\mathsf{I},\mathsf{BLP},0} \Vdash \bigwedge_{i=1}^{n} (F_i \wedge v_i{:}F_i)^0$. By Theorem 4.4, we have $\mathsf{K}^{\mathsf{I},\mathsf{BLP},0} \Vdash G^0$. Lemma 4.3 implies $\mathsf{K}^{\mathsf{I},\mathsf{BLP}} \Vdash G$.

□

4.3 Completeness of iLP and iBLP

In this section we shall prove the following statement.

Theorem 4.7.

1. *Suppose a set* $\Gamma \cup \{A\} \subset \mathcal{ILP}$ *is finite and* $\Gamma \nvdash_{\mathbf{iLP}} A$. *Then there exists an* $\mathcal{ILP}$*-model* $\mathsf{K}^{\mathsf{I},\mathsf{ILP}}$ *such that* $\mathsf{K}^{\mathsf{I},\mathsf{ILP}} \nVdash \bigwedge \Gamma \to A$.

2. *Suppose a set* $\Gamma \cup \{A\} \subset \mathcal{BLP}$ *is finite and* $\Gamma \nvdash_{\mathbf{iBLP}} A$. *Then there exists an* $\mathcal{BLP}$*-model* $\mathsf{K}^{\mathsf{I},\mathsf{BLP}}$ *such that* $\mathsf{K}^{\mathsf{I},\mathsf{BLP}} \nVdash \bigwedge \Gamma \to A$.

4.3.1 Projective formulas

Formula $A \in \mathcal{PC}$ is called *projective* (in IPC) if there exists a *projective unifier* of A, i. e., a propositional substitution σ such that

$$\vdash_{\mathbf{IPC}} \sigma(A) \text{ and}$$

$$\forall B \in \mathcal{PC}\ (\mathrm{Var}(B) \subseteq \mathrm{Var}(A) \Rightarrow A \vdash_{\mathbf{IPC}} B \leftrightarrow \sigma(B)) .$$

For a propositional formula A we put

$$S_A \rightleftharpoons \{B \in \mathcal{PC} \mid \mathrm{Var}(B) \subseteq \mathrm{Var}(A) , B \text{ is projective}, B \vdash_{\mathbf{IPC}} A, c(B) \leqslant c(A)\} ,$$

where $c\colon \mathcal{PC} \to \omega$ is the *implicational complexity*:

- $c(F) = 0$ if $F \in \mathrm{At}$;
- $c(F * G) = \max\{c(F) , c(G)\}$ if $* \in \{\vee, \wedge\}$;
- $c(F \to G) = \max\{c(F) , c(G)\} + 1$.

Projective approximation of $A \in \mathcal{PC}$ is a set $\Pi_A \subseteq S_A$ such that

- if $\{C_1, C_2\} \subseteq \Pi_A$ and $C_1 \vdash_{\mathbf{IPC}} C_2$, then $C_1 \leftrightharpoons C_2$;
- if $C \in S_A$, then there is $B \in \Pi_A$ such that $C \vdash_{\mathbf{IPC}} B$.

Also put $\bar{\Pi}_A \rightleftharpoons \{B \mid B \vdash_{\mathbf{IPC}} A$ and B is projective$\}$. For a finite set $\Phi \subset \mathcal{PC}$ we put $\Pi_\Phi \rightleftharpoons \Pi_{\bigwedge \Phi}$ and $\bar{\Pi}_\Phi \rightleftharpoons \bar{\Pi}_{\bigwedge \Phi}$.

Theorem 4.8 ([6])**.**

1. *The projectivity property is decidable.*
2. *If $A \nvdash_{\mathbf{IPC}} \bot$, then A has a finite non-empty projective approximation Π_A, which is unique up to provable equivalence and can be effectively computed.*

Theorem 4.9 ([6])**.** *$A \rhd_{\mathbf{IPC}} B$ iff $\bigvee \Pi_A \vdash_{\mathbf{IPC}} B$. In particular, $A \rhd_{\mathbf{IPC}} \bigvee \Pi_A$.*

Theorem 4.10 ([6])**.** *A formula F is projective iff the class $\mathcal{K}(F)$ is extensible and non-empty.*

Lemma 4.11 ([4])**.** *If A is projective and $A \wedge \neg p \nvdash_{\mathbf{IPC}} \bot$ for some $p \in \mathrm{Var}$, then $A \wedge \neg p$ is also projective.*

4.3.2 Saturation

Suppose $\Phi \subseteq \mathcal{ILP}$. By definition, put:

$$\Phi_{-1} \rightleftharpoons \{B \mid t\colon B \in \Phi\}$$

$$\Phi_0 \rightleftharpoons \Phi_{-1} \cup \{u\colon B \mid u\colon B \in \Phi\}$$

$$\Phi_1 \rightleftharpoons \Phi_0 \cup \{\neg u\colon B \mid \neg u\colon B \in \Phi\}$$

An implication $\bigwedge \Theta \to A \in \mathcal{ILP}$ is called *projectively saturated* if the following conditions hold:

1. $\Theta^0 \cap \bar{\Pi}_{(\Theta_1)^0} \neq \varnothing$;

2. if $t{:}F \in \mathrm{SFm}(\Theta \cup \{A\})$, then either $t{:}F \in \Theta$ or $\neg t{:}F \in \Theta$;

3. $\Theta \nvdash_{\mathbf{iLP}} A$.

For iBLP, the definitions are similar.

Put $\|t\| \rightleftharpoons |STm(t)|$ and let $\Phi \subset \mathcal{ILP}$ be finite. Put also $\|\Phi\|_{\mathrm{Tm}} \rightleftharpoons \max\{\|t\| \mid t{:}F \in SFm(\Phi)\}$. A proof term $t \in \mathrm{Tm}_{\mathcal{ILP}}$ is called Φ-*bounded* if $\mathrm{PAt}(t) \subseteq \mathrm{PAt}(\Phi)$ and $\|t\| \leqslant \|\Phi\|_{\mathrm{Tm}}$. A finite set $\Theta \subset \mathcal{ILP}$ is *operationally complete* if the following conditions hold:

- if $s{:}\,(F \to G) \in \Theta$, $t{:}F \in \Theta$, and $s \cdot t$ is Θ-bounded, then $s \cdot t{:}G \in \Theta$;
- if $t{:}F \in \Theta$ and $!\,t$ is Θ-bounded, then $!\,t{:}t{:}F \in \Theta$;
- if $t{:}F \in \Theta$ and $s + t$ is Θ-bounded, then $s + t{:}F \in \Theta$ and $t + s{:}F \in \Theta$;
- if $t{:}F \in \Theta$, $\dfrac{F}{G} \in \mathrm{V}_n$, and $\mathfrak{f}_n\, t$ is Θ-bounded, then $\mathfrak{f}_n\, t{:}G \in \Theta$;

Lemma 4.12. *Suppose a set $\Gamma \cup \{A\} \subset \mathcal{ILP}$ is finite and $\Gamma \nvdash_{\mathbf{iLP}} A$. Then there exists an operationally complete set $\Theta \supseteq \Gamma$ such that $\bigwedge \Theta \to A$ is projectively saturated.*

Proof. Construct a finite set $\Delta \supseteq \Gamma$ such that $\nvdash_{\mathbf{iLP}} \Delta \to A$ and if $t{:}F \in SFm(\Delta \cup \{A\})$, then either $t{:}F \in \Delta$ or $\neg t{:}F \in \Delta$ and if also $t \in \mathrm{PCnst}$ and B is an axiom of iLP, then $t{:}F \in \Delta$. For each $t_i{:}F_i \in SFm(\Gamma \cup \{A\})$ add $t_i{:}F_i$ or $\neg t_i{:}F_i$ to Γ so that $\Gamma, \{t_i{:}F_i\} \nvdash_{\mathbf{iLP}} A$. This is always possible since $\Gamma', t{:}F \vdash_{\mathbf{iLP}} A$ and $\Gamma', \neg t{:}F \vdash_{\mathbf{iLP}} A$ imply, by Axiom A3, $\Gamma' \vdash_{\mathbf{iLP}} A$. Now construct $\Lambda \supseteq \Delta$ using the following algorithm $\mathcal{A}$.

1. $\Lambda := \Delta$; $n := 1$; $(m := \|\Delta\|_{\mathrm{Tm}}) = \|\Gamma \cup \{A\}\|_{\mathrm{Tm}}$; $(V := \mathrm{PAt}(\Delta)) = \mathrm{PAt}(\Gamma \cup \{A\})$.

2. Stop if Λ contains no quasiatoms.

3. Mark all the quasiatoms belonging to Λ; $\Lambda^n_* := \varnothing$.

4. Consider a marked quasiatom $t{:}F \in \Lambda$; if there is no one, then go to 9.

5. If $\|!\,t\| \leqslant m$, then $\Lambda^n_* := \Lambda^n_* \cup \{!\,t{:}t{:}F\}$.

6. $\Lambda' := \{s \mid \|s + t\| \leqslant m \text{ and } \mathrm{PAt}(s) \subseteq V\}$;
   $\Lambda^n_* := \Lambda^n_* \cup \{t + s{:}F, s + t{:}F \mid s \in \Lambda'\}$.

7. If F is the premise of a rule from V_k, G is the corresponding conclusion, and $\|\mathfrak{f}_k\, t\| \leqslant m$, then $\Lambda^n_* := \Lambda^n_* \cup \{\mathfrak{f}_k\, t{:}G\}$.

8. Unmark the quasiatom $t{:}F$ and go to 4.

9. Mark all the pairs of quasiatoms belonging to $\Lambda \times \Lambda$.

10. Consider a marked pair $(H, J) \in \Lambda \times \Lambda$; if there is no one, then go to 14.

11. If $H \rightleftharpoons t{:}F$, $J \rightleftharpoons s{:}\,(F \to G)$, and $\|t \cdot s\| \leqslant m$, then $\Lambda^n_* := \Lambda^n_* \cup \{t \cdot s{:}G\}$.

12. Unmark the pair (H, J) and go to 10.

13. $\Lambda^n := \Lambda^n_* \setminus \Lambda$; stop if $\Lambda^n = \varnothing$.

14. $\Lambda := \Lambda \cup \Lambda^n$; $n := n+1$; go to 3.

It is easy to see that the following statements are invariants of the loop 3-14:

I_1 Λ^n is finite;

I_2 $\|\Lambda^n\|_{\mathrm{Tm}} \leqslant m$;

I_3 $\min\{\|t\| \mid t{:}F \in \Lambda_{n+1}\} > \min\{\|t\| \mid t{:}F \in \Lambda_n\}$ if $\Lambda_{n+1} \neq \varnothing$;

I_4 if $t{:}F \in SFm(\Lambda^n)$, then either $t{:}F \in \Lambda^n$ or $t{:}F \in SFm\Big(\bigcup_{k=0}^{n-1} \Lambda^k \cup \Delta\Big)$;

I_5 $\Lambda \vdash_{\mathbf{iLP}} \Lambda^n$.

Statements I_1 - I_3 secure the finite termination of $\mathcal{A}$ and the finiteness of Λ. Clearly, $\Lambda \supseteq \Delta$ is operationally complete.

Statement I_5 yields that $\Delta \vdash_{\mathbf{iLP}} \bigwedge \Lambda$. If $\Lambda \vdash_{\mathbf{iLP}} A$, then $\Delta \vdash_{\mathbf{iLP}} A$. Therefore $\Lambda \nvdash_{\mathbf{iLP}} A$.

Lemma 4.13. *If $t{:}F \in SFm(\Lambda \cup \{A\})$, then either $t{:}F \in \Lambda$ or $t{:}F \in SFm(\Delta)$.*

Proof. If $t{:}F \in SFm(\Delta \cup \{A\})$, then $t{:}F \in SFm(\Delta)$ by them construction of Δ. Otherwise, the quasiatom $t{:}F$ was added to Δ by $\mathcal{A}$. Hence there is $n \geqslant 1$ such that $t{:}F \in SFm(\Lambda^n)$. By I_4, we get $t{:}F \in \Lambda^k$ for some $k \in \{1, \dots, n\}$. Therefore $t{:}F \in \Lambda$. □

Lemma 4.14. $\Lambda_0 \vdash_{\mathbf{iLP}} \left(\bigvee \Pi_{(\Lambda_0)^0}\right)^{-0}$

Proof. By Axiom A2, $\Lambda \vdash_{\mathbf{iLP}} \bigwedge \Lambda_0$. Therefore $\Lambda \nvdash_{\mathbf{iLP}} A$ and Lemma 3.3 imply $(\Lambda_0)^0 \nvdash_{\mathbf{IPC}} \bot$. Hence, by Theorem 4.8, there exists finite non-empty projective approximation $\Pi_{(\Lambda_0)^0}$. Using Theorems 4.9 and 3.5, we have $(\Lambda_0)^0 \vdash_{\mathbf{IPC}+\bigcup\limits_{\mathbf{n}\in\omega}\mathbf{V_n}} \bigvee \Pi_{(\Lambda_0)^0}$. By Lemma 3.3, this yields that $\Lambda_0 \vdash_{\mathbf{iLP}+\bigcup\limits_{\mathbf{n}\in\omega}\mathbf{V_n}} \left(\bigvee \Pi_{(\Lambda_0)^0}\right)^{-0}$. We shall prove the claim by induction on the derivation in $\mathrm{iLP} + \bigcup\limits_{\mathrm{n}\in\omega} \mathrm{V_n}$. Suppose $\Lambda_0 \vdash_{\mathbf{iLP}} F$ and $\dfrac{F}{G} \in \{\mathrm{V}_n\}$. Let us show that $\Lambda_0 \vdash_{\mathbf{iLP}} G$. By Axiom A2, we get $\Lambda_0 \setminus \Lambda_{-1} \vdash_{\mathbf{iLP}} F$. It follows from Lemma 3.4 that there are ground terms g and h such that

$$\vdash_{\mathbf{iLP}} g{:}\,((\Lambda_0 \setminus \Lambda_{-1}) \to F)$$

and

$$\vdash_{\mathbf{iLP}} h{:}\,\Big(t_1{:}F_1 \to \dots \to \Big(t_k{:}F_k \to \bigwedge(\Lambda_0 \setminus \Lambda_{-1})\Big)\dots\Big)$$

if $\Lambda_0 \setminus \Lambda_{-1} = \{t_j{:}F_j \mid j \in \{1, \dots, k\}\}$. By Axioms A4 and A5, we get

$$\Lambda_0 \setminus \Lambda_{-1} \vdash_{\mathbf{iLP}} g \cdot (h \cdot !\,t_1 \cdot \dots \cdot !\,t_k){:}F.$$

From Axioms $\mathrm{A7_n}$ and A2 it follows that $\Lambda_0 \setminus \Lambda_{-1} \vdash_{\mathbf{iLP}} G$ and, finally, $\Lambda_0 \vdash_{\mathbf{iLP}} G$. □

There exists a formula $C \in \Pi_{(\Lambda_0)^0}$ such that $\Lambda \wedge C^{-0} \nvdash_{\mathbf{iLP}} A$. Indeed, assume the converse. Then $\Lambda \wedge (\bigvee \Pi_{(\Lambda_0)^0})^{-0} \vdash_{\mathbf{iLP}} A$. By Axiom A2, $\Lambda \vdash_{\mathbf{iLP}} \bigwedge \Lambda_0$. Lemma 4.14 implies $\Lambda_0 \vdash_{\mathbf{iLP}} (\bigvee \Pi_{(\Lambda_0)^0})^{-0}$. Hence $\Lambda \vdash_{\mathbf{iLP}} A$. So we get a contradiction. Put $D \rightleftharpoons C^{-0} \wedge \bigwedge (\Lambda_1 \setminus \Lambda_0)$ and $\Theta \rightleftharpoons \Lambda \cup \{D\}$. It is obvious that $\Theta \nvdash_{\mathbf{iLP}} A$ and $\Theta_1 = \Lambda_1$. The latter yields that Θ is operationally complete.

Let us prove that the implication $\bigwedge \Theta \to A$ is projectively saturated. Suppose $t{:}F \in SFm(\Theta \cup \{A\}) = SFm(\Lambda \cup \{D, A\})$. By the construction of Λ and the definition of projective approximation, we have $t{:}F \in SFm(\Lambda)$. From Lemma 4.13 it follows that either $t{:}F \in \Lambda \subseteq \Theta$ or $\neg t{:}F \in \Lambda$ and if $t \in PCnst$ and F is an axiom a iLP, then $t{:}F \in \Lambda$. By Lemma 3.3, $\Theta^0 \nvdash_{\mathbf{IPC}} A^0$. Therefore Lemma 4.11 yields that D^0 is projective. We shall show that $D^0 \vdash_{\mathbf{IPC}} (\Theta_1)^0$. Suppose $t{:}F \in \Theta$; then $t{:}F \in \Lambda$. Hence $\{F, t{:}F\} \subseteq \Lambda_0$. By the definition of projective approximation, $C \vdash_{\mathbf{IPC}} (\bigwedge \Lambda_0)^0$. Consequently $C \vdash_{\mathbf{IPC}} (F \wedge t{:}F)^0$. Thus $D^0 \vdash_{\mathbf{IPC}} (F \wedge t{:}F)^0$. Since $\Theta_1 = \Lambda_1$, from $\neg t{:}F \in \Theta$ it follows that $D^0 \vdash_{\mathbf{IPC}} (\neg t{:}F)^0$. □

The similar statement holds for iBLP.

Lemma 4.15 ([4])**.** *Suppose a set $\Gamma \cup \{A\} \subset \mathcal{BLP}$ is finite and $\Gamma \nvdash_{\mathbf{iBLP}} A$. Then there is a finite set $\Theta \supseteq \Gamma$ such that the implication $\bigwedge \Theta \to A$ is projectively saturated.*

4.3.3 Construction of countermodel

Lemma 4.16.

1. *Suppose $\bigwedge \Theta \to A$ is projectively saturated. Then there exists an $\mathcal{L}_{\mathbf{iLP}}$-model $\mathsf{K}^{\mathbf{iLP}}$ such that $\mathsf{K}^{\mathbf{iLP}} \nVdash \bigwedge \Theta \to A$.*

2. *Suppose $\bigwedge \Theta \to A$ is projectively saturated. Then there exists an $\mathcal{L}_{\mathbf{iBLP}}$-model $\mathsf{K}^{\mathbf{iBLP}}$ such that $\mathsf{K}^{\mathbf{iBLP}} \nVdash \bigwedge \Theta \to A$.*

Proof. Let us prove the first claim. Due to Lemma 3.3 and Theorem 4.1, there exists a rooted model K such that $\mathsf{K} \nVdash \bigwedge \Theta^0 \to A^0$. We have $\mathsf{K}, x \Vdash \bigwedge \Theta^0$ and $\mathsf{K}, x \nVdash A^0$ for a certain node x. Since $\bigwedge \Theta \to A$ is projectively saturated, there is a projective formula $P \in \Theta^0$ such that $P \vdash_{\mathbf{IPC}} \bigwedge(\Theta_1)^0$. By Theorem 4.10, the class $\mathcal{K}(P)$ is extensible. We also have $\mathsf{K}_x \in \mathcal{K}(P)$. Therefore according to Lemma 4.2, we can construct an Iemhoff model K^{I} such that $\mathsf{K}_x \in Cone(\mathsf{K}^{\mathsf{I}}) \subseteq \mathcal{K}(P)$. So we have $\mathsf{K}^{\mathsf{I}} \Vdash \bigwedge(\Theta_1)^0$, $\mathsf{K}^{\mathsf{I}}, x \Vdash \bigwedge \Theta^0$, and $\mathsf{K}^{\mathsf{I}}, x \nVdash (A)^0$. Without loss of generality, we assume that $\mathsf{K}, w \Vdash q$ implies $q \in \mathrm{Var}(\Theta^0 \cup A^0)$ for any node w. From the definition of Smorynski closure and the monotony property, it follows that K^{I} forces any other variables at no nodes.

Lemma 4.17. *If $\mathsf{K}^{\mathsf{I}}, w \Vdash (t{:}F)^0$, then $t{:}F \in \Theta$.*

Proof. By the above assumption, $t{:}F \in \mathrm{SFm}(\Theta \cup \{A\})$. Then for $\bigwedge \Theta \to A$ being saturated, we have $t{:}F \in \Theta$ or $\neg t{:}F \in \Theta$. Assume the latter. Then $\neg t{:}F \in \Theta_1$ and consequently, $\neg(t{:}F)^0 \in (\Theta_1)^0$. This yields that $\mathsf{K}^{\mathsf{I}} \Vdash \neg(t{:}F)^0$. The contradiction proves the lemma. □

Put $\mathcal{E}'(t) = \{F \mid \mathsf{K}^{\mathsf{I}} \Vdash (t{:}F)^0\}$. Let us define the evidence function $\mathcal{E}$ for the required countermodel.

- $\mathcal{E}(u) = \mathcal{E}'(u)$;
- $\mathcal{E}(c) = \{F \mid F \text{ is an axiom of iLP}\} \cup \mathcal{E}'(c)$;
- $\mathcal{E}(s \cdot r) = \{G \mid F \to G \in \mathcal{E}(s) \text{ and } F \in \mathcal{E}(r)\} \cup \mathcal{E}'(s \cdot r)$;
- $\mathcal{E}(!\,s) = \{s\colon F \mid F \in \mathcal{E}(s)\} \cup \mathcal{E}'(!\,s)$;
- $\mathcal{E}(s + r) = \mathcal{E}(s) \cup \mathcal{E}(r) \cup \mathcal{E}'(s + r)$;
- $\mathcal{E}(\mathfrak{f}_n\, s) = \{G \mid F \in \mathcal{E}(s) \text{ and } \frac{F}{G} \in \mathbf{V_n}\} \cup \mathcal{E}'(\mathfrak{f}_n\, s)$.

We shall show that the pair $(\mathsf{K}^{\mathsf{I}}, \mathcal{E})$ is the required countermodel $\mathsf{K}^{\mathbf{iLP}}$. Obviously, the evidence function is well-defined. Hence $(\mathsf{K}^{\mathsf{I}}, \mathcal{E})$ is a model of $\mathcal{L}_{\mathbf{iLP}}$.

Lemma 4.18. *If $F \in \mathcal{E}(t)$, then $\Theta \vdash_{\mathbf{iLP}} t\colon F$.*

Proof. Induction on the construction of the term t. Consider only the case of application since the others are treated similarly. Assume that $G \in \mathcal{E}(s \cdot r)$. By definition, we must consider two cases. If $G \in \mathcal{E}'(s \cdot r)$, then $\mathsf{K}^{\mathsf{I}} \Vdash (s \cdot r\colon G)^0$. From Lemma 4.17, it follows that $s \cdot r\colon G \in \Theta$. Now consider the second case. We have $F \to G \in \mathcal{E}(s)$ and $F \in \mathcal{E}(r)$ for some formula F. By the induction hypothesis, $\Theta \vdash_{\mathbf{iLP}} s\colon(F \to G)$ and $\Theta \vdash_{\mathbf{iLP}} r\colon F$. Using Axiom A4, obtain $\Theta \vdash_{\mathbf{iLP}} s \cdot r\colon G$. □

Lemma 4.19. *Suppose $H \in \mathrm{SFm}(\bigwedge \Theta \to A)$. Then $\mathsf{K}^{\mathsf{I}}, y \Vdash H^0$ and $\mathsf{K}^{\mathbf{iLP}}, y \Vdash H$ are equivalent for all y.*

Proof. Induction on the construction of H. Suppose $H \coloneqq F \to G$; then $\mathsf{K}^{\mathsf{I}}, y \Vdash H^0$ iff for all z such that $y \preccurlyeq z$ either $\mathsf{K}^{\mathsf{I}}, z \nVdash F^0$ or $\mathsf{K}^{\mathsf{I}}, z \Vdash G^0$. By the induction hypothesis, we equivalently have $\mathsf{K}^{\mathbf{iLP}}, z \nVdash F$ or $\mathsf{K}^{\mathbf{iLP}}, z \Vdash G$ for all z such that $y \preccurlyeq z$. This is equivalent to $\mathsf{K}^{\mathbf{iLP}}, y \Vdash F \to G$. The cases of the other propositional connectives and atoms are similar. Consider $H \coloneqq t\colon F$. Suppose $\mathsf{K}^{\mathsf{I}}, y \Vdash (t\colon F)^0$. For Lemma 4.17, $t\colon F \in \Theta$. Then $\{t\colon F, F\} \subseteq \Theta_1$. We obtain $\mathsf{K}^{\mathsf{I}} \Vdash (t\colon F)^0$ and $\mathsf{K}^{\mathsf{I}} \Vdash F^0$. By the induction hypothesis, $\mathsf{K}^{\mathbf{iLP}} \Vdash F$. By definition, we have $\mathcal{E}(t) \ni F$. Thus $\mathsf{K}^{\mathbf{iLP}}, y \Vdash t\colon F$. Vice versa, suppose $\mathsf{K}^{\mathbf{iLP}}, y \Vdash t\colon F$. By definition, $F \in \mathcal{E}(t)$. Since $t\colon F \in \mathrm{SFm}(\Theta \cup \{A\})$, we obtain either $t\colon F \in \Theta$ or $\neg t\colon F \in \Theta$. In the latter case, Lemma 4.18 yields that $\Theta \vdash_{\mathbf{iLP}} t\colon F$, and we have a contradiction here since $\Theta \nvdash_{\mathbf{iLP}} A$. Thus $t\colon F \in \Theta$. As it was shown in the above, this implies $\mathsf{K}^{\mathsf{I}} \Vdash (t\colon F)^0$. □

Now from K^{I}, $x \nVdash \bigwedge \Theta^0 \to A^0$, it follows that $\mathsf{K}^{\mathbf{iLP}} \nVdash \bigwedge \Theta \to A$. The second claim can be proved in the similar way provided that $\mathcal{E}(t) = \mathcal{E}'(t)$. □

Thus, Theorem 4.7 is completely proved.

4.4 Corollaries

The following statement is similar to Glivenko's theorem with respect to LP [1].

Theorem 4.20. *If $\vdash_{\mathbf{LP}} F$, then $\vdash_{\mathbf{iLP}} \neg\neg F$.*

Proof. We can assume without loss of generality that $\mathcal{LP} \subseteq \mathcal{ILP}$, where $\mathcal{LP}$ is the language of LP. It is easy to see that all the axioms of LP are forced at each maximal node of any $\mathcal{ILP}$-model. Hence if $\vdash_{\mathbf{LP}} F$, then no node forces $\neg F$, i. e., $\mathsf{K}^{\mathsf{I},\mathsf{ILP}} \Vdash \neg\neg F$ for all $\mathcal{ILP}$-models $\mathsf{K}^{\mathsf{I},\mathsf{ILP}}$. Now use Theorem 4.7. □

4.4.1 Conservativity

Theorem 4.21. *Suppose* $F \in \mathcal{BLP}$*. Then* $\vdash_{\mathbf{iLP}} F$ *implies* $\vdash_{\mathbf{iBLP}} F$*.*

Proof. Suppose $\nvdash_{\mathbf{iBLP}} F$. Let us show that $\nvdash_{\mathbf{iLP}} F$. By Theorem 4.7, there exists a model $\mathsf{K}^{\mathsf{I},\mathsf{BLP}} = \big(\mathsf{K}^{\mathsf{I}}, \mathcal{E}_0\big)$ such that $\mathsf{K}^{\mathsf{I},\mathsf{BLP}} \nVdash F$. Consider $\big(\mathsf{K}^{\mathsf{I}}, \mathcal{E}\big)$, where

- $\mathcal{E}(t) = \mathcal{E}_0(t)$ if $t \in \mathrm{PVar}$;
- $\mathcal{E}(t) = \{B \mid B \text{ is an axiom of iLP}\}$ if $t \in \mathrm{PCnst}$;
- $\mathcal{E}(s \cdot r) = \{G \mid F \to G \in \mathcal{E}(s) \text{ and } F \in \mathcal{E}(r)\}$;
- $\mathcal{E}(!\, s) = \{s\!:\!F \mid F \in \mathcal{E}(s)\}$;
- $\mathcal{E}(s + r) = \mathcal{E}(r + s) = \mathcal{E}(s) \cup \mathcal{E}(r)$;
- $\mathcal{E}(\mathfrak{f}_n\, s) = \big\{G \mid F \in \mathcal{E}(s) \text{ and } \frac{F}{G} \in \mathrm{V}_n\big\}$.

Clearly, the pair $(\mathsf{K}^{\mathsf{I}}, \mathcal{E})$ is a model of $\mathcal{ILP}$. We denote this model by $\mathsf{K}^{\mathsf{I},\mathsf{ILP}}$. Note that $\mathcal{E}(t) = \mathcal{E}_0(t)$ for any t such that $t\!:\!G \in SFm(F)$. Therefore, by induction on the construction of F, one can easily prove that $\mathsf{K}^{\mathsf{I},\mathsf{ILP}}, x \Vdash F$ is equivalent to $\mathsf{K}^{\mathsf{I},\mathsf{ILP}}, x \Vdash F$ for any x. Thus we have $\mathsf{K}^{\mathsf{I},\mathsf{ILP}} \nVdash F$. From Theorem 4.6 it follows that $\nvdash_{\mathbf{iLP}} F$. ☐

Theorem 4.22. *Suppose* $F \in \mathcal{PC}$*. Then* $\vdash_{\mathbf{iBLP}} F$ *implies* $\vdash_{\mathbf{IPC}} F$*.*

Proof. Suppose $\nvdash_{\mathbf{IPC}} F$. Let us show that $\nvdash_{\mathbf{iBLP}} F$. According to Remark 4.5, there exists a model K^{I} such that $\mathsf{K}^{\mathsf{I}} \nVdash F$. Consider the pair $(\mathsf{K}^{\mathsf{I}}, \varnothing)$. Clearly, it is a model of $\mathcal{BLP}$. We denote this model by $\mathsf{K}^{\mathsf{I},\mathsf{BLP}}$. By induction on the construction of F, one can easily prove that $\mathsf{K}^{\mathsf{I},\mathsf{BLP}}, x \Vdash F$ is equivalent to $\mathsf{K}^{\mathsf{I},\mathsf{BLP}}, x \Vdash F$ for any x. Thus we have $\mathsf{K}^{\mathsf{I},\mathsf{BLP}} \nVdash F$. From Theorem 4.6 it follows that $\nvdash_{\mathbf{iBLP}} F$. ☐

Theorem 4.23. *Suppose* $F \in \mathcal{PC}$*. Then* $\vdash_{\mathbf{iLP}} F$ *implies* $\vdash_{\mathbf{IPC}} F$*.*

Proof. Note that $\mathcal{ILP} \cap \mathcal{PC} = \mathcal{BLP} \cap \mathcal{PC}$. We can now use Theorems 4.21 and 4.22. ☐

5 Arithmetical semantics

We use the primitive recursive Gödel numbering $\ulcorner\cdot\urcorner$ of $\mathcal{A}$ as it is constructed in [13]. Extend the numbering to $\mathrm{Tm}_{\mathcal{ILP}} \cup \mathrm{Var}$. Let us fix the following properties:

- the numbering is injective;
- the code of a finite sequence of syntactical objects is not less than the codes of its members, which are primitive recursively computable from it;
- 0 is the code of the empty sequence.

Formula $\varphi(y_1,\ldots,y_n)$ is called *primitive recursive* if $\varphi \rightleftharpoons (\bar{f}y_1\ldots y_n = \bar{0})$, where $\bar{f}$ is a symbol for a certain primitive recursive function $f\colon \omega^n \to \omega$ and $y_1,\ldots,y_n$ are arbitrary variables. It is clear that

$$\begin{array}{lcl} f(k_1,\ldots,k_n)=0 & \Leftrightarrow & \vdash_{\mathbf{HA}} \varphi\Big(\overline{k_1},\ldots,\overline{k_n}\Big) \text{ and} \\ f(k_1,\ldots,k_n)\neq 0 & \Leftrightarrow & \vdash_{\mathbf{HA}} \neg\varphi\Big(\overline{k_1},\ldots,\overline{k_n}\Big) \end{array}$$

for any $k_i \in \omega$.

Let t_i be ground arithmetical terms and $\tilde{t}_i$ their interpretations in $\mathbb{N}$. We say that the formula $\psi(\tilde{t}_1,\ldots,\tilde{t}_n)$ holds, is true or say simply that $\psi(\tilde{t}_1,\ldots,\tilde{t}_n)$ if $\mathbb{N} \models \psi[x_1/t_1,\ldots,x_n/t_n]$. Thus if formula ψ is primitive recursive, then $\psi[x_1/t_1,\ldots,x_n/t_n]$ is decidable in HA; moreover, $\psi[x_1/t_1,\ldots,x_n/t_n]$ is deducible in HA if $\psi(\tilde{t}_1,\ldots,\tilde{t}_n)$ and refutable otherwise.

5.1 Arithmetical interpretation of iLP

A *proof predicate* is a primitive recursive arithmetical formula $\mathrm{Prf}[x,y]$ such that $\vdash_{\mathbf{HA}} \varphi$ iff there is $n \in \omega$ such that $\mathbb{N} \models \mathrm{Prf}\Big(\overline{n},\overline{\ulcorner\varphi\urcorner}\Big)$ for any $\varphi \in \mathcal{A}_0$.

A proof predicate $\mathrm{Prf}(x,y)$ is *normal* if the following conditions hold:

(finiteness) for any $k \in \omega$ the set $T(k) \rightleftharpoons \Big\{ l \mid \mathbb{N} \models \mathrm{Prf}\Big(\overline{k},\overline{l}\Big)\Big\}$ is finite and the function $\lambda k.\ulcorner\{\bar{l} \mid l \in T(k)\}\urcorner$ is general recursive;

(conjoinability) for any $\{k,l\} \subset \omega$ there is $n \in \omega$ such that $T(k) \cup T(l) \subseteq T(n)$.

Any primitive recursive formula expressing the following primitive recursive [13] predicate is a normal proof predicate:

$\mathrm{Proof}(k,l) \rightleftharpoons$ "k is the code of a derivation in HA containing a formula with the code l"

Lemma 5.1. *If* $\mathrm{Prf}(x,y)$ *is a normal proof predicate, then there exist general recursive functions* $m\colon \omega^2 \to \omega$, $a\colon \omega^2 \to \omega$, $c\colon \omega \to \omega$ *and* $f_n\colon \omega \to \omega$ *for all* $n \in \omega$ *such that for any* $\{k,l\} \subset \omega$ *and any* $\{\varphi,\psi\} \subset \mathcal{A}_0$ *the following formulas hold:*

$$\begin{array}{l} \mathrm{Prf}(k,\ulcorner\varphi\to\psi\urcorner) \to (\mathrm{Prf}(l,\ulcorner\varphi\urcorner) \to \mathrm{Prf}(m(k,l)\,,\ulcorner\psi\urcorner))\,; \\ \mathrm{Prf}(k,\ulcorner\varphi\urcorner) \to \mathrm{Prf}(a(k,l)\,,\ulcorner\varphi\urcorner)\,; \quad \mathrm{Prf}(l,\ulcorner\varphi\urcorner) \to \mathrm{Prf}(a(k,l)\,,\ulcorner\varphi\urcorner)\,; \\ \mathrm{Prf}(k,\ulcorner\varphi\urcorner) \to \mathrm{Prf}\Big(c(k)\,,\ulcorner\mathrm{Prf}\Big(\overline{k},\overline{\ulcorner\varphi\urcorner}\Big)\urcorner\Big)\,; \\ \mathrm{Prf}(k,\ulcorner\varphi\urcorner) \to \mathrm{Prf}(f_n(k)\,,\ulcorner\psi\urcorner) \ \text{if } \dfrac{\varphi}{\psi} \in \mathrm{V}_n. \end{array}$$

Proof. Let $MP\colon \omega^2 \to \omega$ and $V_n\colon \omega \to \omega$ for all $n \in \omega$ be the functions such that:

$$MP(k,l) = \begin{cases} \ulcorner\eta\urcorner & \text{if } k = \ulcorner\xi\to\eta\urcorner,\ l = \ulcorner\xi\urcorner, \text{ and } \{\xi,\eta\} \subset \mathcal{A}_0; \\ 0 & \text{otherwise;} \end{cases}$$

$$V_n(k) = \begin{cases} \ulcorner\eta\urcorner & \text{if } k = \ulcorner\xi\urcorner,\ \frac{\xi}{\eta} \in \mathrm{V}_n, \text{ and } \{\xi,\eta\} \subset \mathcal{A}_0; \\ 0 & \text{otherwise.} \end{cases}$$

The functions MP and V_n are general recursive; V_n is such since the conclusion of a Visser's rule is constructed from the subformulas of its premise. Let us put

$$m(k,l) = \mu z.\, \forall x \in T(k)\, \forall y \in T(l)\, (MP(x,y) = 0 \vee \mathrm{Prf}(z, MP(x,y)))\,;$$
$$a(k,l) = \mu z.\, T(k) \cup T(l) \subseteq T(z)\,;$$
$$c(k) = \mu z.\, \forall x \in T(k)\, \mathrm{Prf}\Big(z, \ulcorner \mathrm{Prf}\big(\overline{k}, \dot{x}\big) \urcorner\Big)\,;$$
$$f_n(k) = \mu z.\, \forall x \in T(k)\, (V_n(x) = 0 \vee \mathrm{Prf}(z, V_n(x)))\,.$$

Since Prf is normal, $T(n)$ is finite and computable. Thus we apply μ to the decidable relations; hence the constructed functions are partial recursive. We shall show that they are general recursive, i. e., there are the appropriate $z \in \omega$.

Consider the function c. We have $\mathrm{Prf}(k, x_i)$ for any $x_i \in T(k)$. Since Prf is primitive recursive, we get $\vdash_{\mathbf{HA}} \mathrm{Prf}\big(\overline{k}, \overline{x_i}\big)$. Therefore $\mathrm{Prf}\Big(z_i, \ulcorner \mathrm{Prf}\big(\overline{k}, \overline{x_i}\big) \urcorner\Big)$ for a certain z_i. By the conjoinability property, there is z such that for all x_i $\mathrm{Prf}\Big(z, \ulcorner \mathrm{Prf}\big(\overline{k}, \overline{x_i}\big) \urcorner\Big)$.

Consider f_n. Suppose $x_i \in T(k)$ and $V_n(x_i) \neq 0$. Then $\frac{\xi_i}{\eta_i} \in \mathrm{V}_n$ for $\ulcorner \xi_i \urcorner = x_i$ and $\ulcorner \eta_i \urcorner = V_n(x_i)$. We also have $\vdash_{\mathbf{HA}} \xi_i$. Since ξ_i and η_i are sentences, then there are $\{F_i, G_i\} \subset \mathcal{PC}$ and an arithmetical substitution τ such that $\frac{F_i}{G_i} \in \mathrm{V}_n$, $\tau(F_i) \leftrightharpoons \xi_i$ and $\tau(G_i) \leftrightharpoons \eta_i$. By Theorems 3.5 and 3.8, we obtain $\vdash_{\mathbf{HA}} \eta_i$. Hence there is z_i such that $\mathrm{Prf}(z_i, \ulcorner \eta_i \urcorner)$. Now use the conjoinability property again. The function m is considered similarly.

In the case of the function a we only need to use the conjoinability property. It is easy to see now, that the above formulas hold. □

Suppose $\mathrm{Prf}(x,y)$ is a normal proof predicate; m, a, c, and f_n for all $n \in \omega$ are functions as in Lemma 5.1; $\cdot^* |_{\mathrm{Var}} \colon \mathrm{Var} \to \mathcal{A}_0$ and $\cdot^* |_{\mathrm{PAt}} \colon \mathrm{PAt} \to \omega$ are arbitrary maps. For these parameters the *arithmetical interpretation* of $\mathcal{ILP}$ is the map $\cdot^* \colon \mathcal{ILP} \cup \mathrm{Tm}_{\mathcal{ILP}} \to \mathcal{A}_0 \cup \omega$ such that

- $\bot^* \leftrightharpoons \bot$;
- $\cdot^*$ commutes with the propositional connectives;
- $(s \cdot t)^* = m(s^*, t^*)$; $(s + t)^* = a(s^*, t^*)$; $(!\, t)^* = c(t^*)$; $(\mathfrak{f}_n\, t)^* = f_n(t^*)$ for all $n \in \omega$;
- $(t \colon F)^* \leftrightharpoons \mathrm{Prf}\Big(\overline{t^*}, \overline{\ulcorner F^* \urcorner}\Big)$.

The logic iLP_0 in $\mathcal{ILP}$ has the same axiom schemata as iLP and the inference rule MP.

Lemma 5.2. *If* $\vdash_{\mathbf{iLP_0}} F$*, then* $\vdash_{\mathbf{HA}} F^*$ *for any interpretation* $\cdot^*$*.*

Proof. Induction on the derivation of F in iLP_0. In the case of Axioms A4, A5, A6 and $\mathrm{A7_n}$ the claim holds since the functions m, a, c and f_n satisfy the conditions of Lemma 5.1. For A3 note that $(t \colon F)^* \leftrightharpoons \mathrm{Prf}\Big(\overline{t^*}, \overline{\ulcorner F^* \urcorner}\Big)$ is decidable in HA since Prf is primitive recursive. Consider the case of A2. If $\mathrm{Prf}(t^*, \ulcorner F^* \urcorner)$, then $\vdash_{\mathbf{HA}} F^*$. Hence $\vdash_{\mathbf{HA}} (t \colon F)^* \to F^*$. If $\mathrm{Prf}(t^*, \ulcorner F^* \urcorner)$ is false, then $\vdash_{\mathbf{HA}} \neg \mathrm{Prf}\Big(\overline{t^*}, \overline{\ulcorner F^* \urcorner}\Big)$ for Prf is primitive recursive. Therefore $\vdash_{\mathbf{HA}} \mathrm{Prf}\Big(\overline{t^*}, \overline{\ulcorner F^* \urcorner}\Big) \to F^*$. □

5.1.1 Completeness

Theorem 5.3. *Suppose a set $\Gamma \cup \{A\} \subset \mathcal{ILP}$ is finite and $\Gamma \nvdash_{\mathbf{iLP}} A$. Then there exists an interpretation $\cdot^*$ such that $\Gamma^* \nvdash_{\mathbf{HA}} A^*$.*

Lemma 5.4. *Suppose $\Theta \cup \{A\} \subseteq \mathcal{ILP}$ and the implication $\bigwedge \Theta \to A$ is projectively saturated. Then $\Theta \nvdash_{\mathbf{iLP}} A$ yields that there exists a propositional substitution σ such that the following conditions hold:*

1. $\vdash_{\mathbf{IPC}} \sigma\big(F^0\big) \wedge \sigma\big((t{:}F)^0\big)$ *for any* $t{:}F \in \Theta$;
2. $\vdash_{\mathbf{IPC}} \neg\sigma\big((t{:}F)^0\big)$ *for any* $\neg t{:}F \in \Theta$;
3. $\sigma\big(\Theta^0\big) \nvdash_{\mathbf{IPC}} \sigma\big(A^0\big)$.

Proof. Since $\bigwedge \Theta \to A$ is projectively saturated, there is $P \in \Theta^0 \cap \bar{\Pi}_{(\Theta_1)^0}$. Let σ be a projective unifier of P. Since $\vdash_{\mathbf{IPC}} \sigma(P)$, $P \vdash_{\mathbf{IPC}} C$ implies $\vdash_{\mathbf{IPC}} \sigma(C)$ for any $C \in \mathcal{PC}$. If $\mathrm{Var}(C) \subseteq \mathrm{Var}(P)$, then $\vdash_{\mathbf{IPC}} \sigma(C)$ implies $P \vdash_{\mathbf{IPC}} C$ because $P \vdash_{\mathbf{IPC}} C \leftrightarrow \sigma(C)$. Let us put $\sigma\,|_{\mathrm{Var}\setminus\mathrm{Var}(P)} = \mathrm{id}_{\mathrm{Var}\setminus\mathrm{Var}(P)}$. Then $P \vdash_{\mathbf{IPC}} q \leftrightarrow \sigma(q)$ for all $q \in \mathrm{Var}$. Now $P \vdash_{\mathbf{IPC}} C$ and $\vdash_{\mathbf{IPC}} \sigma(C)$ are equivalent for any $C \in \mathcal{PC}$.

If $P \vdash_{\mathbf{IPC}} \bigwedge (\Theta_1)^0$ and $t{:}F \in \Theta$, then $\Big\{F^0, (t{:}F)^0\Big\} \subseteq (\Theta_1)^0$. Therefore $P \vdash_{\mathbf{IPC}} F^0 \wedge (t{:}F)^0$. Consequently $\vdash_{\mathbf{IPC}} \sigma\big(F^0\big) \wedge \sigma\Big((t{:}F)^0\Big)$. The second claim can be proved similarly. By Lemma 3.3, we have $\Theta^0 \nvdash_{\mathbf{IPC}} A^0$. Since $P \in \Theta^0$, $P \nvdash_{\mathbf{IPC}} \bigwedge \Theta^0 \to A^0$. Therefore $\nvdash_{\mathbf{IPC}} \sigma\big(\bigwedge \Theta^0 \to A^0\big)$, i. e., $\sigma\big(\Theta^0\big) \nvdash_{\mathbf{IPC}} \sigma\big(A^0\big)$. ☐

Theorem 3.8 implies de Jongh's theorem:

Theorem 5.5 ([12])**.** *Suppose $A \in \mathcal{PC}$. Then $\vdash_{\mathbf{IPC}} A$ iff $\vdash_{\mathbf{HA}} \tau(A)$ for any arithmetical substitution τ.*

Proof of Theorem 5.3. By Lemma 4.12, there exists an operationally complete set $\Theta \supseteq \Gamma$ such that the implication $\bigwedge \Theta \to A$ is projectively saturated and $\Theta \nvdash_{\mathbf{iLP}} A$. Below we denote by Θ only this set.

Lemma 5.6. *There exists an arithmetical substitution τ such that the following conditions hold:*

1. $\vdash_{\mathbf{HA}} \tau\big(B^0\big) \wedge \tau\Big((t{:}B)^0\Big)$ *for any* $t{:}B \in \Theta$;
2. $\vdash_{\mathbf{HA}} \neg\tau\Big((t{:}B)^0\Big)$ *for any* $\neg t{:}B \in \Theta$;
3. $\tau\big(\Theta^0\big) \nvdash_{\mathbf{HA}} \tau\big(A^0\big)$;
4. *for any $p \in \mathrm{Var}$ there exists a formula φ_p such that* $\tau(p) = \varphi_p \wedge (\overline{\ulcorner p \urcorner} = \overline{\ulcorner p \urcorner})$.

Proof. By Lemma 5.4, there exists a propositional substitution σ such that $\sigma\big(\Theta^0\big) \nvdash_{\mathbf{IPC}} \sigma\big(A^0\big)$. Using Theorem 5.5, we get an arithmetical substitution τ'' such that $\tau''\big(\sigma\big(\Theta^0\big)\big) \nvdash_{\mathbf{HA}} \tau''\big(\sigma\big(A^0\big)\big)$. By τ' denote the composition $\tau'' \circ \sigma$, which is an arithmetical substitution. So we have $\tau'\big(\Theta^0\big) \nvdash_{\mathbf{HA}} \tau'\big(A^0\big)$. The first and second claims for τ' follow from Lemma 5.4 and Theorem 5.5. Finally, put $\tau(p) = \tau'(p) \wedge (\overline{\ulcorner p \urcorner} = \overline{\ulcorner p \urcorner})$ for all $p \in \mathrm{Var}$. Since $\vdash_{\mathbf{HA}} \tau(p) \leftrightarrow \tau'(p)$, the claims 1-3 hold for τ. $\square$

For the sequel we suppose τ is an arithmetical substitution as in Lemma 5.6. Let $\psi[x, y]$ be an arbitrary formula. Consider the map $\cdot^{\dagger \ulcorner \psi \urcorner} \colon \mathcal{ILP} \cup \mathrm{Tm}_{\mathcal{ILP}} \to \mathcal{A}_0 \cup \omega$ (also denoted by $\cdot^{\dagger}$) such that

- $p^{\dagger} = \tau(p)$ if $p \in \mathrm{Var}$;
- $\bot^{\dagger} = \bot$;
- $\cdot^{\dagger}$ commutes with the propositional connectives;
- $t^{\dagger} = \ulcorner t \urcorner$ if $t \in \mathrm{Tm}_{\mathcal{ILP}}$;
- $(t \colon F)^{\dagger} = \psi\Big(\overline{t^{\dagger}}, \overline{\ulcorner F^{\dagger} \urcorner}\Big)$.

Lemma 5.7. *If a formula $\psi[x, y]$ is primitive recursive, then the map $\cdot^{\dagger \ulcorner \psi \urcorner}$ is injective.*

Proof. Assume that $F^{\dagger} = G^{\dagger}$. We shall prove that $F = G$ by induction on the construction of F. Suppose $F = p \in \mathrm{Var}$. Then we get $F^{\dagger} = \varphi_p \wedge (\overline{\ulcorner p \urcorner} = \overline{\ulcorner p \urcorner}) = G^{\dagger}$. Therefore either $G = G_1 \wedge G_2$ or $G = p' \in \mathrm{Var}$ by definition and Lemma 5.6 . For there is no such formula G_2 that $G_2^{\dagger} = (\overline{\ulcorner p \urcorner} = \overline{\ulcorner p \urcorner})$, the first case is impossible. So, $G = p'$, which yields $\ulcorner p \urcorner = \ulcorner p' \urcorner$. Since the numbering is injective, we conclude that $p = p'$.

Now suppose $F = t \colon H$. Then $F^{\dagger} = (\bar{f}_{\psi}\, \overline{\ulcorner t \urcorner}\, \overline{\ulcorner H^{\dagger} \urcorner} = \bar{0}) = G^{\dagger}$ for a certain functional symbol $\bar{f}_{\psi}$ since ψ is primitive recursive. The formula G is a quasiatom: $G = t' \colon H'$. Indeed, if it is not, then $G^{\dagger} = \bot$ or $G^{\dagger}$ has a propositional main connective. From injectivity of the numbering and the induction hypothesis, it follows that $t = t'$ and $H = H'$.

The case when $F = F_1 \wedge F_2$ is symmetrical to the first one. The other cases are trivial. $\square$

From this Lemma, it follows that for a primitive recursive formula ψ one can construct a primitive recursive function $f^{\dagger} \colon \omega^2 \to \omega$ such that

$$f^{\dagger}(k, l) \quad = \quad \begin{cases} \ulcorner F \urcorner, & \text{if } k = \ulcorner F^{\dagger \ulcorner \psi \urcorner} \urcorner \text{ and } l = \ulcorner \psi \urcorner; \\ 0 & \text{otherwise.} \end{cases}$$

The proof of the diagonalization lemma for PRA in [13] will be also valid for the following statement.

Lemma 5.8. *Suppose a formula $\varphi[y, y_1, \ldots, y_n]$ is primitive recursive. Then there exists a primitive recursive formula $\psi[y_1, \ldots y_n]$ such that*

$$\vdash_{\mathbf{HA}} \psi(y_1, \ldots y_n) \leftrightarrow \varphi\Big(y \big/ \overline{\ulcorner \psi(y_1, \ldots y_n) \urcorner}\, , y_1, \ldots, y_n\Big)\, .$$

In the sequel, a Θ-bounded proof term is called *bounded* while any other is called *unbounded*. Consider the normal proof predicate $\mathrm{Proof}(x, y)$ defined above and the respective functions m', a', c', and f'_n given by Lemma 5.1. Without loss of generality, we assume that $\mathrm{Proof}(\ulcorner t \urcorner, k)$ is false for all bounded terms t and all $k \in \omega$. By Lemma 5.8, there is a primitive recursive formula $\mathrm{Prf}(x, y)$ such that

$$\vdash_{\mathbf{HA}} \mathrm{Prf}(x,y) \leftrightarrow \mathrm{Proof}(x,y) \vee \\ \text{“}x = \ulcorner t \urcorner \text{ and } y = \ulcorner F^{\dagger \ulcorner \mathrm{Prf}(x,y) \urcorner} \urcorner \text{ for some quasiatom } t\colon F \in \Theta\text{”}. \quad (1)$$

Indeed, the expression in quotation marks describes a primitive recursive predicate: given x', y', and $\ulcorner \mathrm{Prf}(x, y) \urcorner$ recover t and F such that $x' = \ulcorner t \urcorner$ and $y' = \ulcorner F^{\dagger \ulcorner \mathrm{Prf}(x,y) \urcorner} \urcorner$ or find out that they don't exist—this procedure is primitive recursive since $f^{\dagger}$ is such; then check if $t\colon F \in \Theta$ using the finiteness of Θ. Hence the expression can be represented by a primitive recursive formula. Since the formula $\mathrm{Proof}(x, y)$ is a proof predicate, it is also primitive recursive. Using the lemma about definition by cases [13] we conclude that the right part of equivalence (1) is primitive recursive. Obviously, for any $\varphi \in \mathcal{A}_0$ if $\vdash_{\mathbf{HA}} \varphi$, then $\mathrm{Prf}(k, \ulcorner \varphi \urcorner)$ for some $k \in \omega$.

Below we denote $\cdot^{\dagger \ulcorner \mathrm{Prf}(x,y) \urcorner}$ by $\cdot^{\dagger}$. Now define functions m, a, c, and f_n for all $n \in \omega$.

$$\begin{aligned} m(k,l) = \mu z.\, (& \\ & (\text{“}k = \ulcorner s \urcorner \text{ and } l = \ulcorner t \urcorner \text{ for some bounded } s \cdot t\text{”} \wedge \\ & \qquad z = \ulcorner s \cdot t \urcorner) \vee \\ & (\text{“}k = \ulcorner s \urcorner \text{ for some bounded } s \text{ and } l \neq \ulcorner t \urcorner \text{ for any bounded } t\text{”} \wedge \\ & \qquad \exists v\, [v = \mu w.\, (\textstyle\bigwedge\{\mathrm{Proof}\big(w, \ulcorner C^{\dagger} \urcorner\big) \mid s\colon C \in \Theta\}) \wedge \\ & \qquad z = m'(v,l)]) \vee \\ & (\text{“}k \neq \ulcorner s \urcorner \text{ for any bounded } s \text{ and } l = \ulcorner t \urcorner \text{ for some bounded } t\text{”} \wedge \\ & \qquad \exists v\, [v = \mu w.\, (\textstyle\bigwedge\{\mathrm{Proof}\big(w, \ulcorner B^{\dagger} \urcorner\big) \mid t\colon B \in \Theta\}) \wedge \\ & \qquad z = m'(k,v)]) \vee \\ & (\text{“}k = \ulcorner s \urcorner \text{ and } l = \ulcorner t \urcorner \text{ for some bounded } s,\, t \text{ while } s \cdot t \text{ is unbounded”} \wedge \\ & \qquad \exists v_1\, \exists v_2\, [v_1 = \mu w.\, (\textstyle\bigwedge\{\mathrm{Proof}\big(w, \ulcorner C^{\dagger} \urcorner\big) \mid s\colon C \in \Theta\}) \wedge \\ & \qquad v_2 = \mu w.\, (\textstyle\bigwedge\{\mathrm{Proof}\big(w, \ulcorner B^{\dagger} \urcorner\big) \mid t\colon B \in \Theta\}) \wedge z = m'(v_1, v_2)]) \vee \\ & (\text{“}k \neq \ulcorner s \urcorner \text{ and } l \neq \ulcorner t \urcorner \text{ for any bounded } s \text{ and } t\text{”} \wedge z = m'(k,l))); \end{aligned}$$

$$
\begin{aligned}
a(k,l) = \mu z.\,(& \\
& (\text{“}k = \ulcorner s \urcorner \text{ and } l = \ulcorner t \urcorner \text{ for some bounded } s+t\text{”} \wedge \\
& \quad z = \ulcorner s+t \urcorner) \vee \\
& (\text{“}k = \ulcorner s \urcorner \text{ for some bounded } s \text{ and } l \neq \ulcorner t \urcorner \text{ for any bounded } t\text{”} \wedge \\
& \quad \exists v\,[v = \mu w.\,(\textstyle\bigwedge\{\mathrm{Proof}\big(w, \ulcorner C^{\dagger} \urcorner\big) \mid s{:}C \in \Theta\}) \wedge \\
& \quad z = a'(v,l)]) \vee \\
& (\text{“}k \neq \ulcorner s \urcorner \text{ for any bounded } s \text{ and } l = \ulcorner t \urcorner \text{ for some bounded } t\text{”} \wedge \\
& \quad \exists v\,[v = \mu w.\,(\textstyle\bigwedge\{\mathrm{Proof}\big(w, \ulcorner B^{\dagger} \urcorner\big) \mid t{:}B \in \Theta\}) \wedge \\
& \quad z = a'(k,v)]) \vee \\
& (\text{“}k = \ulcorner s \urcorner \text{ and } l = \ulcorner t \urcorner \text{ for some bounded } s,\, t \text{ while } s+t \text{ is unbounded”} \wedge \\
& \quad \exists v_1\,\exists v_2\,[v_1 = \mu w.\,(\textstyle\bigwedge\{\mathrm{Proof}\big(w, \ulcorner C^{\dagger} \urcorner\big) \mid s{:}C \in \Theta\}) \wedge \\
& \quad v_2 = \mu w.\,(\textstyle\bigwedge\{\mathrm{Proof}\big(w, \ulcorner B^{\dagger} \urcorner\big) \mid t{:}B \in \Theta\}) \wedge z = a'(v_1,v_2)]) \vee \\
& (\text{“}k \neq \ulcorner s \urcorner \text{ and } l \neq \ulcorner t \urcorner \text{ for any bounded } s \;\; t\text{”} \wedge z = a'(k,l)));
\end{aligned}
$$

$$
\begin{aligned}
c(k) = \mu z.\,(& \\
& (\text{“}k = \ulcorner t \urcorner \text{ for some bounded } !\,t\text{”} \wedge z = \ulcorner !\,t \urcorner) \vee \\
& (\text{“}k = \ulcorner t \urcorner \text{ for some bounded } t \text{ while } !\,t \text{ is unbounded”} \wedge \\
& \quad \exists v_1\,\exists v_2\,[v_1 = \mu w.\,(\textstyle\bigwedge\{\mathrm{Proof}\big(w, \ulcorner B^{\dagger} \urcorner\big) \mid t{:}B \in \Theta\}) \wedge \\
& v_2 = \mu w.\,(\textstyle\bigwedge\{\mathrm{Proof}\Big(w, \ulcorner \mathrm{Proof}\Big(\overline{v_1}, \overline{\ulcorner C^{\dagger} \urcorner}\Big) \to \mathrm{Prf}\Big(\overline{k}, \overline{\ulcorner C^{\dagger} \urcorner}\Big) \urcorner\Big) \mid t{:}C \in \Theta\}) \wedge \\
& \quad z = m'(v_2, c'(v_1)))]) \vee \\
& (\text{“}k \neq \ulcorner t \urcorner \text{ for any bounded } t\text{”} \wedge \\
& \exists v\,[v = \mu w.\,(\textstyle\bigwedge\{\mathrm{Proof}\Big(w, \ulcorner \mathrm{Proof}\Big(\overline{k}, \overline{u}\Big) \to \mathrm{Prf}\Big(\overline{k}, \overline{u}\Big) \urcorner\Big) \mid \mathrm{Proof}(k,u)\}) \wedge \\
& \quad z = m'(v, c'(k))]));
\end{aligned}
$$

$$
\begin{aligned}
f_n(k) = \mu z.\,(& \\
& (\text{“}k = \ulcorner t \urcorner \text{ for some bounded } \mathfrak{f}_n\, t\text{”} \wedge z = \ulcorner \mathfrak{f}_n\, t \urcorner) \vee \\
& (\text{“}k = \ulcorner t \urcorner \text{ for some bounded } t \text{ while } \mathfrak{f}_n\, t \text{ is unbounded”} \wedge \\
& \quad \exists v\,[v = \mu w.\,(\textstyle\bigwedge\{\mathrm{Proof}\big(w, \ulcorner B^{\dagger} \urcorner\big) \mid t{:}B \in \Theta\}) \wedge \\
& \quad z = f_n'(v)]) \vee \\
& (\text{“}k \neq \ulcorner t \urcorner \text{ for any bounded } t\text{”} \wedge z = f_n'(k)).
\end{aligned}
$$

Now we define the required interpretation $\cdot^*$. Let Prf and m, a, c, f_n just defined be the proof predicate and the respective functions for $\cdot^*$. Also let it be that $\cdot^* |_{\mathrm{Var}} = \tau$ and $\cdot^* |_{\mathrm{PAt}} = \ulcorner \cdot \urcorner$.

Lemma 5.9. *For any bounded term t and any formula $F \in SFm(\Theta \cup \{A\})$, the following statements hold:*

1. $t^* = t^{\dagger}$;

2. $F^* \rightleftharpoons F^\dagger$.

Proof.

1. Induction on the construction of t. The definitions of $\cdot^*$ and $\cdot^\dagger$ cover the case when $t \in \mathrm{PAt}$ and also yield the formulas

$$(r \cdot s)^* = m(r^*, s^*) = m(\ulcorner r \urcorner, \ulcorner s \urcorner) = \ulcorner r \cdot s \urcorner = (r \cdot s)^\dagger\,;$$
$$(!\,s)^* = c(s^*) = c(\ulcorner s \urcorner) = \ulcorner !\,s \urcorner = (!\,s)^\dagger$$

as well as the similar ones for $+$ and $\mathfrak{f}_n$.

2. Induction on the construction of F. If $F \in \mathrm{Var}$, the definitions are sufficient. Suppose $F \rightleftharpoons t{:}G$. Then $(t{:}G)^* \rightleftharpoons \mathrm{Prf}\Big(\overline{t^*}, \overline{\ulcorner G^* \urcorner}\Big)$. By claim 1 and the induction hypothesis, $t^* = t^\dagger$ and $G^* \rightleftharpoons G^\dagger$. Hence $\mathrm{Prf}\Big(\overline{t^*}, \overline{\ulcorner G^* \urcorner}\Big) \rightleftharpoons \mathrm{Prf}\Big(\overline{t^\dagger}, \overline{\ulcorner G^\dagger \urcorner}\Big) \rightleftharpoons (t{:}G)^\dagger$. The other cases are trivial.

□

Lemma 5.10. *If $F \in SFm(\Theta \cup \{A\})$, then $\vdash_{\mathbf{HA}} F^* \leftrightarrow \tau\big(F^0\big)$.*

Proof. Induction on the construction of F. If $F \in \mathrm{Var}$, the definition is sufficient. Consider the case when $F \rightleftharpoons t{:}G$. Then $t{:}G \in \Theta$ or $\neg t{:}G \in \Theta$ because $\bigwedge \Theta \to A$ is projectively saturated. If $t{:}G \in \Theta$, we get $\vdash_{\mathbf{HA}} \tau\Big((t{:}G)^0\Big)$ by Lemma 5.6. From (1) it follows that $\mathrm{Prf}\big(\ulcorner t \urcorner, \ulcorner G^\dagger \urcorner\big)$. Then $\vdash_{\mathbf{HA}} \mathrm{Prf}\Big(\overline{\ulcorner t \urcorner}, \overline{\ulcorner G^\dagger \urcorner}\Big)$ since Prf is primitive recursive. By definition,

$$\mathrm{Prf}\Big(\overline{\ulcorner t \urcorner}, \overline{\ulcorner G^\dagger \urcorner}\Big) \rightleftharpoons \mathrm{Prf}\Big(\overline{t^\dagger}, \overline{\ulcorner G^\dagger \urcorner}\Big) \rightleftharpoons (t{:}G)^\dagger\,.$$

Hence $\vdash_{\mathbf{HA}} (t{:}G)^\dagger$. By Lemma 5.9, this implies that $\vdash_{\mathbf{HA}} (t{:}G)^*$. Thus $\vdash_{\mathbf{HA}} (t{:}G)^* \leftrightarrow \tau\Big((t{:}G)^0\Big)$. If $\neg t{:}G \in \Theta$, then Lemma 5.6 yields that $\vdash_{\mathbf{HA}} \neg\tau\Big((t{:}G)^0\Big)$. Since $\Theta \nvdash_{\mathbf{iLP}} A$, obtain $t{:}G \notin \Theta$. By (1) and Lemma 5.7, $\mathrm{Prf}\big(\ulcorner t \urcorner, \ulcorner G^\dagger \urcorner\big)$ is false because Proof is false for the numbers of proof terms. Using Lemma 5.9, we have $\vdash_{\mathbf{HA}} \neg\mathrm{Prf}\Big(\overline{t^*}, \overline{\ulcorner G^* \urcorner}\Big)$, i.e. $\vdash_{\mathbf{HA}} \neg\,(t{:}G)^*$. Thus $\vdash_{\mathbf{HA}} (t{:}G)^* \leftrightarrow \tau\Big((t{:}G)^0\Big)$. The other cases are trivial. □

Lemma 5.11. *Suppose $\varphi \in \mathcal{A}_0$; then $\vdash_{\mathbf{HA}} \varphi$ iff $\mathrm{Prf}(k, \ulcorner \varphi \urcorner)$ for some $k \in \omega$.*

Proof. We only need to consider the direction from right to left. From (1) it follows that there are two possible cases: $\mathrm{Proof}(k, \ulcorner \varphi \urcorner)$ or $k = \ulcorner t \urcorner$ and $\ulcorner \varphi \urcorner = \ulcorner F^\dagger \urcorner$ for some quasiatom $t{:}F \in \Theta$. In the former case we get $\vdash_{\mathbf{HA}} \varphi$ since Proof is a proof predicate. Consider the latter one. By Lemma 5.6, $\vdash_{\mathbf{HA}} \tau\big(F^0\big)$. From Lemmas 5.10 and 5.9 it follows that $\vdash_{\mathbf{HA}} F^\dagger$. For the Gödel numbering is injective, $\varphi \rightleftharpoons F^\dagger$. Thus $\vdash_{\mathbf{HA}} \varphi$. □

Lemma 5.12. *The functions m, a, c and f_n are general recursive and for any $\{k, l\} \subset \omega$ and any $\{\varphi, \psi\} \subset \mathcal{A}_0$ the following formulas are true:*

1. $\mathrm{Prf}(k, \ulcorner \varphi \to \psi \urcorner) \to (\mathrm{Prf}(l, \ulcorner \varphi \urcorner) \to \mathrm{Prf}(m(k,l)\,, \ulcorner \psi \urcorner))$;

2. $\mathrm{Prf}(k, \ulcorner\varphi\urcorner) \to \mathrm{Prf}(a(k,l), \ulcorner\varphi\urcorner)$; $\mathrm{Prf}(l, \ulcorner\varphi\urcorner) \to \mathrm{Prf}(a(k,l), \ulcorner\varphi\urcorner)$;

3. $\mathrm{Prf}(k, \ulcorner\varphi\urcorner) \to \mathrm{Prf}\Big(c(k), \ulcorner\mathrm{Prf}\big(\overline{k}, \overline{\ulcorner\varphi\urcorner}\big)\urcorner\Big)$;

4. $\mathrm{Prf}(k, \ulcorner\varphi\urcorner) \to \mathrm{Prf}(f_n(k), \ulcorner\psi\urcorner)$ *if* $\dfrac{\varphi}{\psi} \in \mathrm{V}_n$.

Proof.

1. Suppose $\mathrm{Prf}(k, \ulcorner\varphi \to \psi\urcorner)$ and $\mathrm{Prf}(l, \ulcorner\varphi\urcorner)$. From the definition of bounded proof term it follows that only the following cases are possible.

   (a) Neither k nor l are numbers of bounded terms. By (1), the formulas $\mathrm{Proof}(k, \ulcorner\varphi \to \psi\urcorner)$ and $\mathrm{Proof}(l, \ulcorner\varphi\urcorner)$ hold. The definition of m' implies that $\mathrm{Proof}(m'(k,l), \ulcorner\psi\urcorner)$ does also. As $m(k,l) = m'(k,l)$, we get $\mathrm{Prf}(m(k,l), \ulcorner\psi\urcorner)$ by (1).

   (b) $k = \ulcorner s\urcorner$ and $l = \ulcorner t\urcorner$ for some bounded term $s \cdot t$. By (1) and Lemma 5.7, we have $\varphi \doteq F^\dagger$ and $\psi \doteq G^\dagger$ for some quasiatoms $s\colon (F \to G) \in \Theta$ and $t\colon F \in \Theta$. Then $s \cdot t\colon G \in \Theta$ since Θ is operationally complete. By (1), $\mathrm{Prf}\big(\ulcorner s \cdot t\urcorner, \ulcorner G^\dagger\urcorner\big)$. As $m(k,l) = \ulcorner s \cdot t\urcorner$, we get $\mathrm{Prf}(m(k,l), \ulcorner\psi\urcorner)$.

   (c) k is the number of no bounded term and $l = \ulcorner t\urcorner$ for some bounded term t. By (1), the formula $\mathrm{Proof}(k, \ulcorner\varphi \to \psi\urcorner)$ holds and $\varphi \doteq F^\dagger$ for some quasiatom $t\colon F \in \Theta$. Compute the number

   $$v \quad = \quad \mu w.\,\big(\textstyle\bigwedge\{\mathrm{Proof}\big(w, \ulcorner B^\dagger\urcorner\big) \mid t\colon B \in \Theta\}\big)$$

   by the following method. Let B_i for $i \in \{1, \ldots, j\}$ be all the formulas such that $t\colon B_i \in \Theta$. By Lemmas 5.6 and 5.10, we have $\vdash_{\mathbf{HA}} B_i^*$. Using Lemma 5.9, get $\vdash_{\mathbf{HA}} B_i^\dagger$. By the conjoinability property of Proof, there is a number w such that $\mathrm{Proof}\Big(w, \ulcorner B_i^\dagger\urcorner\Big)$ for all $i \in \{1, \ldots, j\}$. Let v be the least such w. In particular, $\mathrm{Proof}(v, \ulcorner\varphi\urcorner)$. By the definition of m', $\mathrm{Proof}(m'(k,v), \ulcorner\psi\urcorner)$. Since $m(k,l) = m'(k,v)$, we get $\mathrm{Prf}(m(k,l), \ulcorner\psi\urcorner)$ by (1).

   (d) l is the number of no bounded term and $k = \ulcorner s\urcorner$ for some bounded term s. This case is similar to the previous one.

   (e) $k = \ulcorner s\urcorner$ and $l = \ulcorner t\urcorner$ for some bounded terms s, t and unbounded $s \cdot t$. By (1) and Lemma 5.7, we have $\varphi \doteq F^\dagger$ and $\psi \doteq G^\dagger$ for some quasiatoms $s\colon (F \to G) \in \Theta$ and $t\colon F \in \Theta$. Compute the numbers

   $$\begin{aligned} v_1 &= \mu w.\,\big(\textstyle\bigwedge\{\mathrm{Proof}\big(w, \ulcorner C^\dagger\urcorner\big) \mid s\colon C \in \Theta\}\big), \\ v_2 &= \mu w.\,\big(\textstyle\bigwedge\{\mathrm{Proof}\big(w, \ulcorner B^\dagger\urcorner\big) \mid t\colon B \in \Theta\}\big) \end{aligned}$$

   in the same way as in case 1c. We obtain $\mathrm{Proof}(v_1, \ulcorner\varphi \to \psi\urcorner)$ and $\mathrm{Proof}(v_2, \ulcorner\varphi\urcorner)$. From the definition of m' it follows that the formula $\mathrm{Proof}(m'(v_1,v_2), \ulcorner\psi\urcorner)$ holds. By (1), $\mathrm{Prf}(m(k,l), \ulcorner\psi\urcorner)$ does also since $m(k,l) = m'(v_1,v_2)$.

2. This case is considered similarly to case 1.

3. Suppose $\mathrm{Prf}(k, \ulcorner\varphi\urcorner)$. Only the following cases are possible.

(a) $k = \ulcorner t \urcorner$ for some bounded term $!t$. By (1), we have $\varphi \coloneqq F^\dagger$ for some quasiatom $t{:}F \in \Theta$. As Θ is operationally complete, $!t{:}t{:}F \in \Theta$. By (1), we get $\mathrm{Prf}\left(\ulcorner !t \urcorner, \ulcorner (t{:}F)^\dagger \urcorner\right)$. Since $c(k) = \ulcorner !t \urcorner$, we conclude by the definition of $\cdot^\dagger$ that $\mathrm{Prf}\left(c(k), \ulcorner \mathrm{Prf}\left(\overline{k}, \overline{\ulcorner \varphi \urcorner}\right) \urcorner\right)$.

(b) k is the number of no bounded term. By (1), $\mathrm{Proof}(k, \ulcorner \varphi \urcorner)$ holds. From the definition of c' it follows that

$$\mathrm{Proof}\left(c'(k), \ulcorner \mathrm{Proof}\left(\overline{k}, \overline{\ulcorner \varphi \urcorner}\right) \urcorner\right).$$

By (1), $\vdash_{\mathbf{HA}} \mathrm{Proof}\left(\overline{k}, y\right) \rightarrow \mathrm{Prf}\left(\overline{k}, y\right)$. From the finiteness and conjoinability properties of Proof it follows that there exists v such that

$$v = \mu w. \left(\bigwedge \{\mathrm{Proof}\left(w, \ulcorner \mathrm{Proof}\left(\overline{k}, \overline{u}\right) \rightarrow \mathrm{Prf}\left(\overline{k}, \overline{u}\right) \urcorner\right) \mid \mathrm{Proof}(k, u)\}\right).$$

Using (1) and the definition of c we obtain

$$\begin{aligned}
&\mathrm{Proof}\left(v, \ulcorner \mathrm{Proof}\left(\overline{k}, \overline{\ulcorner \varphi \urcorner}\right) \rightarrow \mathrm{Prf}\left(\overline{k}, \overline{\ulcorner \varphi \urcorner}\right) \urcorner\right);\\
&\mathrm{Proof}\left(m'(v, c'(k)), \ulcorner \mathrm{Prf}\left(\overline{k}, \overline{\ulcorner \varphi \urcorner}\right) \urcorner\right);\\
&\mathrm{Prf}\left(m'(v, c'(k)), \ulcorner \mathrm{Prf}\left(\overline{k}, \overline{\ulcorner \varphi \urcorner}\right) \urcorner\right);\\
&\mathrm{Prf}\left(c(k), \ulcorner \mathrm{Prf}\left(\overline{k}, \overline{\ulcorner \varphi \urcorner}\right) \urcorner\right).
\end{aligned}$$

(c) $k = \ulcorner t \urcorner$ for some bounded term t and unbounded $!t$. By (1), $\varphi \coloneqq F^\dagger$ for some quasiatom $t{:}F \in \Theta$. In the same way as in case 1c, we compute

$$v_1 = \mu w. \left(\bigwedge \{\mathrm{Proof}\left(w, \ulcorner B^\dagger \urcorner\right) \mid t{:}B \in \Theta\}\right).$$

So we get $\mathrm{Proof}(v_1, \ulcorner \varphi \urcorner)$. By the definition of c', the formula

$$\mathrm{Proof}\left(c'(v_1), \ulcorner \mathrm{Proof}\left(\overline{v_1}, \overline{\ulcorner \varphi \urcorner}\right) \urcorner\right)$$

holds. Consider a formula C such that $t{:}C \in \Theta$. From (1) it follows that $\mathrm{Prf}\left(\overline{k}, \overline{\ulcorner C^\dagger \urcorner}\right)$. Hence $\vdash_{\mathbf{HA}} \mathrm{Prf}\left(\overline{k}, \overline{\ulcorner C^\dagger \urcorner}\right)$, as Prf is a primitive recursive formula. Then $\vdash_{\mathbf{HA}} \mathrm{Proof}\left(\overline{v_1}, \overline{\ulcorner C^\dagger \urcorner}\right) \rightarrow \mathrm{Prf}\left(\overline{k}, \overline{\ulcorner C^\dagger \urcorner}\right)$. Now compute v_2 similarly to case 3b

$$v_2 = \mu w. \left(\bigwedge \{\mathrm{Proof}\left(w, \ulcorner \mathrm{Proof}\left(\overline{v_1}, \overline{\ulcorner C^\dagger \urcorner}\right) \rightarrow \mathrm{Prf}\left(\overline{k}, \overline{\ulcorner C^\dagger \urcorner}\right) \urcorner\right) \mid t{:}C \in \Theta\}\right).$$

We obtain

$$\mathrm{Proof}\left(v_2, \ulcorner \mathrm{Proof}\left(\overline{v_1}, \overline{\ulcorner \varphi \urcorner}\right) \rightarrow \mathrm{Prf}\left(\overline{k}, \overline{\ulcorner \varphi \urcorner}\right) \urcorner\right).$$

The same way as in case 3b we conclude that

$$\mathrm{Prf}\left(m'(v_2, c'(v_1)), \ulcorner \mathrm{Prf}\left(\overline{k}, \overline{\ulcorner \varphi \urcorner}\right) \urcorner\right),$$

whence

$$\mathrm{Prf}\left(c(k), \ulcorner \mathrm{Prf}\left(\overline{k}, \overline{\ulcorner \varphi \urcorner}\right) \urcorner\right).$$

4. Suppose $\mathrm{Prf}(k, \ulcorner\varphi\urcorner)$ and $\dfrac{\varphi}{\psi} \in \mathrm{V}_n$ for some formula ψ. Only the following cases are possible.

   (a) $k = \ulcorner t \urcorner$ for some bounded term $\mathfrak{f}_n\, t$. By (1), $\varphi \mathrel{=\!\!=} F^\dagger$ for some quasiatom $t\colon F \in \Theta$. The conclusion of a Visser's rule is constructed from the subformulas of its premise. Therefore, by the definition of $\cdot^\dagger$, there exists a formula $G \in \mathcal{ILP}$ such that $G^\dagger \mathrel{=\!\!=} \psi$ and $\dfrac{F}{G} \in \mathrm{V}_n$. Since Θ is operationally complete, we get $\mathfrak{f}_n\, t\colon G \in \Theta$. By (1), $\mathrm{Prf}\big(\ulcorner \mathfrak{f}_n\, t \urcorner, \ulcorner G^\dagger \urcorner\big)$. Thus $\mathrm{Prf}(f_n(k), \ulcorner\psi\urcorner)$ as $f_n(k) = \ulcorner \mathfrak{f}_n\, t \urcorner$.

   (b) k is the number of no bounded term. The formula $\mathrm{Proof}(k, \ulcorner\varphi\urcorner)$ holds by (1). From the definition of f'_n, we obtain $\mathrm{Proof}(f'_n(k), \ulcorner\psi\urcorner)$. Now (1) yields $\mathrm{Prf}(f'_n(k), \ulcorner\psi\urcorner)$. Thus $\mathrm{Prf}(f_n(k), \ulcorner\psi\urcorner)$ as $f_n(k) = f'_n(k)$.

   (c) $k = \ulcorner t \urcorner$ for some bounded term t and unbounded $\mathfrak{f}_n\, t$. By (1), we have $\varphi \mathrel{=\!\!=} F^\dagger$ for some quasiatom $t\colon F \in \Theta$. Compute

   $$v = \mu w.\, \big(\textstyle\bigwedge\{\mathrm{Proof}\big(w, \ulcorner B^\dagger \urcorner\big) \mid t\colon B \in \Theta\}\big)$$

   in the same way as in case 1c. We get $\mathrm{Proof}(v, \ulcorner\varphi\urcorner)$. By the definition of f'_n, the formula

   $$\mathrm{Proof}(f'_n(v), \ulcorner\psi\urcorner)$$

   holds. From (1), we obtain $\mathrm{Prf}(f_n(k), \ulcorner\psi\urcorner)$ since $f_n(k) = f'_n(v)$.

□

Lemma 5.13. *The proof predicate* Prf *is normal.*

Proof.

finiteness By (1), from the finiteness of Θ and Proof it follows the set $T(k) = \{l \mid \mathrm{Prf}(k, l)\}$ is finite and the function $k \mapsto \ulcorner\{\bar{l} \mid l \in T(k)\}\urcorner$ is general recursive.

conjoinability By lemma 5.12, $T(k) \cup T(l) \subseteq T(a(k, l))$ for all $\{k, l\} \subset \omega$

□

Now we shall finalize the proof of Theorem 5.3. From Lemmas 5.11, 5.12, and 5.13 it follows that $\cdot^*$ is an arithmetical interpretation. Lemma 5.6 yields that $\tau\big(\Theta^0\big) \nvdash_{\mathbf{HA}} \tau\big(A^0\big)$. By Lemma 5.10, we get $\Theta^* \nvdash_{\mathbf{HA}} A^*$. Hence $\Gamma^* \nvdash_{\mathbf{HA}} A^*$ as $\Gamma \subseteq \Theta$. □

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