Finality Regained

A Coalgebraic Study of Scott-sets and Multisets

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Abstract

In this paper we study iterated circular multisets in a coalgebraic framework. We will produce two essentially different universes of such sets. The unisets of the first universe will be shown to be precisely the sets of the Scott universe. The unisets of the second universe will be precisely the sets of the AFA-universe. We will have a closer look into the connection of the iterated circular multisets and arbitrary trees.

Key words: multiset, non-wellfounded set, Scott-universe, AFA, coalgebra, modal logic, graded modalities

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Contents

1 Introduction

1.1 Multisets on a Given Domain

Multisets are very natural objects: they can model a number of different situations in different contexts, like the store of a shop or the bag of a housewife. A multiset —a *bag* in Computer Science— is like a set, except that an element can have multiple occurrences in it. For example, a grocery shop with 3 apples, 2 pears, 1 banana, and 0 kiwi in store can be modeled by the multiset

$$[\![\, \mathsf{apple}, \mathsf{apple}, \mathsf{apple}, \mathsf{pear}, \mathsf{pear}, \mathsf{banana} \,]\!] \, .$$

In proof theory sequents are often modeled as pairs of multisets (see e.g. [8]).

Multiple occurrences of an object $d \in D$ in a multiset can be described by a family of relations $\{\in_k \colon k \in \mathsf{Card}^+\}$: e.g., if α is the grocery shop above and $a := \mathsf{apple}$, we have $a \in_i \alpha$, for $i \in \{1, 2, 3\}$, $a \notin_j \alpha$, for $j \geq 4$; we say that the multiplicity of a in the multiset α is 3 or that $m_\alpha(a) = 3$.

We should distinguish between the platonic idea of a multiset and set-theoretic representations of it. Different representations of the same platonic idea should be expected to be naturally isomorphic (in a sense to be specified later). We will see that for certain purposes certain representations are better than others *even if they are naturally isomorphic*. We will also meet *two* salient platonic ideas of multiset. Before explaining the fact that there are at least two notions of multiset, let's first look at some set-theoretic representations.

The first choice that comes to mind is to represent a multiset on a domain D as a function that associates to each $d \in D$ a cardinal number, which says how many times the element d is present in the multiset. Since we allow domains that are large classes, but we want to represent multisets on any domain by *sets*, we represent a multiset on a class D by a (small) partial function from D to Card^+, where Card^+ is the class of all strictly positive cardinals. In other words, we replace zero by undefined. For example, in the grocery shop above the domain D is given by the set $\{a, p, b, k\}$ and the shop is represented by a function f with $f : D \to \mathsf{Card}^+$, $f(a) = 3$, $f(p) = 2$, $f(b) = 1$, $f(k) = \uparrow$.

Let's look at a second representation of multiset. A first approximation is to say that a multiset of elements of D is a function f from a set I, the set of items, to D. Here D can be viewed as the set of types. In our example of the grocery store the items could be taken to be the concrete fruits apple_1, apple_2, apple_3, pear_1, pear_2, banana_1; the types could be taken apple, pear, banana, kiwi. We would have $f(\mathsf{apple}_1) = \mathsf{apple}$, etcetera. This first approximation is not quite right. It fails to capture the level of abstraction that we aim at in speaking of multisets. The point is that we want to abstract away from the concrete individuality of the items. The only thing that interests us about the items is the type they have and the fact that they differ amongst each other. We do not want to know about properties they might have that are not included in the selected set of types D. The way to implement this is to say that $f : I \to D$ and $g : J \to D$ stand for the same multiset of elements of D if there is a bijection h between I and J such that $f = g \circ h$.

It is easy to see how to translate back and forth between the two representations. A disadvantage of the second representation is that the equivalence class representing a multiset will be a proper class. We will sidestep this problem by stipulating that the item set I will always be a cardinal. Our second representation will be the basis of the functor Γ introduced in Section 3.1.

Up to this point we have been looking at multisets as inert objects. Unless we have some relations between them and some operations on them, they do not truly qualify as first class citizens of the realm of mathematics. Let's ask ourselves: what are the proper morphisms on mulitsets? Reflecting on our second presentation, we quickly arrive at the following proposal: a morphism ϕ from the D-multiset α to the E-multiset β is a function from D to E, a translation of types, such that we can find $f : I \to D$, representing α, $g : J \to E$, representing β, and an injection $h : I \to J$ with $g \circ h = \phi \circ f$. The basic idea in our choice of h is that morphisms preserve items.

Let's translate our definition of morphism to the terms of our first set-theoretic representation. ϕ is a morphism from α considered as a partial function from D to Card^+ to β considered as a partial function from E to Card^+ if, for all $e \in E$, $\sum_{\phi(d)=e} \alpha(d) \leq \beta(e)$. (Here we treat 'undefined' as if it were zero and we treat the empty sum as zero/undefined.)

Upon reflection, we see that our morphisms have a natural factorization. We can split ϕ into the 'image'-mapping from α to $\phi[\alpha]$, where $\phi[\alpha]$ is given as the function on E with $\phi[\alpha](e) := \sum_{\phi(d)=e} \alpha(d)$, and an extension mapping from $\phi[\alpha]$ to β. Extension mappings simply increase the cardinalities of the elements of a multiset. The image mapping will play a major role in this paper: it will take the form of the functor Γ.

We could view morphisms as follows. A multiset is an infon representing how many items of certain types are in a certain store. (We could choose e.g. between saying that it represents how many items there are *precisely* and how many elements there are *at least*.) The image mapping corresponds to 'retyping'. E.g. ϕ could map apple, pear, banana, and kiwi to fruit. The image of our sample multiset is now:

$$[\![\, \mathsf{fruit}, \mathsf{fruit}, \mathsf{fruit}, \mathsf{fruit}, \mathsf{fruit}, \mathsf{fruit} \,]\!] \, .$$

In other words, given that there are (precisely/at least) 3 apples, 2 pears and 1 banana in store and given that apples, pears, bananas and kiwis are fruits, then there are (precisely/at least) 6 fruits in store. The extension mappings could correspond to real extensions of the store, in case of the 'precisely' variant of our interpretation, or to epistemic extensions, in case of the 'at least' variant: we learn that there are more items than we originally knew.

The definition of morphism on the cardinality representation has a surprising aspect. Shouldn't the morphisms simply have been defined as follows?

> ϕ is a morphism from α considered as a partial function from D to Card^+ to β considered as a partial function from E to Card^+ if, for all $d \in D$, $\alpha(d) \leq \beta(\phi(d))$.

Why did we get the sum in the definition? The reason is our implicit treatement of the types as exclusive: an item witnessing the presence of an apple cannot at the same time witness the presence of a pear. If we switch to possibly overlapping types, we will get the second notion of morphism. We will look at our multisets as follows. They are pieces of information or *infons* concerning a store to the effect that there are *at least* so many items of this, so many items of that, ... It is essential for our present interpretation that the information is open-ended: there could turn out to be more of each kind. E.g. our types could have been apple and rotten. The grocery store could have been described by $[\![\mathsf{apple}, \mathsf{apple}, \mathsf{rotten}, \mathsf{rotten}]\!]$. This means that there are at least two apples and at least two rotten things. The description could be taken to be compatible with there being three items in store: two apples, of which one rotten, and a rotten pear.

What about morphisms? If we would have a function ϕ sending both apple and rotten to fruit, we can view it as the information that both apples and rotten things are fruits. The ϕ-image of $[\![\mathsf{apple}, \mathsf{apple}, \mathsf{rotten}, \mathsf{rotten}]\!]$ will be $[\![\mathsf{fruit}, \mathsf{fruit}]\!]$, since we learn from the 'information' ϕ that there are at least two fruits. However, there is also a morphism $\phi : [\![\mathsf{apple}, \mathsf{apple}, \mathsf{rotten}, \mathsf{rotten}]\!] \to [\![\mathsf{fruit}, \mathsf{fruit}, \mathsf{fruit}]\!]$, since, via ϕ and 'extension', our information could grow to the knowledge that there are at least three fruits in store.

It is clear that as before we could split our morphisms into two stages. There is a new image-mapping defined by $\phi[\alpha](e) := \mathsf{sup}\{\alpha(d) \mid \phi(d) = e\}$. Extension mappings are as before. This alternative image mapping will lead to our functor Δ. We discover here our second Platonic idea of multiset: multisets with overlapping types.

Can we find a third representation, in the style of the second representation, that models the possibility of overlapping types? Here is one way to do it. A representation of a multiset on D is a binary relation r (modeled as a set of pairs) such that $\mathsf{dom}(r) \subseteq D$. Two representations r and s of multisets on D are the same if there is a bijection h between r and s considered as sets of pairs, such that $\pi_1 \circ h = \pi_1$. (Here $\pi_1(d, i) = d$.)

A morphism ϕ from the D-multiset α to the E-multiset β is a function from D to E, such that we can find r, representing α, and s, representing β, and a function $h : r \to s$ such that π_2 and h are jointly injective and $\pi_2 \circ h = \phi \circ \pi_2$. (The functions p and q on P are *jointly injective* if $\lambda x {\in} P \cdot (px, qx)$ is injective.) It is easy to see that we did indeed define a category and that our earlier notion of sameness coincides with isomorphism in this category.

Our third representation has again the disadvantage that the equivalence classes are proper classes. We will sidestep this problem by working with standard representatives. This modified version of the third representation will lead to the uniform form of the functor Δ introduced in Section 3.2. It is easy to see that our third representation yields precisely our second notion of morphism if we switch back to the cardinal representation.

1.2 Iterated and Circular Multisets

As in the case of ordinary sets we want to represent *iterated* multisets, where multisets contain (various occurrences) of other multisets. In this case the domain D is made of multisets. By epsilon recursion it is easy to define the class of wellfounded multisets, but here we are interested in circular situations, like in non-wellfounded set theory: we want to have the possibility for a multiset x to be a member of itself, repeated any number of times, i.e. we want to guarantee the existence of multisets satisfying equations like $x = [\![x, x]\!]$.

Circular multisets can be modeled using the theory of coalgebras. In the simpler case of sets, the AFA-universe is described as a final coalgebra for the powerset functor Pow. This functor sends a class A to the class $\mathsf{Pow}(A)$ of all subsets of A and a class-function $f : A \to B$ to the class-function $\mathsf{Pow}(f) : \mathsf{Pow}(A) \to \mathsf{Pow}(B)$, which sends a subset A' of A to its image via f: $\mathsf{Pow}(f)(A') = \{fx : x \in A'\}$. Analogously, we need a multiset functor to describe the multiset universe. This functor F should send a class A to the class $\mathsf{F}(A)$ of all A-multisets (represented using one of the various possibilities we gave in the introduction), and a function $f : A \to B$ to a function $\mathsf{F}(f) : \mathsf{F}(A) \to \mathsf{F}(B)$. Hence, if $\alpha \in \mathsf{F}(A)$ is the representation of an A-multiset, we should define a B-multiset $\beta = \mathsf{F}(f)(\alpha) \in \mathsf{F}(B)$, representing the action of the class-function f on the multiset α. It is quite clear that β must contain elements of type fa with $a \in A$, but what about multiplicity? The discussion made in the preceding section leads to two different Platonic ideas of this action. The first one arises by considering our types as exclusive, or non-overlapping, and gives $m_\beta(b) = \sum_{f(x)=b} m_\alpha(x)$. On the other hand, if we follow the idea of overlapping types we get $m_\beta(b) = \mathsf{sup}\{m_\alpha(x) : f(x) = b\}$. We use the exclusive types idea in Section 3.1 to define the multiset functor Γ, while the overlapping types idea suggests in Section 3.2 a different functor, which we denote by Δ. Suppose for example that $A = \{x, y\}$ and f sends both x, y to z; if $\alpha = [\![x, x, y]\!]$ then $\Gamma(f)(\alpha) = [\![z, z, z]\!]$, while $\Delta(f)(\alpha) = [\![z, z]\!]$. We can then apply the general theory of coalgebras to the functors Γ and Δ, obtaining a definition of Γ-coalgebra and Δ-coalgebra, of Γ- and Δ-bisimulation, of Γ- and Δ-collapse, and prove the existence of a Γ-final coalgebra and a Δ-final coalgebra. We then use these two final coalgebras to define two non-isomorphic multiset universes: the Γ and Δ-multiuniverses. The difference between these two multiuniverses can be already appreciated at the level of simple (uni-)sets (i.e. the multisets containing *hereditarely* at most one occurrence of each element): the unisets inside the Δ-universe are a model of the well-known non-wellfounded set theory $\mathsf{ZFC}^- + \mathsf{AFA}$ (Zermelo-Fraenkel with choice, with foundation replaced with the anti-foundation axiom AFA), while the unisets inside the Γ-universe are a model of $\mathsf{ZFC}^- + \mathsf{Scott}$. This kind of sets was first proposed by Scott in [7] and later reconsidered by Aczel in [1]. Using the notion of Scott-bisimulation, Aczel compared the theory $\mathsf{ZFC}^- + \mathsf{Scott}$ with $\mathsf{ZFC}^- + \mathsf{AFA}$. Both were obtained from ZFC^- (Zermelo-Fraenkel with choice, without foundation) by using a strengthening of the extensionality axiom, defined in terms of bisimulation: the maximal bisimulation for $\mathsf{ZFC}^- + \mathsf{AFA}$, Scott-bisimulation for $\mathsf{ZFC}^- + \mathsf{Scott}$.

In this confrontation, the Scott-sets seemed to be less natural and manageable than the AFA-sets (e.g. in $\mathsf{ZFC}^-\mathsf{AFA}$ any graph has a *unique* decoration, while in $\mathsf{ZFC}^- + \mathsf{Scott}$ graphs can have more than one decoration; in $\mathsf{ZFC}^-\mathsf{AFA}$ any set can be represented by a collapsed graph, while a similar notion of collapse is not available in $\mathsf{ZFC}^- + \mathsf{Scott}$).

In this work we claim that the natural context of the Scott-bisimulation is the multiset-context (defined via the Γ-operator above). We show that Scott-bisimulation (in its generalization to multigraphs) corresponds precisely to Γ-bisimulation. Hence, by moving from the set to the multiset context we acquire the possibility of working with coalgebras having a natural notion of collapse, decoration, and so on, which were missed in the set-context.

Using the general theory of coalgebras we see that the Γ-multiuniverse can be modeled using the class of pointed Γ-collapsed multigraphs or, in categorical terms, by using a final Γ-coalgebra. In the case of the Γ-multiuniverse we prove that another description is possible which is not generally available using the theory of coalgebras: Γ-multisets correspond exactly to rooted trees.

We continue our investigation of the Δ- and Γ-multiuniverses with a description of multisets by way of logics. In this context the Γ-multiuniverse appears to be more natural, since the corresponding logic is a fragment of the well-known and much used logic of graded modalities.

Finally, we turn to the problem of enriching the structure of our multiuniverses. We proceed by introducing singleton and unary unions, using the categorical notion of monad. In the case of the functor Γ the corresponding Kleisli category allows to define a product of coalgebras having the same domain, and this product is shown to be representable by matrix multiplication.

1.3 Organization of the Paper

This paper is organized as follows. In Section 2 we give some notation concerning multigraphs and review the fundamentals of the theory of coalgebras, which are used throughout the paper. In Section 3.1 we introduce a multiset functor Γ, working out the results we can obtain by applying the general theory of coalgebras. In this way we are able to introduce our first multiuniverse, the one of exclusive types. Section 3.2 deals with the definition and properties of the functor Δ of overlapping types and of the corresponding Δ-multiuniverse. We then return to exclusive types, and in Section 4 we study the special role of rooted trees inside the category of Γ-coalgebras and morphisms. In Section 5 we deal with the identification of multisets with formulae of infinitary logics and finally, in Section 6 we enrich the structure of our multiuniverses by using monads.

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2 Prerequisites

To understand the paper the reader is supposed to be familiar with the theory of coalgebras. For an introduction to the subject see e.g. [6]. In the following we briefly review some of the necessary materials from [1],[3], and [6].

2.1 Coalgebras and Morphisms

In this paper we largely use the theory of coalgebras. A primary example of coalgebra is given by considering the functor Pow on the category of classes and functions. We start by briefly discussing this example and its relations with non-wellfounded sets.

2.1.1 A Prototype: Pow

Let $\mathcal{C}$ be the category of classes and class-functions between them. For the moment we assume only that our universe satisfies ZFC^-, that is, Zermelo Fraenkel Set Theory with choice and without foundation. The powerset operator Pow can be turned into a functor from $\mathcal{C}$ to $\mathcal{C}$ by defining it on a class A as:

$$\mathsf{Pow}(A) = \{x : \ x \text{ is a set and } \ x \subseteq A\},$$

and on a function $f : A \to B$ as the function $\mathsf{Pow}(f) : \mathsf{Pow}(A) \to \mathsf{Pow}(B)$ which assigns to every $x \subseteq A$ the following subset of B: $\mathsf{Pow}(f)(x) = \{f(y) : y \in x\}$. We also use the simpler notation $f[x]$ for $\mathsf{Pow}(f)(x)$.

Consider now a *directed graph* $\mathcal{G}$. When possible, we would like to associate to $\mathcal{G}$ a *decoration* with sets, that is, a function d from the nodes of the graph to sets such that if v is a node then $\mathsf{d}(v) = \{\mathsf{d}(v') : \langle v, v'\rangle \text{ is an edge of } \mathcal{G}\}$. Depending on the set theory under consideration the class of graphs having a decoration can change, and a graph can have zero, one, or various decorations. For example, under *foundation* any graph has at most one decoration, and wellfounded graphs have exactly one, while a version of the well-known *anti-foundation* axiom AFA just says that *any* graph has a unique decoration. These differences between the various set theories can also be expressed *categorically* as follows. Directed graphs (possibly with class domains) can be identified with coalgebras of the functor Pow, that is, with pairs $\langle A, e\rangle$ consisting of a class A and a function $e : A \to \mathsf{Pow}(A)$ (in this identification, the set $e(a)$ corresponds to $\{a' : \langle a, a'\rangle \text{ is an edge of } \mathcal{G}\}$). Coalgebras are the objects of a new category $\mathsf{Coal}_{\mathsf{Pow}}$, having as maps *coalgebra morphisms*, where a function $f : A \to A'$ is a **morphism** between the coalgebras $\langle A, e\rangle, \langle A', e'\rangle$ iff the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & \mathsf{Pow}(A) \arrow[d, "\mathsf{Pow}(f)"] \\ A' \arrow[r, "e'"] & \mathsf{Pow}(A') \end{tikzcd}$$

We also consider the category $\mathsf{Alg}_{\mathsf{Pow}}$, whose objects are pairs $\langle A, e\rangle$ with $e : \mathsf{Pow}(A) \to A$ and whose maps are the *algebra morphisms*, i.e. functions $f : A \to A'$ for which the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Pow}(A) \arrow[r, "e"] \arrow[d, "\mathsf{Pow}(f)"'] & A \arrow[d, "f"] \\
\mathsf{Pow}(A') \arrow[r, "e'"] & A'
\end{tikzcd}$$

Notice that if V denotes the universe of all sets, then $\mathsf{V} = \mathsf{Pow}(\mathsf{V})$, so that the pair $(\mathsf{V}, \mathsf{id})$ is a Pow-coalgebra, where id is the identity function. The categorical description of the set theories above is then given by the following equivalences:

- V is a model of $\mathsf{ZFC} \Leftrightarrow \langle \mathsf{V}, \mathsf{id}\rangle$ is an initial object in the category $\mathsf{Alg}_{\mathsf{Pow}}$,
- V is a model of $\mathsf{ZFC}^- + \mathsf{AFA} \Leftrightarrow \langle \mathsf{V}, \mathsf{id}\rangle$ is a final object in the category $\mathsf{Coal}_{\mathsf{Pow}}$.

2.1.2 Scott's Sets.

In [7] a model of ZFC^- is constructed using rooted irredundant trees modulo isomorphism, where a tree $\mathcal{T}$ is irredundant if it has no proper automorphism, or, equivalently: for all $u \in \mathcal{T}$, $u', v' \in \mathsf{Succ}(u)$, if $\langle \mathcal{T}, u'\rangle$ is isomorphic to $\langle \mathcal{T}, v'\rangle$ then $u' = v'$. The rooted irredundant trees modulo isomorphism are also a model for the Scott-axiom, which roughly says that a rooted tree is isomorphic to the unraveling of a set iff it is irredundant. This axiom is compared with the AFA-axiom in [1]. First, the definition of Scott-bisimulation is given: two rooted Pow-coalgebras $\langle \mathcal{A}, a\rangle$, $\langle \mathcal{A}', a'\rangle$ are said to be **Scott-bisimilar** if their unravelings are isomorphic (for a definition of unraveling see Section 4.2), and a coalgebra is said to be **Scott-extensional** if two different nodes in the coalgebra are never Scott-bisimilar. A pointed coalgebra $\langle \mathcal{A}, a\rangle$ is said to be an **exact picture** of a set if there exists an injective morphism from $\langle \mathcal{A}, a\rangle$ to the coalgebra $\langle \mathsf{V}, \mathsf{id}\rangle$. Using these notions one can give a categorical formulation of the Scott-axiom:

- V is a model of $\mathsf{ZFC}^- + \mathsf{Scott} \Leftrightarrow$ the exact pictures are exactly the Scott-extensional pointed coalgebras.

A consequence is that using $\mathsf{ZFC}^- + \mathsf{Scott}$ as underlying set theory we lose the finality of the coalgebra $\langle \mathsf{V}, \mathsf{id}\rangle$: it is still true that for any coalgebra $\mathcal{A}$ there exists a morphism from $\mathcal{A}$ to $\langle \mathsf{V}, \mathsf{id}\rangle$, but unicity is lost. In Section 3.1 we will see that the Scott-axiom has a natural interpretation in the context of multisets, where the finality of the (multi-)universe can be regained.

2.1.3 Coalgebraic Theory

The above example using the functor Pow can be generalized by considering a generic endofunctor F on the category $\mathcal{C}$ of classes and class functions. The resulting general theory of coalgebras has been extensively used in Theoretical Computer Science: coalgebras are used to model automata and transition or dynamical systems, or, in the semantics of programs, final coalgebras have been used to deal with infinite data types (see e.g. [6] for useful examples).

If F is an endofunctor of $\mathcal{C}$ (that is, a functor from $\mathcal{C}$ to $\mathcal{C}$), then a **coalgebra** $\mathcal{A}$ is a pair $\langle A, e\rangle$ where A is a class and e is a class-function from A to $F(A)$ (we sometimes specify it by giving a function $e : A \to F(A)$ and use the subscript notation e_a instead of $e(a)$). The coalgebra $\mathcal{A}$ is **small** if the domain A is a set. Notice that $F(A)$ might still be a proper class. A **morphism** between two coalgebras $\mathcal{A} = \langle A, e\rangle$ and $\mathcal{A}' = \langle A', e'\rangle$ is a class-function f from A to A' such that the diagram on the right below is commutative.

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & F(A) \arrow[d, "F(f)"] \\ A' \arrow[r, "e'"] & F(A') \end{tikzcd}$$

An **isomorphism** of coalgebras in a bijective morphism for which the inverse is also a morphism. One can show that a bijective morphism is always an isomorphism. If $\langle A', e'\rangle$, $\langle A, e\rangle$ are coalgebras with $A' \subseteq A$, then $\langle A', e'\rangle$ is a **sub-coalgebra** of $\langle A, e\rangle$ if the injection $inj : A' \to A$ is a morphism.

Coalgebras and coalgebra-morphisms form a category that we denote by Coal_F. A coalgebra $\mathcal{P}$ is **final** if it is a terminal object in the category Coal_F, i.e. if for any coalgebra $\mathcal{A}$ there exists a unique morphism from $\mathcal{A}$ to $\mathcal{P}$.

A relation $Z \subseteq A \times A'$ is an F**-bisimulation** between coalgebras $\mathcal{A} = \langle A, e\rangle$, $\mathcal{A}' = (A', e')$ if there exists a coalgebra (Z, z) with domain Z for which the projections $\pi_1 : Z \to A, \pi_2 : Z \to A'$ are morphisms (this notion generalizes the classical notion of bisimulation between graphs, which is obtained for $F = \mathsf{Pow}$). An F-bisimulation of a coalgebra $\mathcal{A}$ is defined as an F-bisimulation between $\mathcal{A}$ and itself. Since F-bisimulations are closed under unions, in any coalgebra $\mathcal{A}$ the relation $\sim_{\mathcal{A}} = \cup\{Z : Z \text{ is an } F\text{-bis. on } \mathcal{A}\}$ is the maximal F-bisimulation on $\mathcal{A}$. The relation $\sim_{\mathcal{A}}$ is an equivalence relation on $\mathcal{A}$. A **pointed coalgebra** is a pair $\langle \mathcal{A}, a\rangle$, where $\mathcal{A} = \langle A, e\rangle$ is a coalgebra and $a \in A$. A morphism of pointed coalgebras $\langle \mathcal{A}, a\rangle$, $\langle \mathcal{A}', a'\rangle$ is a morphism f from $\mathcal{A}$ to $\mathcal{A}'$ with $f(a) = a'$. An F-bisimulation of pointed coalgebras $\langle \mathcal{A}, a\rangle$, $\langle \mathcal{A}', a'\rangle$ is an F-bisimulation of $\mathcal{A}, \mathcal{A}'$ such that $\langle a, a'\rangle \in Z$. The **collapse** $\overline{\mathcal{A}} = \langle \overline{A}, \overline{e}\rangle$ of a small F-coalgebra $\mathcal{A} = \langle A, e\rangle$ is defined as follows. Its domain $\overline{A}$ is the set of the equivalence classes of A modulo the maximal F-bisimulation on A. The class-function $\overline{e}$ is defined by $\overline{e}([a]) = F(\pi)(e(a))$, where π is the canonical projection from A to

A^*. One can show that $\overline{e}$ is well-defined on $\overline{A}$ and the projection π is a surjective morphism.

All the above properties hold for a generic endofunctor F. However, there are useful statements like: there exists a final coalgebra, the kernel of a morphism is a bisimulation equivalence, the composition of two bisimulations is again a bisimulation, and the greatest fixed point of F is a final coalgebra, which cannot be proved without assuming additional properties of the endofunctor F.

2.1.4 Four Important Properties

A functor F is **set-based** if for all classes C and all $a \in F(C)$, there is some *set* $c \subseteq C$ and some $a_0 \in F(c)$ such that $a = (Fi)a_0$, where i is the inclusion of c in C. It is possible to prove that any set-based functor has a final coalgebra (see [2]).

A functor F is **standard**, if whenever $f : A \to B$ is an inclusion, then $F(f) : F(A) \to F(B)$ is also an inclusion. If a functor F is standard then it is monotone as an operator on classes and hence it has a greatest fixed point F^*. Since $F(F^*) = F^*$, the pair $\mathcal{F}^* = (F^*, \mathsf{id})$ is a coalgebra, where id is the identity function on F^*. We call it the **greatest fixed point coalgebra** (g.f.p. coalgebra, for short).

The third property we mention here regards the preservation of certain commutative diagrams. A commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

is a **weak pullback square** if whenever we have two functions $i : X \to B$, $j : X \to C$ with $v \circ i = g \circ j$, there exists a (not necessarily unique) function $l : X \to A$ with $u \circ l = j$, $f \circ l = i$. Under the axiom of choice, this is equivalent to: for any $b \in B$ and $c \in C$ such that $v(b) = g(c)$, we can find an $a \in A$ with $f(a) = b, u(a) = c$.

A functor F **preserves weak pullbacks** if the image of every weak pullback square is itself a weak pullback square:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "F(u)"'] & F(B) \arrow[d, "F(v)"] \\ F(C) \arrow[r, "F(g)"] & F(D) \end{tikzcd}$$

Standardness, set-basedness, and preservation of weak pullbacks represent the minimal properties we require of a functor F to have a well-behaved coalgebraic theory. Functors satisfying these properties are called *well-behaved* in the following; well-behaved functors have final coalgebras and F-bisimulation, morphisms, and final maps are related as follows.

Proposition 2.1 *If F is a standard, set based functor that preserves weak pullbacks, then the following hold.*

1. *If $f : \langle A_1, e_1\rangle \to \langle A, e\rangle$, $g : \langle A_2, e_2\rangle \to \langle A, e\rangle$ are morphisms then the pullback of f and g, that is, the relation $P = \{\langle a, b\rangle \in A_1 \times A_2 : f(a) = g(b)\}$ is an F-bisimulation between $\langle A_1, e_1\rangle$ and $\langle A_2, e_2\rangle$. In particular, if f is a morphism from $\langle A, e\rangle$ to $\langle A', e'\rangle$, then the kernel of f, that is, the relation $R = \{\langle a, b\rangle \in A \times A : f(a) = f(b)\}$, is an F-bisimulation of the coalgebra $\langle A, e\rangle$. Conversely, any bisimulation equivalence is the kernel of a morphism.*

2. *Final coalgebras exist in Coal_F. If $\mathcal{P}$ is a final coalgebra, $\mathcal{A}$ is a coalgebra, and $s : \mathcal{A} \to \mathcal{P}$ is the unique morphism from $\mathcal{A}$ to $\mathcal{P}$, then for all $a, a' \in \mathcal{A}$ it holds*
$$a \sim_{\mathcal{A}} a' \Leftrightarrow s(a) = s(a').$$
In particular, the maximal bisimulation on a final coalgebra $\mathcal{P}$ is the diagonal $\Delta_{\mathcal{P}} = \{\langle a, a\rangle : a \in \mathcal{P}\}$.

One can easily see that if $\mathcal{P} = (P, \pi)$ is final then π is an isomorphism between $\mathcal{P}$ and $F(\mathcal{P})$. We would like to strengthen this property by asking that the map π is the identity map, as it is the case in the greatest fixed point coalgebra $\langle F^*, \mathsf{id}\rangle$, because this would greatly simplify calculations with the elements of the final coalgebra. Unfortunately, there exist well-behaved functors for which the greatest fixed point coalgebra is not a final coalgebra. This is the reason we look for another property implying the finality of $\langle F^*, \mathsf{id}\rangle$. To describe this property we need to construct a universe of sets *on top* of a class of indeterminates X, i.e. a universe in which the elements of X are considered as *atoms*. This can be done by considering the functor $\mathsf{Pow}(X + -)$, sending a class A to the class $\mathsf{Pow}(X + A)$, where $X + A$ is the disjoint union of X and A (which we represent by $X + A = \{0\} \times X \cup \{1\} \times A$). We denote the greatest fixed point of $\mathsf{Pow}(X + -)$ by V_X. Given a function $f : X \to \mathsf{V}$, there exists a unique function $\hat{f} : \mathsf{V}_X \to \mathsf{V}$ such that, for every $v \in \mathsf{V}_X$,
$$\hat{f}(v) = \{f(x) : \langle\langle 0, x\rangle \in v\} \cup \{\hat{f}(v') : \langle 1, v'\rangle \in v\}.$$
In other words, the function $\hat{f}$ is obtained by *shifting* f *inside* v as long as one reaches an atom in X. This can be proved by recursion in ZFC, while in $\mathsf{ZFC}^- + \mathsf{AFA}$ it is known as the Substitution Lemma (see [1]).

Definition 2.2 *A functor F is* uniform on maps *(see [1, 9]) if for every class A there exists a map $\phi_A : F(A) \to \mathsf{V}_A$ such that for every function $f : A \to \mathsf{V}$ the following diagram commutes:*

$$\begin{array}{ccc} F(A) & \xrightarrow{\phi_A} & \mathsf{V}_A \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle \hat{f}} \\ F(\mathsf{V}) & \xrightarrow{\iota} & \mathsf{V} \end{array}$$

where ι denotes the injection of $F(\mathsf{V})$ into V.

Working in $\mathsf{ZFC}^- + \mathsf{AFA}$ it is possible to prove that if F is uniform on maps then $\langle F^*, \mathsf{id}\rangle$ is a final coalgebra.

In order to compare coalgebras of different functors we use natural transformations. A natural transformation ν from F to G consists of a family of functions $\{\nu_A\}_{A\in\mathcal{C}}$ where $\nu_A : F(A) \to G(A)$, such that, for any function $f : A \to B$, the following diagram is commutative.

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle \nu_A}\downarrow & & \downarrow{\scriptstyle \nu_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

The natural transformation ν induces a map from the F-coalgebras to the G-coalgebras that preserves bisimulation. This function sends the F-coalgebra $\mathcal{A} = \langle A, e\rangle$ to the G-coalgebra $\nu(\mathcal{A}) = \langle A, \nu_A \circ e\rangle$; it is easy to see that an F-bisimulation Z on $\mathcal{A}$ is a G-bisimulation of $\nu(\mathcal{A})$ (see [6]). If ν is a natural isomorphism (that is, every ν_A is a bijection), the converse is also true and moreover:

Proposition 2.3 *Suppose F, G are standard, set based functors that preserve weak pullbacks and suppose ν is natural isomorphism from F to G. Then ν induces a functor (still denoted by ν) between the category Coal_F of F-coalgebras and the category of Coal_G of G-coalgebras, which is faithful on bisimulations and final coalgebras.*

2.2 Multigraphs and Multigraph Notation

In this paper, a **multigraph** is like a directed graph, but we allow a pair of nodes to be linked by more than one arrow. Formally, a multigraph can be described as a pair $\langle A, \rho\rangle$ where A is a *set* and ρ is a partial *multiplicity function* $\rho : A \times A \to \mathsf{Card}^+$. If $a \in A$ we denote by $\mathsf{Succ}(a)$ the set $\{b \in A : \rho(a, b) \geq 1\}$.

A **pointed multigraph** is a pair $\langle \mathcal{A}, a\rangle$, where $\mathcal{A} = \langle A, \rho\rangle$ is a multigraph and $a \in A$. A pointed multigraph $\langle \mathcal{A}, a\rangle$ is **rooted** if for each $a' \in A$ there exists a finite sequence $a_0 = a, \ldots, a_n = a'$ with $a_{i+1} \in \mathsf{Succ}(a_i)$, for all $i = 1, \ldots, n-1$.

In this paper we consider two different notions of multigraph homomorphisms, the sup- and sum-homomorphisms. A sup-**homomorphism** between multigraphs $\langle A, \rho\rangle$ and $\langle A', \rho'\rangle$ is a function $f : A \to A'$ with $\rho(a, b) \leq \rho'(fa, fb)$, where $\uparrow$ (i.e. the 'value' *undefined*) is to be considered as smaller than any positive cardinals. We use this convention throughout the paper. Equivalently, a sup-**homomorphism** is a function f such that, if $a, b \in A$, then

$$\mathsf{sup}\{\rho(a, c) : c \in f^{-1}(fb)\} \leq \rho'(fa, fb).$$

A sum-**homomorphism** between multigraphs $\langle A, \rho\rangle$ and $\langle A', \rho'\rangle$ is a function $f : A \to A'$ with $\sum_{c\in f^{-1}(fb)} \rho(a, c) \leq \rho'(fa, fb)$. An **isomorphism** between $\langle A, \rho\rangle$ and $\langle A', \rho'\rangle$ is a bijective function $f : A \to A'$ with $\rho(a, b) = \rho'(fa, fb)$. It is easy to see that our isomorphisms are both precisely the isomorphisms of the sup-category and precisely the isomorphisms of the sum-category. A multigraph $\langle A, \rho\rangle$ is represented by a picture where nodes $a, b \in A$ are linked with $\rho(a, b)$ arrows (counting $\uparrow$ as zero). A morphism (isomorphism) of the pointed coalgebras $\langle \mathcal{A}, a\rangle$, $\langle \mathcal{A}', a'\rangle$ is a morphism (isomorphism) f between the multigraphs $\mathcal{A}, \mathcal{A}'$ with $f(a) = a'$.

3 Multisets of Exclusive and Overlapping Types

In this section we apply the general theory of coalgebras to present our universes of *circular multisets*. The idea is to define two different well-behaved functors which can be used to model, via their greatest fixed point coalgebra, the non-wellfounded multisets of exclusive and overlapping types described in the introduction. This requires a definition of the functors on class and class-functions. We shall see that the simpler definition that comes to mind for defining the functors on objects is not adequate, because the g.f.p. coalgebra is not a final coalgebra. To solve the problem, we use more elaborated definitions on objects, so that the resulting functors are well-behaved *and* uniform on maps. This will imply the finality of the g.f.p. coalgebras for both functors.

Once our universes of multisets are correctly defined, we look at the unisets inside the two multiuniverses, that is, at those multisets that hereditarily contain only elements with at most multiplicity one. We will easily prove in Section 3.2 that the unisets inside the multiuniverse of overlapping types are AFA-sets. The same question for the multiuniverse of exclusive types is postponed until Section 4, where an anwer is obtained as a corollary of a deeper study on the role of trees inside the category of Γ-coalgebras. By using trees we prove that the unisets inside the multiuniverse of exclusive types are Scott's sets.

3.1 Multisets and Sums

We first construct a multiuniverse based on the idea of exclusive types: a function $f : A \to B$ transforms an A-multiset α in a B-multiset $f[\alpha]$, where the multiplicity of an element $y = fx$ in $f[\alpha]$ is the sum of all the multiplicities in A of the elements in $f^{-1}(y)$. E.g. if $A = \{a, b\}, B = \{c\}, f(a) = f(b) = c$, and $\alpha = [\![a, a, b]\!]$, then $f[\alpha] = [\![c, c, c]\!]$. Following this idea, we define an endofunctor $\breve{\Gamma}$ on the category $\mathcal{C}$ of classes and class-functions: for each class A, $\breve{\Gamma}(A)$ represents the class of all A-multisets and for each function $f : A \to B$, $\breve{\Gamma}(f)(\alpha)$ is the B-multiset $f[\alpha]$ as above. The natural choice for representing the A-multisets is to consider $\breve{\Gamma}(A)$ as the class of all partial functions from A to positive cardinals.

Definition 3.1 The $\breve{\Gamma}$-functor
Let $\mathcal{C}$ be the category of classes and class-functions. The endofunctor $\breve{\Gamma}$ on $\mathcal{C}$ is defined as follows.

- *if $A \in \mathcal{C}$, then*

$$\breve{\Gamma}(A) := \{\alpha : \alpha \textit{ is a small function}, \mathsf{dom}(\alpha) \subseteq A, \mathsf{range}(\alpha) \subseteq \mathsf{Card}^+\}.$$

- *If $f : A \to B$, then $\breve{\Gamma}(f) : \breve{\Gamma}(A) \to \breve{\Gamma}(B)$ is defined by*

$$\breve{\Gamma}(f)(\alpha) := \{\langle fx, \sum_{fx'=fx} \alpha(x')\rangle : x \in \mathsf{dom}(\alpha)\},$$

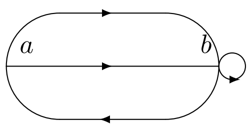
(with the convention that an empty sum counts as $\uparrow$ and that $\uparrow + \ k \ = \ k$, for any positive cardinal k).

Notice that the notion of a small $\breve{\Gamma}$-coalgebra can be identified with that of a multigraph: the multigraph corresponding to the $\breve{\Gamma}$-coalgebra $e : A \to \breve{\Gamma}(A)$ has the same domain A and multiplicity function equal to $\rho(a, a') := e_a(a')$. In view of this correspondence, when considering a $\breve{\Gamma}$-coalgebra we will use indifferently the coalgebraic or the multigraph notation.

A function $h : A \to A'$ is a $\breve{\Gamma}$-morphism between the small $\breve{\Gamma}$-coalgebras $\langle A, e\rangle$ and $\langle A', e'\rangle$ iff (using the multigraph notation): $\rho'(ha, hb) = \sum\{\rho(a, b') : h(b') = h(b)\}$. By comparing the notion of coalgebraic morphism with the one of homomorphism between multigraphs we see that coalgebraic morphisms are the sum-multigraph homomorphisms for which the value of $\rho'(ha, hb)$ is the smallest possible. Notice that the classes of $\breve{\Gamma}$-isomorphisms and multigraph isomorphisms coincide.

It is possible to prove that the functor $\breve{\Gamma}$ is well-behaved and hence it has, *up to a certain point*, a good coalgebraic theory. However, its greatest fixed point coalgebra is not a final coalgebra, assuming ZFC or even $ZFC^- + AFA$. This is easy to see in the case of ZFC because there cannot be any morphism from a loop to $\langle\breve{\Gamma}^*, \mathsf{id}\rangle$, while in the case of $ZFC^- + AFA$ we can prove that uniqueness is lost, as the following example shows.

Example. Consider the following $\breve{\Gamma}$-coalgebra $\mathcal{A}$ (represented by drawing the corresponding multigraph):



We prove (assuming AFA) that there are two different morphisms from $\mathcal{A}$ to $\langle \breve{\Gamma}^*, \mathsf{id}\rangle$. The first is given by considering an AFA set γ such that $\gamma = \{\langle \gamma, 2\rangle\}$. One can easily check that $\gamma \in \breve{\Gamma}^*$ and that we get a morphism $\phi : \mathcal{A} \to \langle \breve{\Gamma}^*, \mathsf{id}\rangle$ by defining $\phi(a) = \phi(b) = \gamma$. The second morphism is given by considering two AFA-sets α, β with $\alpha = \{\langle \beta, 2\rangle\}$ and $\beta = \{\langle \beta, 1\rangle, \langle \alpha, 1\rangle\}$. It is then clear that $\alpha \neq \beta$ and $\alpha, \beta \in \breve{\Gamma}^*$. We get a morphism $\psi : \mathcal{A} \to \langle \breve{\Gamma}^*, \mathsf{id}\rangle$ by putting $\psi(a) = \alpha, \psi(b) = \beta$. Notice that $\phi \neq \psi$.

The above example shows that using $\breve{\Gamma}(A)$ we are not allowed to view non-wellfounded multisets as elements of the greatest fixed point coalgebra of $\breve{\Gamma}$, although we can look at them as elements of a final coalgebra. We can solve this inconvenience by choosing a different representation of A-multisets, given by a functor Γ, wich is naturally isomorphic to $\breve{\Gamma}$, but which is uniform on maps. The natural isomorphism between Γ and $\breve{\Gamma}$ provides a bijective correspondence between Γ-coalgebras and $\breve{\Gamma}$-coalgebras that preserves bisimulation and final coalgebras. Moreover, since Γ is uniform on maps, the g.f.p. coalgebra $\langle \Gamma^*, \mathsf{id}\rangle$ is final and we can represent non-wellfounded multisets using its elements. By moving from $\breve{\Gamma}$ to Γ we maintain all good properties of $\breve{\Gamma}$, but we also acquire the possibility of working with the greatest fixed point coalgebra instead of working just with a generic final coalgebra.

It turns out to be convenient to introduce an auxiliary functor Γ_0 before we give Γ. The elements of $\Gamma(A)$ will be equivalence classes of elements of $\Gamma_0(A)$. We will see that Γ_0-coalgebras are quite useful in studying Γ-coalgebras. We take $\Gamma_0(A)$ to be the set of *numbered multisets* of elements of A. This means two things: (i) we do not yet abstract away from the individuality of the underlying items of the multisets and (ii) we 'normalize' the item sets to cardinals.[1] The second move is just a convenient trick to make sure that the equivalence classes leading to Γ will be sets. Here is the definition of Γ_0.

Definition 3.2 The Γ_0-multiset functor
Let $\mathcal{C}$ be the category of classes and class functions. The endofunctor Γ_0 on $\mathcal{C}$ is defined by:

- *if $A \in \mathcal{C}$ then $\Gamma_0(A) = \{l : l \text{ is a function}, \ \mathsf{dom}(l) \in \mathsf{Card}, \ \mathsf{range}(l) \subseteq A\}$;*
- *if $f : A \to B$, then $\Gamma_0(f)$ from $\Gamma_0(A)$ to $\Gamma_0(B)$ is defined by $\Gamma_0(f)(l) := f \circ l$.*

[1] We employ the usual representation of cardinals as initial ordinals. Thus, a cardinal κ is the set of ordinals of cardinality strictly less than κ.

We call the coalgebras of Γ_0 *numbered multigraphs*, and denote them with letters $\mathfrak{A}, \mathfrak{A}', \mathfrak{B} \ldots$. By definition, a Γ_0-morphism between Γ_0-coalgebras $\mathfrak{A} = \langle A, e \rangle$ and $\mathfrak{A}' = \langle A', e' \rangle$ is a function $h : A \to A'$ such that $e'_{ha} = h \circ e_a$.

In order to go from numbered A-multisets to A-multisets, we consider an equivalence relation on Γ_0 whose classes are defined as follows: if $l \in \Gamma_0$, then

$$[l] = \{l' \in A^{\mathsf{dom}(l)} : \text{ there exists a permutation } p \text{ on } \mathsf{dom}(l) \text{ with } l' = l \circ p\}.$$

In this way we abstract away from the individuality of the underlying items of the multisets: the only relevant information left is how many items of a certain types belongs to the given multiset.

Definition 3.3 The Γ-multiset functor
Let $\mathcal{C}$ be the category of classes and class functions. The endofunctor Γ on $\mathcal{C}$ is defined by:

- *if $A \in \mathcal{C}$, then $\Gamma(A) = \{[l] : l \in \Gamma_0(A)\}$;*
- *if $f : A \to B$, then $\Gamma(f) : \Gamma(A) \to \Gamma(B)$ is defined by $\Gamma(f)([l]) := [f \circ l]$.*

Notice that the definition of Γ on functions does not depend on the representative l of the A-multiset $[l]$, and that the multiplicity $m_{\Gamma(f)([l])}(fx)$ of fx as an element of the B-multiset $\Gamma(f)([l])$ is given by $\sum_{fx'=fx} m_{[l]}(x')$. Hence the definition of the operator Γ on functions corresponds to the idea presented in the introduction that the action of a function on a multiset is obtained by summing the multiplicity of all elements having the same image.

There exists a natural transformation $[\cdot]$ from Γ_0 to Γ sending $l \in \Gamma_0(A)$ to $[l] \in \Gamma(A)$. This natural transformation yields a functor —par abus de langage again $[\cdot]$— from numbered multigraphs to multigraphs.

Γ-coalgebras are denoted with the letters $\mathcal{A}, \mathcal{A}', \mathcal{B} \ldots$. Given a Γ_0-coalgebra $\mathfrak{A} = \langle A, e \rangle$ we can consider the corresponding *unnumbered* version, which is a Γ-coalgebra and is denoted by $[\mathfrak{A}]$:

$$[\mathfrak{A}] = \langle A, [e] \rangle,$$

where $[e](a) = [e(a)]$, for all $a \in A$. If $h : A \to B$ is a Γ_0-morphism from the Γ_0-coalgebra $\mathfrak{A} = \langle A, e \rangle$ to $\mathfrak{B} = \langle B, f \rangle$, then it is easily seen that h is also a Γ-morphism between $[\mathfrak{A}]$ and $[\mathfrak{B}]$. The converse is not generally true, but it is possible to *tune* the Γ_0-coalgebra $\mathfrak{A}$ so that h becomes a Γ_0-morphism as well. This is stated in the following lemma, which will be used to transfer results from the category of Γ_0-coalgebras to the category of Γ-coalgebras.

Lemma 3.4 *Let $\mathfrak{A} = \langle A, e \rangle$ and $\mathfrak{B} = \langle B, f \rangle$ be Γ_0-coalgebras. If h is a Γ-morphism between $[\mathfrak{A}]$ and $[\mathfrak{B}]$, then there exists a Γ_0-coalgebra $\mathfrak{C}$ with $[\mathfrak{C}] = [\mathfrak{A}]$ such that h is a Γ_0-morphism between $\mathfrak{C}$ and $\mathfrak{B}$.*

Proof.
Since h is a Γ-morphism we know that $[f_{ha}] = [h \circ e_a]$. For any $a \in A$, fix a

permuatation p_a on $\mathsf{dom}(f_{ha}) = \mathsf{dom}(e_a)$ such that $f_{ha} = h \circ e_a \circ p_a$. Then the Γ_0-coalgebra $\mathfrak{C} = \langle A, e \star p \rangle$, where $(e \star p)(a) = e_a \circ p_a$, is such that $[\mathfrak{C}] = [\mathfrak{A}]$. From $f_{ha} = h \circ e_a \circ p_a$, we see that the function h is a Γ_0-morphism between $\mathfrak{C}$ and $\mathfrak{B}$. $\square$

To apply the general theory of coalgebras to our functors Γ_0 and Γ we prove that they are well-behaved functors:

Lemma 3.5 *The endofunctors Γ_0 and Γ are standard, set-based, preserve weak-pullbacks, and are uniform on maps.*

Proof.
We only prove that Γ_0, Γ preserve weak pullbacks, leaving the rest to the reader. Consider first Γ_0. Suppose we have a weak pullback

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

To prove that

$$\begin{tikzcd} \Gamma_0(A) \arrow[r, "\Gamma_0(f)"] \arrow[d, "\Gamma_0(u)"'] & \Gamma_0(B) \arrow[d, "\Gamma_0(v)"] \\ \Gamma_0(C) \arrow[r, "\Gamma_0(g)"] & \Gamma_0(D) \end{tikzcd}$$

is also a weak pullback, suppose that β, γ are such that $\Gamma_0(v)(\beta) = \Gamma_0(g)(\gamma)$, where $\beta \in \Gamma_0(B)$ and $\gamma \in \Gamma_0(C)$. In particular, $\mathsf{dom}(\beta) = \mathsf{dom}(\gamma)$. We have to find a $\alpha \in \Gamma_0(A)$ with $\Gamma_0(f)(\alpha) = \beta$ and $\Gamma_0(u)(\alpha) = \gamma$. This can be done as follows. Fix a $d \in D$. Since the first diagram is a weak pullback square, by the axiom of choice for any pair $\langle b, c \rangle \in v^{-1}(d) \times g^{-1}(d)$ we can select an element $a_{b,c} \in A$ with $f(a_{b,c}) = b, u(a_{b,c}) = c$. Since $\Gamma_0(v)(\beta) = \Gamma_0(g)(\gamma)$, we have $\langle \beta(k), \gamma(k) \rangle \in v^{-1}(d) \times g^{-1}(d)$, for $d = v(\beta(k)) = u(\gamma(k))$. Define $\alpha(k) = a_{\beta k, \gamma k}$. Then $\Gamma_0(f)(\alpha)(k) = f \circ \alpha(k) = f(a_{\beta k, \gamma k}) = \beta k$, while $\Gamma_0(u)(\alpha)(k) = u \circ \alpha(k) = u(a_{\beta k, \gamma k}) = \gamma k$.

Consider now the corresponding diagram for Γ and suppose that $[\beta], [\gamma]$ are such that $\Gamma(v)[\beta] = \Gamma(g)[\gamma]$, where $\beta \in \Gamma_0(B)$ and $\gamma \in \Gamma_0(C)$. It follows that $[v \circ \beta] = [g \circ \gamma]$ and hence that, for some permutation p of $\mathsf{dom}(\beta)$, $v \circ \beta \circ p = g \circ \gamma$. Let $\beta' := \beta \circ p$. We find: $\Gamma_0(v)(\beta') = \Gamma_0(g)(\gamma)$. By our previous result, there is an $\alpha \in \Gamma_0(A)$ such that $\Gamma_0(f)(\alpha) = \beta'$ and $\Gamma_0(u)(\alpha) = \gamma$. Then $\Gamma(f)[\alpha] = [\beta'] = [\beta]$ and $\Gamma(u)[\alpha] = [\gamma]$. $\square$

Since the functor Γ is uniform on maps, assuming $ZFC^- + AFA$ we know that the coalgebra $\langle \Gamma^*, \mathsf{id} \rangle$ is a final coalgebra. We can then consider Γ^* as the domain of a multiuniverse of exclusive types (or Γ-multiuniverse), with relations $\in_k$ defined by: $x \in_k y$ iff $|l^{-1}(x)| \geq k$ for a function l with $y = [l]$. We can define the **Γ-unisets** as the Γ-multisets x such that whenever we have a descending sequence of multi-memberships $x_n \in_{k_n} x_{n-1} \in_{k_{n-1}} \ldots x_1 \in_{k_1} x$, then $k_n = k_{n-1} = \ldots = k_1 = 1$. In Section 4 we shall investigate this new multiuniverse and prove that the unisets inside it are, modulo isomorphism, the standard model of the Scott-universe. This will justify the new name of Scott-multisets for Γ-multisets.

3.2 Multisets and Sups

We now present our second universe of *circular multisets*, taking the overlapping types point of view: a function $f : A \to B$ transforms an A-multiset α in a B-multiset $f[\alpha]$, where the multiplicity of an element $y = fx$ in $f[\alpha]$ is the sup of all the multiplicities in A of the elements in $f^{-1}(y)$. E.g. if $A = \{a, b\}, B = \{c\}, f(a) = f(b) = c$, and $\alpha = [\![a, a, b]\!]$, then $f[\alpha] = [\![c, c]\!]$. Following this idea, we define an endofunctor $\check{\Delta}$ on the category $\mathcal{C}$ of classes and class-functions: for each class A, $\check{\Delta}(A)$ represents the class of all A-multisets and for each function $f : A \to B$, $\check{\Delta}(f)(\alpha)$ is the B-multiset $f[\alpha]$ as above.

Definition 3.6 The $\check{\Delta}$-functor
Let $\mathcal{C}$ be the category of classes and class-functions. The endofunctor $\check{\Delta}$ on $\mathcal{C}$ is defined as follows.

- *if $A \in \mathcal{C}$, then*

$$\check{\Delta}(A) := \{\alpha : \alpha \textit{ is a small function}, \mathsf{dom}(\alpha) \subseteq A, \mathsf{range}(\alpha) \subseteq \mathsf{Card}^{+}\}.$$

- *If $f : A \to B$, then $\check{\Delta}(f) : \check{\Delta}(A) \to \check{\Delta}(B)$ is defined by*

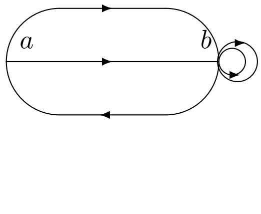
$$\check{\Delta}(f)(\alpha) := \{\langle fx, \mathsf{sup}\{\alpha(x') : fx' = fx\}\rangle : x \in \mathsf{dom}(\alpha)\},$$

(with the convention that $\uparrow$ is smaller than any positive cardinal k).

Notice that the notion of a small $\check{\Delta}$-coalgebra can be identified with that of a multigraph: the multigraph corresponding to the $\check{\Delta}$-coalgebra $e : A \to \check{\Delta}(A)$ has the same domain A and multiplicity function equal to $\rho(a, a') := e_a(a')$. In view of this correspondence, when considering a $\check{\Delta}$-coalgebra we will use indifferently the coalgebraic or the multigraph notation. A function $h : A \to A'$ is a $\check{\Delta}$-morphism between the small $\check{\Delta}$-coalgebras $\langle A, e\rangle$ and $\langle A', e'\rangle$ iff (using the multigraph notation): $\rho'(ha, hb) = \mathsf{sup}\{\rho(a, b') : h(b') = h(b)\}$. Hence coalgebraic morphisms are the sup-multigraph homomorphisms for which the value of $\rho'(ha, hb)$ is the smallest possible. Notice that the classes of $\check{\Delta}$-isomorphisms and multigraph isomorphisms coincide.

To define $\check{\Delta}(A)$ we used the natural representation of A-multisets as functions from A to Card^{+}, but as in the case of exclusive types we can prove that the g.f.p. coalgebra of $\check{\Delta}$ is not final: as before, this is easy to see if we assume ZFC, while the following example proves the same in the case of $ZFC^{-} + AFA$.

Example. Consider the following $\check{\Delta}$-coalgebra $\mathcal{A}$ (represented by drawning the corresponding multigraph):



Assuming AFA we can prove that there are two different morphisms from $\mathcal{A}$ to $\langle \breve{\Delta}^*, \mathsf{id} \rangle$. The first is given by considering an AFA set γ such that $\gamma = \{\langle \gamma, 2 \rangle\}$. One can easily check that $\gamma \in \breve{\Delta}^*$ and that we get a morphism $\phi : \mathcal{A} \to \breve{\Delta}^*$ by defining $\phi(a) = \phi(b) = \gamma$. The second morphism is given by considering two AFA-sets α, β with $\alpha = \{\langle \beta, 2 \rangle\}$ and $\beta = \{\langle \beta, 2 \rangle, \langle \alpha, 1 \rangle\}$. It is then clear that $\alpha \neq \beta$ and hence that $\alpha, \beta \in \breve{\Delta}^*$. We get a morphism $\psi : \mathcal{A} \to \langle \breve{\Delta}^*, \mathsf{id} \rangle$ by putting $\psi(a) = \alpha, \psi(b) = \beta$. Notice that $\phi \neq \psi$.

Hence, using $\breve{\Delta}$ we are not allowed to view non-wellfounded multisets as elements of the greatest fixed point coalgebra of $\breve{\Delta}$, although we can look at them as elements of a final coalgebra. As in the case of exclusive types we will solve this problem by defining a functor Δ which is naturally isomophic to $\breve{\Delta}$ and it is uniform on maps, but first we prove:

Lemma 3.7 *The endofunctor $\breve{\Delta}$ is standard, set-based, and preserves weak-pullbacks.*

We only prove that $\breve{\Delta}$ preserves weak pullbacks. Suppose we have a weak pullback square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ C \arrow[r, "g"] & D \end{tikzcd} \quad \text{To prove that} \quad \begin{tikzcd} \breve{\Delta}(A) \arrow[r, "\breve{\Delta}(f)"] \arrow[d, "\breve{\Delta}(u)"'] & \breve{\Delta}(B) \arrow[d, "\breve{\Delta}(v)"] \\ \breve{\Delta}(C) \arrow[r, "\breve{\Delta}(g)"] & \breve{\Delta}(D) \end{tikzcd}$$

is also a weak pullback, suppose that β, γ are such that $\breve{\Delta}(v)(\beta) = \breve{\Delta}(g)(\gamma)$, where $\beta \in \breve{\Delta}(B)$ and $\gamma \in \breve{\Delta}(C)$. We have to find an A-multiset α with $\breve{\Delta}(f)(\alpha) = \beta$ and $\breve{\Delta}(u)(\alpha) = \gamma$. We define:

$$\alpha(x) = \begin{cases} \mathsf{min}\{\beta(fx), \gamma(ux)\} & \text{if } v(fx) = g(ux); \\ \uparrow & \text{otherwise} \end{cases}$$

Using the definition of the functor $\breve{\Delta}$ on functions one can easily show that $\breve{\Delta}(f)(\alpha) = \beta$ and $\breve{\Delta}(u)(\alpha) = \gamma$. $\Box$

Notice that if $\mathcal{A} = \langle A, e \rangle$ is a graph (i.e. if $e_a(a') = 1$, whenever defined), then the concepts of $\breve{\Delta}$-bisimulation and $\breve{\Delta}$-collapse on $\mathcal{A}$ coincide with the classical notions of bisimulation and collapse of $\mathcal{A}$ as a graph.

Returning to the problem of finality of the g.f.p. coalgebra, we now define a functor Δ which is naturally isomorphic to $\breve{\Delta}$ and it is uniform on maps.

Definition 3.8 The Δ functor
Let $\mathcal{C}$ be the category of classes and class-functions. The endofunctor Δ on the category $\mathcal{C}$ of classes and class functions is defined as follows.

- *if $A \in \mathcal{C}$ then a Δ-multiset on A (i.e. an element of $\Delta(A)$) is a small relation $r \subseteq A \times \mathsf{Ord}$ with the property: if $a \in A$ and h, k are ordinals such that the cardinality of h is smaller or equal than the cardinality of k, then $ark \Rightarrow arh$.*
- *If $f : A \to B$, then $\Delta(f) : \Delta(A) \to \Delta(B)$ is defined as*

$$\Delta(f)(r) := \{\langle fx, k\rangle : \langle x, k\rangle \in r\}.$$

It is easy to see that $\Delta(f)(r)$ is a Δ-multiset on B [2]. There is a natural isomorphism ν between the functors $\check{\Delta}$ and Δ which sends a function $\alpha \in \check{\Delta}(A)$ to the relation $\nu_A(\alpha) = \{\langle a, k\rangle : a \in \mathsf{dom}(\alpha), k < \alpha(a)\} \in \Delta(A)$. The advantage of the Δ functor is that its definition on functions commutes with substitution: Δ is uniform on maps (see Definition 2.2). The natural isomorphism between $\check{\Delta}$ and Δ provides a bijective correspondence between $\check{\Delta}$-coalgebras and Δ-coalgebras which preserves bisimulation and final coalgebras. Moreover, since Δ is uniform on maps, assuming AFA we know that the coalgebra $\langle \Delta^*, \mathsf{id}\rangle$ is final. By moving from $\check{\Delta}$ to Δ we acquire the possibility of working with the greatest fixed point coalgebra instead of working with a generic final coalgebra.

The results of this section allow us to give the formal definition of the multiset universe of overlapping types, that we call the **AFA-multiuniverse** (this name will have a formal justification in the next theorem). Its domain is given by the g.f.p. of the functor Δ^*, while we define $s \in_k r$ iff $\langle s, h\rangle \in r$ for each ordinal $h < k$.

The **unisets** inside the AFA-multiuniverse are defined as those AFA-multisets x such that whenever we have a descending sequence of multi-memberships $x_n \in_{k_n} x_{n-1} \in_{k_{n-1}} \dots x_1 \in_{k_1} x$, then $k_n = k_{n-1} = \dots = k_1 = 1$. As we already pointed out, the notion of Δ-bisimulation generalizes the classical notion of bisimulation on graphs, in the sense that if a multigraph $\mathcal{A}$ is a graph, then a binary relation on $\mathcal{A}$ is a Δ-bisimulation if and only if it is a bisimulation of the graph. This implies that the unisets inside the AFA-multiuniverse are AFA-sets, i.e. they are a model for the theory $\mathsf{ZFC}^- + \mathsf{AFA}$. We give a formal coalgebraic proof of this in the following theorem.

Theorem 3.9 *(Assuming $ZFC^- + AFA$)*
The unisets inside the AFA-multiuniverse are a model of ZFC^- +AFA.

Proof.
By definition, the class U of unisets can be described as:

$$U = \{x \in \Delta^* : \langle x_n, h_n\rangle \in x_{n-1}, \dots \langle x_1, h_1\rangle \in x \Rightarrow h_n = \dots = h_1 = 0\}.$$

[2] The relation between our present definition and the representation in the introduction of multisets with possibly overlapping items as equivalence classes of relations is as follows. The equivalence classes of the introduction were proper classes, so we cannot use them directly to define Δ, but it is easy to see that every equivalence class contains precisely one small relation as in the definition of Δ. Our witnessing element normalizes the range of the relation to a cardinal and maximizes the overlap of the items. The $\Delta(f)$ can be shown to correspond to the image-mappings discussed in the introduction.

Consider a new functor P, defined on classes by $\mathsf{P}(A) = \{r \in \Delta(A) : r \subseteq A \times \{0\}\}$ and on functions: if $f : A \to B$, then $\mathsf{P}(f) : \mathsf{P}(A) \to \mathsf{P}(B)$ is $\mathsf{P}(f)(r) := \{\langle f(x), 0\rangle : \langle x, 0\rangle \in r\}$. One can easily show that P is uniform on maps and hence its g.f.p. coalgebra $\langle \mathsf{P}^*, \mathsf{id}\rangle$ is a final coalgebra. Moreover, the unisets U inside the AFA-multiuniverse Δ^* are exactly given by the g.f.p. coalgebra of the functor P: one can easily check that any element of P^* belongs to U and that $U \subseteq \mathsf{P}^*(U)$. This implies $U = \mathsf{P}^*$. On the other hand, there is an obvious natural isomorphism ν between the functor P and the functor Pow: if A is a class and $r \in \mathsf{P}(A)$ we define $\nu_A(r) = \{a \in A : \langle a, 0\rangle \in r\}$. Any Pow-coalgebra $\mathcal{A} = \langle A, e\rangle$ provides an interpretation $\in^{\mathcal{A}}_{\mathsf{Pow}}$ of the membership relation on the domain A (where $a \in^{\mathcal{A}}_{\mathsf{Pow}} a'$ iff $a \in e(a)$) and this interpretation is a model of $\mathsf{ZFC}^- + \mathsf{AFA}$, if the coalgebra $\mathcal{A}$ is final. Similarly, in the domain of a P-coalgebra $\mathcal{A} = \langle A, e\rangle$ we interpret the membership relation as $a \in^{\mathcal{A}}_{\mathsf{P}} a'$ iff $\langle a, 0\rangle \in e(a')$. By Proposition 2.3 we know that the natural isomorphism ν induces a bijective correspondence between P and Pow-coalgebras in which a P-coalgebra $\mathcal{A} = \langle A, e\rangle$ is sent to the Pow-coalgebra $\nu(\mathcal{A}) = \langle A, \nu_A \circ e\rangle$; one can easily verify that for all $a, a' \in A$ it holds:

$$a \in^{\mathcal{A}}_{\mathsf{P}} a' \Leftrightarrow a \in^{\nu_{\mathcal{A}}}_{\mathsf{Pow}} a',$$

so that $\langle A, \in^{\mathcal{A}}_{\mathsf{P}}\rangle$ and $\langle A, \in^{\nu_{\mathcal{A}}}_{\mathsf{Pow}}\rangle$ are isomorphic interpretations of the language of set theory. Consider then the Pow-coalgebra $\nu_{\mathcal{P}^*} = \langle \mathsf{P}^*, \nu_{\mathsf{P}^*}\rangle$ that corresponds via ν to the g.f.p. coalgebra $\mathcal{P}^* = \langle \mathsf{P}^*, \mathsf{id}\rangle$ of the functor P. Since P is uniform on maps, $\langle \mathsf{P}^*, \mathsf{id}\rangle$ is a final P-coagebra, hence $\nu_{\mathcal{P}^*}$ is a final Pow-coalgebra and $\langle \mathsf{P}^*, \in^{\nu_{\mathcal{P}^*}}_{\mathsf{Pow}}\rangle$ is a model of $\mathsf{ZFC}^- + \mathsf{AFA}$; by the preceding discussion it follows that $\langle \mathsf{P}^*, \in^{\mathcal{P}^*}_{\mathsf{P}}\rangle$ (i.e. the unisets inside the AFA-multiuniverse) are a model of $\mathsf{ZFC}^- + \mathsf{AFA}$ as well. $\square$

4 Trees and Exclusive Types

We now return to the exclusive types multiuniverse. In Section 4.1 we prove that trees and numbered trees play a special role in the class of Γ and Γ_0-coalgebras: the embedding functor from numbered trees to numbered multigraphs has a right adjoint, the unraveling *functor*, and this adjunction is used in Section 4.2 to prove that Γ-unisets are Scott-sets.

4.1 An Adjunction

A pointed numbered multigraph $e : A \to \Gamma_0(A)$ with point a is a *rooted numbered tree* with root a if:

1. For any $a' \in A$, $e(a')$ is an injection, i.e. a numbered tree is a numbered unigraph,
2. Let's write $c \prec_1 c'$ for $e(c)(k) = c'$, for some k in the domain of $e(c)$. Then for any $a' \in A$ there is precisely one sequence $a = b_1 \prec_1 b_2 \cdots \prec_1 b_n = a'$.

The rooted numbered trees can be considered as the objects of a category $\mathcal{T}$ having as morphisms tree isomorphisms. $\mathcal{T}$ is a subcategory of the pointed numbered multigraphs (actually, a full subcategory, because the results of this section prove that a Γ_0-morphism between rooted numbered trees is always an isomorphism). Let emb be the corresponding embedding functor. We want to prove that emb has a right adjoint. To this end, we define an unraveling functor from Γ_0-coalgebras to rooted numbered trees and prove in Theorem 4.2 that it is a right adjoint of emb.

Definition 4.1 *The* unraveling functor unr *from numbered pointed multigraphs to rooted numbered trees is described as follows.*
The unraveling $\langle\mathfrak{A}, a\rangle^u := \mathsf{unr}\langle\mathfrak{A}, a\rangle$ *of a pointed* Γ_0*-coalgebra* $\mathfrak{A} = (e : A \to \Gamma_0(A))$ *with point* a *is the pointed* Γ_0*-coalgebra* $(e' : A' \to \Gamma_0(A'))$ *with as point the empty sequence* ε *and:*

- A' *is a class of sequences of ordinals* $\sigma = k_1 \cdots k_n$*. We will define by simultaneous recursion sets* A'_n *of sequences of ordinals of length* $\leq n$ *and a mapping* $\sigma \mapsto a_\sigma$ *from* A'_n *to* A*.* A' *is the union of the* A'_n*. The union of the mappings* $\sigma \mapsto a_\sigma$ *will be a mapping from* A' *to* A*.*
  - $A'_0 := \{\varepsilon\}$, $a_\varepsilon := a$,
  - $\sigma k \in A'_{i+1}$ *if* $\sigma \in A_i$ *and* $k \in \mathsf{dom}(e(a_\sigma))$, $a_{\sigma k} := e(a_\sigma)(k)$.
- $e'(\sigma)(k) = \sigma k$, *if* $\sigma k \in A'$.

The unraveling $h^u := \mathsf{unr}(h)$ *of a morphism of pointed* Γ_0*-coalgebras*
$h : \langle(e : A \to \Gamma_0 A), a\rangle \to \langle(f : B \to \Gamma_0 B), b\rangle$
is simply:
$h^u := \mathsf{id}_{A'}$.

To see that h^u is indeed a morphism of rooted numbered trees, we only need to verify that $\mathsf{unr}(\langle(e : A \to \Gamma_0 A), a\rangle) = \mathsf{unr}(\langle(f : B \to \Gamma_0 B), b\rangle)$. We prove this by induction on the length of the elements of A', proving simultaneously $b_\sigma = h(a_\sigma)$. For the case of the empty sequence we are easily done. We have

$$\begin{aligned} h(a_{\sigma k}) &= h(e(a_\sigma)(k)) \\ &= f(h(a_\sigma))(k) \\ &= f(b_\sigma)(k) \\ &= b_{\sigma k} \end{aligned}$$

Note that also $\mathsf{dom}(e(a_\sigma)) = \mathsf{dom}(f(b_\sigma))$, so $A' = B'$. So clearly unr is a functor.

We define the natural transformation $\mathsf{end} : \mathsf{emb} \circ \mathsf{unr} \to \mathsf{id}$ as follows.

$$\mathsf{end}_{\langle\mathfrak{A},a\rangle}(\sigma) := a_\sigma.$$

To see that $\mathsf{end}_{\langle\mathfrak{A},a\rangle}$ is indeed a morphism of Γ_0-coalgebras, note that:

$$e(\mathsf{end}_{\langle\mathfrak{A},a\rangle}(\sigma))(k) \quad = \quad e(a_\sigma)(k)$$

$$
\begin{aligned}
. \quad &= \mathsf{end}_{\langle\mathfrak{A},a\rangle}(\sigma k) \\
&= \mathsf{end}_{\langle\mathfrak{A},a\rangle}(e'(\sigma)(k)) \\
&= \Gamma_0(\mathsf{end}_{\langle\mathfrak{A},a\rangle})(e'(\sigma))(k)
\end{aligned}
$$

It is immediate that end is a natural transformation by the fact that if $h : \langle(e : A \to \Gamma_0 A), a\rangle \to \langle(f : B \to \Gamma_0 B), b\rangle$

is a Γ_0-morphism, then $h(a_\sigma) = b_\sigma$. In case our coalgebra is rooted rather than pointed clearly end is surjective.

Theorem 4.2 *The functor* unr *is right adjoint to the functor* emb.

Proof.
We start with the functor emb from the category of the rooted numbered trees to the category of the pointed numbered multigraphs. We assign to a pointed numbered multigraph $\langle\mathfrak{A}, a\rangle$ a rooted numbered tree $\langle\mathfrak{A}, a\rangle^u$ by unraveling as described above. We have to show that $\mathsf{end}_{\langle\mathfrak{A},a\rangle} : \mathsf{emb}(\langle\mathfrak{A}, a\rangle^u) \to \langle\mathfrak{A}, a\rangle$ is universal from emb to $\langle\mathfrak{A}, a\rangle$ (see e.g. [4]). We check the relevant universality condition. Consider any rooted numbered tree $\langle\mathcal{T}, t\rangle$ and let $h : \mathcal{T} \to \langle\mathfrak{A}, a\rangle$. We have to show that there is a unique isomorphism $h' : \mathcal{T} \to \langle\mathfrak{A}, a\rangle^u$ such that $h = \mathsf{end}_{\langle\mathfrak{A},a\rangle} \circ h'$. Let $\mathcal{T}$ be given by the Γ_0-coalgebra $\tau : T \to \Gamma_0 T$ with root t. Consider any t' in T. Let $t = t_1 \prec_1 t_2 \cdots \prec_1 t_n = t'$ be the unique path from t to t'. Let k_i be the unique ordinal so that $\tau(t_i)(k_i) = t_{i+1}$. We take $h'(t') := k_1 \cdots k_{n-1}$. Since h is a morphism and $\tau(t_i)(k_i) = t_{i+1}$ we have $e(h(t_i))(k_i) = h(t_{i+1})$, so $h'(t')$ is indeed in A'. We show that h' is a morphism:

$$
\begin{aligned}
e'(h'(t'))(k) &= e'(k_1 \cdots k_{n-1})(k) \\
&= k_1 \cdots k_{n-1} k \\
&= h'(\tau(t')(k)) \\
&= \Gamma_0(h')(\tau(t'))(k)
\end{aligned}
$$

Clearly $\mathsf{end}_{\langle\mathfrak{A},a\rangle} \circ h' = h$. The uniqueness of h' is easily shown by induction on the distance of t' from t. We leave to the reader the verification that h is one to one and onto. $\Box$

We now consider the notion of unraveling for (unnumbered) pointed multigraphs.

Definition 4.3 *The* Unraveling *of a pointed multigraph* $\langle\mathcal{A}, a\rangle = \langle\langle A, \rho\rangle, a\rangle$ *(or equivalently, of a pointed* Γ*-coalgebra) is the pointed multigraph* $\mathsf{UNR}\langle\mathcal{A}, a\rangle = (\langle\mathcal{A}, a\rangle)^U := \langle\langle A^U, \rho^U\rangle, a\rangle$ *where:*

(a) A^U *is the set of finite sequences*

$$a_0 k_1 a_1 \ldots k_n a_n,$$

where $a_i \in A$ *for* $i \in \{0, \ldots, n\}$, $a_0 = a$, *and the* k_i*'s are ordinal numbers satisfying:* $k_{i+1} < \rho(a_i, a_{i+1})$, *for all* $i \in \{0, \ldots, n-1\}$ *(in particular, if* $a_0 k_1 a_1 \ldots k_n a_n \in A^U$ *then* $\rho(a_i, a_{i+1}) \geq 1$ *and* $a_{i+1} \in \mathsf{Succ}(a_i)$, *for all* $i < n$*).*

b) The point is the sequence a.

c) $\rho^U(\sigma, \sigma k_{n+1} a_{n+1}) = 1$, *for all* $\sigma k_{n+1} a_{n+1} \in A^U$.

Let us relate this construction with the corresponding one on numbered multigraphs. We can go from rooted numbered trees to ordinary rooted trees by forgetting structure, say the functor is forget. We can go from pointed numbered multigraphs to pointed multigraphs via $[\cdot]$. We have the mapping UNR unraveling pointed multigraphs to rooted trees. Now the point is that $\mathsf{UNR}([\langle \mathfrak{A}, a\rangle])$ is isomorphic to $\mathsf{forget}(\langle \mathfrak{A}, a\rangle^u)$. So UNR can be viewed like this.

1. Start with a pointed multigraph $\langle \mathcal{A}, a\rangle$.
2. Pick a $[\cdot]$-original $\langle \mathfrak{A}, a\rangle$. This choice preprogrammes arbitrary choices in the unraveling.
3. Unravel via unr. You have a rooted numbered tree.
4. Forget structure and you have a rooted tree.
5. Modulo isomorphism this is precisely what we get via UNR.

The mapping UNR is easily defined on the objects but you cannot get it to work on the morphisms, the point being that in the absence of 'numberedness' we don't know which sequence to send to which sequence. For example, consider the multigraph $\langle A, \rho\rangle$: $A = \{a, b, c\}$ and $\rho(a, b) = \rho(a, c) = 2$; the multigraph $\langle A', \rho'\rangle$: $A' = \{a', b'\}$ and $\rho(a', b') = 4$; and the Γ-morphism h sending a to a' and b, c to b'. Considering the two unravelings, there is no intrinsic reason to send e.g. $a1b$ to $a'3b'$, etc.

As in the numbered case, we can easily prove that the function end from A^U to A defined as

$$\mathsf{end}(a_0 k_1 a_1 \ldots k_n a_n) = a_n$$

is a coalgebra morphism from the unraveling $\langle \mathcal{A}, a\rangle^U$ to $\langle \mathcal{A}, a\rangle$; in particular, the pointed Γ-coalgebra $\langle \mathcal{A}, a\rangle^U$ is Γ-bisimilar to $\langle \mathcal{A}, a\rangle$, and can be used as a representative for the Γ-bisimulation class of $\langle \mathcal{A}, a\rangle$. But we can prove more than this. Consider the preorder $\leq$, defined by $\langle \mathcal{A}, a\rangle \leq \langle \mathcal{A}', a'\rangle$ iff there exists a morphism from $\langle \mathcal{A}, a\rangle$ to $\langle \mathcal{A}', a'\rangle$. Since the quotient $\langle \overline{\mathcal{A}}, [a]\rangle$ of a rooted coalgebra $\langle \mathcal{A}, a\rangle$ modulo the maximal Γ-bisimulation is a collapsed multigraph and two rooted coalgebras are bisimilar iff their collapses are isomorphic, we easily obtain: $\langle \overline{\mathcal{A}}, [a]\rangle$ is the unique maximum (modulo isomorphism) whit respect to $\leq$ in the bisimulation class of $\langle \mathcal{A}, a\rangle$. This holds generally, for any well-behaved functor. In the case of Γ (or Γ_0) an easy consequence of the following theorem is that the order $\leq$ also have a unique minimum, given by the unraveling $\langle \mathcal{A}, a\rangle^U$ of an element $\langle \mathcal{A}, a\rangle \in X$. Hence, for these functors we have two natural representatives for a rooted coalgebra, the maximum and the minimum of $\leq$, given respectively by its collapse and its unraveling.

Notice that the existence of a minimum does not generally hold for well-behaved functors. For example, it does not hold for the functor Pow: given a

Pow-coalgebra $\mathcal{A}$ we can always find a smaller coalgebra (with respect to $\leq$) which is not isomorphic to it by using an appropriate k-unraveling of $\mathcal{A}$.

Theorem 4.4 *If $\langle \mathfrak{T}, t\rangle$ is a rooted numbered tree, then any morphism h from a rooted Γ_0-coalgebra $\langle \mathfrak{A}, a\rangle$ to $\langle \mathfrak{T}, t\rangle$ is an isomorphism; vice versa, any rooted numbered coalgebra satisfying this property is isomorphic to a rooted numbered tree. The same is true for rooted trees and rooted multigraphs.*

Proof.
We prove the numbered version and leave to the reader the corresponding proof for Γ, which is easily obtained by applying Lemma 3.4. Consider the natural transformation end from the unraveling functor unr to the identity functor. If $h : \langle \mathfrak{A}, a\rangle \to \langle \mathfrak{T}, t\rangle$ is a morphism, then, since $\mathsf{unr}(h)$ is the identity, we have $\mathsf{end}_{\langle \mathfrak{T}, t\rangle} = h \circ \mathsf{end}_{\langle \mathfrak{A}, a\rangle}$. But $\mathsf{end}_{\langle \mathfrak{T}, t\rangle}$ is an isomorphism and $\mathsf{end}_{\langle \mathfrak{A}, a\rangle}$ is surjective, hence h is a bijection. Vice versa, suppose that $\langle \mathfrak{A}, a\rangle$ is such that any morphism arriving at it is an isomorphism. Then $\mathsf{end} : \langle \mathfrak{A}, a\rangle^u \to \langle \mathfrak{A}, a\rangle$ is an isomorphism and $\langle \mathfrak{A}, a\rangle$ is isomorphic to the rooted numbered tree $\langle \mathfrak{A}, a\rangle^u$. □

4.2 Scott Bisimulation and Trees

In this section we show that the notion of Γ-bisimulation between Γ-coalgebras (that we identify with multigraphs) is the natural generalization of the notion of Scott-bisimulation between graphs (see [1]). Scott-bisimulation on graphs is described by means of the notion of unraveling, that can also be used used to characterize the maximal bisimulation on pointed graphs. The unraveling of a graph produces a rooted tree, in which every node is copied once (in the simple unraveling) or k-times for a cardinal k (in the k-unraveling). It is possible to prove that two graphs are bisimilar if and only if there exists a cardinal k such that the k-unravelings of the graphs are isomorphic. In Definition 4.3 we generalized the notion of simple unraveling to multigraphs and now in Corollary 4.5 we show that two Γ-coalgebras (i.e. two multigraphs) are Γ-bisimilar if and only if their unravelings are isomorphic. Since Scott-bisimulation (in the equivalent definition given in [1]) relates two pointed graphs exactly when their unravelings are isomorphic, we see that the notion of Γ-bisimulation is a generalization, from graphs to multigraphs, of Scott-bisimulation. We then use this result to prove that the unisets of the Γ-multiuniverse are Scott-sets.

Theorem 4.5 *If h is a Γ-morphism between two pointed Γ-coalgebras $\mathcal{A}$, $\mathcal{B}$ then $\mathcal{A}$, $\mathcal{B}$ have isomorphic unravelings. In particular, two pointed Γ-coalgebras are Γ-bisimilar if and only if they have isomorphic unravelings.*

Proof.
Consider a Γ_0-original $\mathfrak{B}$ of $\mathcal{B}$, that is: $\mathcal{B} = [\mathfrak{B}]$. From Lemma 3.4 we know that there exists a Γ_0-original $\mathfrak{A}$ of $\mathcal{A}$ such that h is a Γ_0-morphism from $\mathfrak{A}$ to $\mathfrak{B}$. But then we know that $\mathfrak{A}^u$ is *equal* to $\mathfrak{B}^u$. Since $\mathsf{UNR}(\mathcal{A})$ is isomorphic to $\mathsf{forget}(\mathfrak{A}^u)$ and $\mathsf{UNR}(\mathcal{B})$ is isomorphic to $\mathsf{forget}(\mathfrak{B}^u)$, we are done. □

Theorem 4.5 allows us to identify Γ-multisets and (isomorphism classes of) rooted trees. Define the *canonical rooted tree* $\mathcal{T}(x)$ of a Γ-multiset $x \in \Gamma^*$ as $\langle\langle\Gamma^*, \mathsf{id}\rangle, x\rangle^U$. In this way we pick exactly a rooted tree in any isomorphism class of rooted trees. The rooted trees modulo isomorphism give then an equivalent representation of multisets and using this representation we prove that the unisets inside the Γ-multiuniverse are Scott-sets. This is stated in the following theorem and to prove it we shall use the representation, given in [1], of Scott-sets by means of rooted *irredundant* trees modulo isomorphism: a tree $\mathcal{T}$ is irredundant if it has no proper automorphism, or, equivalently: for all $u \in \mathcal{T}$, $u', v' \in \mathsf{Succ}(u)$, if $\langle\mathcal{T}, u'\rangle$ is isomorphic to $\langle\mathcal{T}, v'\rangle$ then $u' = v'$.

Theorem 4.6 *(Assuming ZFC^-+AFA) The Γ-unisets are a model of* ZFC^-+ Scott.

Proof.
By the preceding discussion we only need to prove that the unraveling of a uniset $x \in \Gamma^*$ is an irredundant rooted tree and that any irredundant rooted tree is isomorphic to such an unraveling. Suppose that $\mathcal{T} = \langle\langle\Gamma^*, \mathsf{id}\rangle, x\rangle^U$ is not irredundant: then there are $u \in \mathcal{T}$, $u', v' \in \mathsf{Succ}(u)$ with $u' \neq v'$ such that $\langle\mathcal{T}, u'\rangle$ is isomorphic to $\langle\mathcal{T}, v'\rangle$. Then $\mathsf{end}(u')$ and $\mathsf{end}(v')$ are Γ-bisimilar nodes in $\langle\Gamma^*, \mathsf{id}\rangle$ and since $\langle\Gamma^*, \mathsf{id}\rangle$ is final, by Proposition 2.1 we obtain $\mathsf{end}(u') = \mathsf{end}(v')$. But u', v' were different successors of u in $\langle\langle\Gamma^*, \mathsf{id}\rangle, x\rangle^U$, hence $\mathsf{end}(u')$ can be equal to $\mathsf{end}(v')$ only if the multiplicity of $\mathsf{end}(u')$ as an element in the multiset $\mathsf{end}(u)$ is greater than one. This is a contradiction because we supposed x to be a uniset.

To prove that any irredundant rooted tree is isomorphic to $\langle\langle\Gamma^*, \mathsf{id}\rangle, x\rangle^U$, for a uniset x, we first show: if the unraveling $\mathcal{T} = \langle\mathcal{A}, a\rangle^U$ of a rooted multigraph is irredundant, then the multigraph must be a graph. Suppose not: then there are two nodes b, c of $\mathcal{A}$ with $\rho(b, c) > 1$; if $t \in \mathcal{T}$ is such that $\mathsf{end}(t) = b$, then $t0c$, $t1c$ are nodes in $\mathcal{T}$, and $\langle\mathcal{T}, t0c\rangle$, $\langle\mathcal{T}, t1c\rangle$ are isomorphic, a contradiction. Suppose then that $\langle\mathcal{T}, t\rangle$ is an irredundant rooted tree. By considering $\mathcal{T}$ as a Γ-coalgebra we can find a Γ-morphism $h : \mathcal{T} \to \langle\Gamma^*, \mathsf{id}\rangle$. By Theorem 4.5 we know that $\langle\mathcal{T}, t\rangle$ is isomorphic to $\langle\langle\Gamma^*, \mathsf{id}\rangle, h(t)\rangle^U$ and by the preceding discussion we know that $h(t)$ must be a uniset. $\square$

5 Multisets and the Logics of Graded Modalities

In this section we give a characterization of Δ and Γ-bisimulation via logic. In the set-context the appropriate logic for describing bisimulation between graphs was proved to be infinitary modal logic ([3]). In the Γ-context we shift to the *graded extension* of this logic by proving that two pointed multigraphs are Γ-bisimilar iff they satisfy the same formulae of infinitary graded modal logic. More than this, we prove that any multigraph can be characterized modulo Γ-bisimulation by a single infinitary graded modal formula, and we isolate a class of formulae that correspond to Scott-multisets. In this way we have three alternative ways for modeling Scott-multisets: as collapsed multigraphs, as trees, or as infinitary formulae. The same can be done for the functor Δ and AFA-multisets, with the difference that now the formulae of infinitary graded modal logic are interpreted in a non-standard way.

Consider the language obtained from infinitary propositional logic by adding the unary operators $\Diamond_h$, for all $h \in \mathsf{Card}^+$. More formally, we define our formulae $\mathcal{F}$ as the smallest class closed under infinitary conjunction (if $\Phi \subseteq \mathcal{F}$ is a *set* then $\bigwedge \Phi \in \mathcal{F}$), negation (if $\phi \in \mathcal{F}$ then $\neg\phi \in \mathcal{F}$), and graded diamonds (if h is a strictly positive cardinal and $\phi \in \mathcal{F}$ then $\Diamond_h \phi \in \mathcal{F}$). In the following, we denote the operator $\Diamond_1$ by the more familiar symbol $\Diamond$ and $\bigwedge \emptyset$ by $\perp$. Since we are dealing with pure multisets, our language does not contain propositional variables. However, the results of the following sections are generalizable to multisets with atoms and in this case our language would contain propositional variables.

5.1 Graded Modalities and Exclusive Types

In the case of the Γ-functor we define the *truth* of a formula ϕ of $\mathcal{F}$ in a pointed multigraph $\langle \mathcal{A}, a\rangle = \langle\langle A, \rho\rangle, a\rangle$ (or, equivalently, in a pointed Γ-coalgebra) by adding the clause below to the inductive definition of truth in infinitary propositional logic:

$$\langle \mathcal{A}, a\rangle \models \Diamond_h \phi \quad \Leftrightarrow \quad \sum_{\langle \mathcal{A}, b\rangle \models \phi} \rho(a,b) \geq h.$$

The resulting logic is denoted by $\mathcal{L}^{grad}_\infty$. If the multigraph $\mathcal{A} = \langle A, \rho\rangle$ is clear from the context, we write $a \models \phi$ instead that $\langle \mathcal{A}, a\rangle \models \phi$.

Notice that our logic $\mathcal{L}^{grad}_\infty$ coincides on pointed graphs with the well-known infinitary graded modal logic. We write $\langle \mathcal{A}, a\rangle \equiv_\Gamma \langle \mathcal{A}', a'\rangle$ (or simply $a \equiv_\Gamma a'$) if $\langle \mathcal{A}, a\rangle, \langle \mathcal{A}', a'\rangle$ are two pointed multigraphs that satisfy the same $\mathcal{L}^{grad}_\infty$-formulae. The following theorem show that $\mathcal{L}^{grad}_\infty$ is the appropriate language for characterizing Γ-bisimulation.

Theorem 5.1 *Two pointed multigraphs $\langle \mathcal{A}, a\rangle = \langle\langle A, \rho\rangle, a\rangle$ and $\langle \mathcal{A}', a'\rangle = \langle\langle A', \rho'\rangle a'\rangle$ are Γ-bisimilar if and only if they satisfy the same $\mathcal{L}^{grad}_\infty$-formulae.*

Proof.
($\Rightarrow$) By an easy induction on the complexity of $\mathcal{L}^{grad}_\infty$-formulae.
($\Leftarrow$) We prove that $\equiv_\Gamma \cap (A \times A')$ is a Γ-bisimulation between $\mathcal{A}$ and $\mathcal{A}'$. Suppose $\langle w, w'\rangle \in \equiv_\Gamma \cap (A \times A')$.

CLAIM 1. For any $v \in \mathsf{Succ}(w) \cup \mathsf{Succ}(w')$, there exists a formula $\phi_v \in \mathcal{L}^{grad}_\infty$ such that for any $z \in \mathsf{Succ}(w) \cup \mathsf{Succ}(w')$ it holds

$$(\star) \quad z \models \phi_v \Leftrightarrow z \equiv_\Gamma v.$$

Suppose $v \in \mathsf{Succ}(w) \cup \mathsf{Succ}(w')$. For any $u \in \mathsf{Succ}(w)$ with $u \not\equiv_\Gamma v$ let $\psi_u \in \mathcal{L}^{grad}_\infty$ be such that $u \not\models \psi_u$ and $v \models \psi_u$. We define

$$\phi_v := \bigwedge \psi_u,$$

and prove that ϕ_v verifies property $(\star)$ above. The direction from right to left is obvious since $\phi_v \in \mathcal{L}^{grad}_\infty$ and $v \models \phi_v$. For the other direction, suppose first that $z \in \mathsf{Succ}(w)$: if $z \not\equiv_\Gamma v$, then the conjunct ψ_z of ϕ_v is such that $z \not\models \psi_z$, and hence $z \not\models \phi_v$. If $z \in \mathsf{Succ}(w')$ and $z \models \phi_v$, we prove that for any formula $\theta \in \mathcal{L}^{grad}_\infty$, if $z \models \theta$ then $v \models \theta$. This is enough to prove that $z \equiv_\Gamma v$. If $z \models \theta$ then $w' \models \Diamond(\theta \wedge \phi_v)$, hence $w \models \Diamond(\theta \wedge \phi_v)$ and there exists $s \in \mathsf{Succ}(w)$ with $s \models \theta \wedge \phi_v$. But for $s \in \mathsf{Succ}(w)$ we already proved that $s \models \phi_v$ implies $s \equiv_\Gamma v$; hence $v \models \theta$.

Notice that the construction of the formula ϕ_v is not symmetric: the point v can be either in $\mathsf{Succ}(w)$ or in $\mathsf{Succ}(w')$, but the formula ϕ_v is in any case the conjunction of formulae ψ_u for $u \in \mathsf{Succ}(w)$. A symmetric construction is possible and even simpler, but using the asymmetric one we will be able to prove the stronger Theorem 5.2 below.

We now use the claim to prove that the equivalence relation $\equiv_\Gamma$ restricted to $\mathcal{A} \times \mathcal{A}'$ satisfies: if $\langle w, w' \rangle \in \equiv_\Gamma \cap (A \times A')$ and $c \in \mathsf{Succ}(w) \cup \mathsf{Succ}(w')$ then:

$$\sum\{\rho(w, a') : a' \equiv_\Gamma c\} = \sum\{\rho'(w', b') : b' \equiv_\Gamma c\}.$$

From this it easily follows that the relation $\equiv_\Gamma \cap (A \times A')$ is a Γ-bisimulation.

Suppose $c \in \mathsf{Succ}(w)$ and $h = \sum\{\rho(w, a') : a' \equiv_\Gamma c\}$. By property $(\star)$ above we have $w \models \Diamond_h \phi_c \wedge \neg\Diamond_{h^+} \phi_c$, where h^+ is the first cardinal greater then h. Then $w' \models \Diamond_h \phi_c \wedge \neg\Diamond_{h^+} \phi_c$, and $\sum\{\rho'(w', b') : b' \equiv_\Gamma c\} = h$.

Suppose $c \in \mathsf{Succ}(w')$. By using the claim again, we can construct a formula ϕ_c such that for $z \in \mathsf{Succ}(w) \cup \mathsf{Succ}(w')$ it holds

$$z \models \phi_c \Leftrightarrow z \equiv_\Gamma c.$$

Since $w' \models \Diamond\phi_c$, we have $w \models \Diamond\phi_c$ and there exists a $v \in \mathsf{Succ}(w)$ such that $v \equiv_\Gamma c$. Then we can reason as above, using v instead of c. $\Box$

We now show how to modify the proof of Theorem 5.1 to achieve a stronger result: any pointed multigraph can be characterized, modulo Γ-bisimulation, by a single formula in $\mathcal{L}^{grad}_\infty$. Given a cardinal h, denote by $\mathcal{L}_h$ the fragment of $\mathcal{L}^{grad}_\infty$ which is obtained by restricting infinitary conjunctions to sets of cardinality strictly smaller than h and graded diamonds to $\Diamond_k$ with $k < h$. Notice that $\mathcal{L}_h$ forms a *set* (while $\mathcal{L}^{grad}_\infty$ is a class which is not a set). Denote the relation *to satisfy the same* $\mathcal{L}_h$*-formulae* by $\equiv_h$. Given a multigraph $\mathcal{A} = \langle A, \rho \rangle$, let $h_\mathcal{A}$ be the smallest cardinal which is strictly greater then $\sum_{v \in \mathsf{Succ}(w)} \rho(w, v)$, for any $w \in A$. Notice that in the preceding proof we always used infinitary conjunctions on sets of cardinality smaller than $h^+_\mathcal{A}$ and graded diamonds $\Diamond_h$ only for $h < h^+_\mathcal{G}$. This means that a similar proof can be exploited to prove that $\equiv_{h^+_\mathcal{A}}$ is a Γ-bisimulation between $\mathcal{A}$ and *any* multigraph $\mathcal{A}'$. Consider then the formula

$$\phi_{\langle \mathcal{A}, w \rangle} = \bigwedge\{\phi \in \mathcal{L}_{h^+_\mathcal{A}} : w \models \phi\}.$$

If $\mathcal{A}'$ is a multigraph and $\langle \mathcal{A}', w' \rangle \models \phi_{\langle \mathcal{A}, w \rangle}$, then $\langle \mathcal{A}, w \rangle \equiv_{h^+_\mathcal{A}} \langle \mathcal{A}', w' \rangle$; by the previous consideration we can deduce that $\langle \mathcal{A}, w \rangle, \langle \mathcal{A}', w' \rangle$ are Γ-bisimilar. This proves:

Theorem 5.2 *For any pointed coalgebra $\langle \mathcal{A}, w\rangle$ there exists a formula $\phi_{\langle \mathcal{A}, w\rangle} \in \mathcal{L}_{\infty}^{grad}$ which characterizes $\langle \mathcal{A}, w\rangle$ modulo Γ-bisimulation, i.e., for any multigraph $\mathcal{A}'$ it holds:*

$$\langle \mathcal{A}', w'\rangle \models \phi_{\langle \mathcal{A}, w\rangle} \Leftrightarrow \langle \mathcal{A}, w\rangle \textit{ is } \Gamma\textit{-bisimilar to } \langle \mathcal{A}', w'\rangle.$$

It follows that any Γ-multiset is characterized by an infinitary graded modal formula. This result suggests another representation of the class of Γ-multisets, in which the domain of the universe is a fragment of the class of infinitary graded modal formulae. We only sketch this in the following. First, we characterize the graded formulae of type $\phi_{\langle \mathcal{A}, a\rangle}$ for a pointed multigraph $\langle \mathcal{A}, a\rangle$ (for a set-analogue, see [3]).

Definition 5.3 *Consider the preorder $\leq$ defined in $\mathcal{L}_{\infty}^{grad}$ by $\psi \leq \phi \Leftrightarrow \models \psi \to \phi$, where $\models \psi \to \phi$ stands for: any pointed multigraph that satisfies ψ, satisfies ϕ as well. Define the class $MS(\mathcal{L}_{\infty}^{grad})$ as the one containing, modulo equivalence, all satisfiable $\mathcal{L}_{\infty}^{grad}$-formulae which are minimal with respect to $\leq$ on satisfiable formulae, that is:*

$$\phi \in MS(\mathcal{L}_{\infty}^{grad})$$
$$\Updownarrow$$
$$\phi \textit{ is satisfiable and for all satisfiable } \psi \textit{ if } \psi \leq \phi \textit{ then } \psi \textit{ is equivalent to } \phi.$$

Lemma 5.4 $\phi \in MS(\mathcal{L}_{\infty}^{grad}) \Leftrightarrow \exists \langle \mathcal{A}, a\rangle$ *with* $\models (\phi_{\langle \mathcal{A}, a\rangle} \leftrightarrow \phi)$.

Then, we identify the class of Γ-multisets with the class $MS(\mathcal{L}_{\infty}^{grad})$, with $\in_k$ given by the relation $\{\langle \phi, \psi\rangle \in MS(\mathcal{L}_{\infty}^{grad}) \times MS(\mathcal{L}_{\infty}^{grad}) :\models \phi \to \Diamond_k \psi\}$.

5.2 Graded Modalities and Overlapping Types

The results of the previous section can be adapted to give a logic description of AFA-multisets, provided we change the interpretation of a formula $\Diamond_h \phi$ in a pointed multigraph $\langle\langle A, \rho\rangle, a\rangle$ (considered as a Δ-coalgebra) as follows:

$$\langle\langle A, \rho\rangle, a\rangle \models \Diamond_h \phi \quad \Leftrightarrow \quad \mathsf{sup}\{\rho(a,b) : b \models \phi\} \geq h.$$

The resulting logic is denoted by $\mathcal{L}_{\infty}^{o\text{-}grad}$. If the multigraph $\langle A, \rho\rangle$ is clear from the context, we write $a \models \phi$ instead that $\langle\langle A, \rho\rangle, a\rangle \models \phi$.

Notice that if $\langle \mathcal{A}, a\rangle$ is a pointed *graph*, then for all $a \in A$ and all $h \geq 2$ we have $a \models \neg\Diamond_2 \phi$, for all $\phi \in \mathcal{F}$. Hence, on pointed graphs the logic $\mathcal{L}_{\infty}^{o\text{-}grad}$ has the same expressive power than infinitary modal logic. We write $\langle \mathcal{A}, a\rangle \equiv_{\Delta} \langle \mathcal{A}', a'\rangle$ (or simply $a \equiv_{\Delta} a'$) if $(\langle \mathcal{A}, a\rangle, \langle \mathcal{A}', a'\rangle$ are two pointed multigraphs that satisfy the same $\mathcal{L}_{\infty}^{o\text{-}grad}$-formulae. Then all the results of the previous section transfer to this context: one only has to substitute Δ for Γ and $\mathcal{L}_{\infty}^{o\text{-}grad}$ for $\mathcal{L}_{\infty}^{grad}$.

5.3 Relations with Coalgebraic Logic.

Given a well-behaved functor F, a general method for constructing a logic characterizing F-coalgebras modulo F-bisimulation is given in [5]. There it is also shown that for certain functors, the *uniform* ones, a single formula of the logic suffices for characterizing a pointed coalgebra. In Section 3 we proved that our functors Γ and Δ are uniform and hence the results of [5] apply to our context. Moreover, one can prove that Moss logics relative to our functors are a fragment of the logics described in the previous sections. The method described in [5] is very general and applies to a large class of functors, but the description of the syntax and semantics of the logics is quite involved; our logics have the advantage of being simply extensions of infinitary modal logics by means of *operators.*

6 Multisets & Monads

The coalgebraic framework that we have employed in the construction of multiset universes has as drawback that it doesn't yield extra structure on the objects produced in an automatic way. E.g. the categorical framework does not provide the desired morphisms or operations between the multisets. We think this drawback can be overcome by enriching the categorical framework. We will not attempt that task in this paper. There is however one enrichment that can be added easily on top of the coalgebraic framework. We can study the dynamics of Γ-coalgebras (or multirelations) by extending the analogues of the powerset functor to monads. This allows us to define composition of Γ-coalgebras in a satisfactory way.

6.1 Singleton and Unary Union

Singleton and unary union are fundamental operators on sets. Together, they fit into the definition of a well-known construction in category theory: the triple $\langle \mathsf{Pow}, \mathsf{sing}, \mathsf{union} \rangle$ is a primary example of a monad.

Definition 6.1 *A* monad *on a category $\mathcal{C}$ is a triple $\langle F, \nu, \mu \rangle$, where F is an endofunctor on $\mathcal{C}$ and $\nu : 1 \to F$, $\mu : F^2 \to F$ are natural transformations satisfying: for all $A \in \mathcal{C}$ the following are commutative diagrams.*

$$\begin{tikzcd}
F(A) \arrow[r, "F(\nu_A)"] \arrow[d, "\nu_{FA}"'] \arrow[dr, "\mathsf{id}_{F(A)}"] & F^2(A) \arrow[d, "\mu_A"] \\
F^2(A) \arrow[r, "\mu_A"'] & F(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
F^3(A) \arrow[r, "\mu_{FA}"] \arrow[d, "F(\mu_A)"'] & F^2(A) \arrow[d, "\mu_A"] \\
F^2(A) \arrow[r, "\mu_A"'] & F(A)
\end{tikzcd}$$

In this section we consider the singleton and unary union operators for multisets. No difference in the definition of the singleton operator arises between exclusive or overlapping types, because given an element $a \in A$ the natural choice for ν_A is simply $\nu_A(a) = [\![a]\!]$. As for the unary union, from our informal definition of exclusive and overlapping types it should be clear that the definition of the union operator is different in the two contexts.

Let us start by defining the unary union μ^e in the exclusive types. We define the unary union of a multiset $\delta \in \Gamma^2(A)$ as the A-multiset in which any element $a \in A$ appears with multiplicity equal to $\sum_{z\in\Gamma(A)} m_\delta(z)m_z(a)$. Hence, the formal definition of the natural transformation μ^e from Γ^2 to Γ can be given as follows. If $\delta \in \Gamma^2(A)$, then $\delta = [\lambda x_1.fx_1]$ with $fx_1 \in \Gamma(A)$, for all $x_1 \in \mathsf{dom}f$; then $fx_1 = [\lambda x_2.f_{x_1}x_2]$ with $f_{x_1}x_2 \in A$. Consider the set of pairs $I = \{\langle x_1, x_2\rangle : x_1 \in \mathsf{dom}(f), x_2 \in \mathsf{dom}(f_{x_1})\}$. We define $\mu^e_A(\delta)$ as the function with domain equal to I and $\mu^e_A(\delta)\langle x_1, x_2\rangle = f_{x_1}x_2$ (this is not entirely correct because the domain of an A-multiset should be a cardinal, but for the sake of simplicity we omit the biunivocal correspondence between I and a cardinal).

Lemma 6.2 *The triple $\langle\Gamma, \nu, \mu^e\rangle$ forms a monad.*

Proof.
We leave to the reader the verification that ν and μ^e are natural transformations. The diagram on the left of the monad definition simply say that if $\alpha \in \Gamma(A)$ is an A-multiset then $\mu^e_A([\![\alpha]\!]) = \alpha = \mu^e_A(\Gamma(\nu_A)(\alpha))$, which is easily checked. As for the diagram on the right, consider $\gamma \in \Gamma^3(A)$. We want to show that

$$\mu_A(\mu_{\Gamma(A)}(\gamma)) = \mu_A(\Gamma_{\mu_A}(\gamma)),$$

where we omitted the superscript e for the sake of readability. Suppose $\gamma = [\lambda x_1.fx_1]$, $fx_1 = [\lambda x_2.f_{x_1}x_2]$, and $f_{x_1}x_2 = [\lambda x_3.f_{x_1,x_2}x_3]$. Then $\mu_{\Gamma(A)}(\gamma) = [\lambda_{\langle x_1,x_2\rangle}.f_{x_1}x_2]$ and $\mu_A(\mu_{\Gamma(A)}(\gamma)) = [\lambda_{\langle\langle x_1,x_2\rangle,x_3\rangle}.f_{x_1,x_2}x_3]$. Hence, $\mu_A(\mu_{\Gamma(A)}(\gamma))$ is equal to $[g]$ with

$$\mathsf{dom}(g) = \{\langle\langle x_1, x_2\rangle, x_3\rangle : x_1 \in \mathsf{dom}(f), x_2 \in \mathsf{dom}(f_{x_1}), x_3 \in \mathsf{dom}(f_{x_1,x_2})\},$$

and $g\langle\langle x_1, x_2\rangle x_3\rangle = f_{x_1,x_2}x_3$. On the other hand,

$$\begin{array}{rcl}
\Gamma_{\mu_A}(\gamma) & = & [\mu_A \circ \lambda x_1.fx_1] \\
& = & [\lambda x_1.\mu_A(fx_1)] \\
& = & [\lambda x_1.\mu_A([\lambda x_2.f_{x_1}x_2])] \\
& = & [\lambda x_1.\mu_A([\lambda x_2.[\lambda x_3.f_{x_1,x_2}x_3]])] \\
& = & [\lambda x_1.[\lambda_{\langle x_2,x_3\rangle}.f_{x_1,x_2}x_3]].
\end{array}$$

Then $\mu_A(\Gamma_{\mu_A}(\gamma)) = [h]$ where

$$\mathsf{dom}(h) = \{\langle x_1, \langle x_2, x_3\rangle\rangle : x_1 \in \mathsf{dom}f, x_2 \in \mathsf{dom}f_{x_1}, x_3 \in \mathsf{dom}f_{x_1,x_2}\}$$

and $h\langle x_1\langle x_2, x_3\rangle\rangle = f_{x_1,x_2}x_3$. It is then clear that $[g] = [h]$ and hence that $\mu_A(\mu_{\Gamma(A)}(\gamma)) = \mu_A(\Gamma_{\mu_A}(\gamma))$. $\Box$

Considering the context of overlapping types, it is easy to see that μ^e is not a natural transformation from Δ^2 to Δ. We define instead a unary union for multiset of overlapping types as follows. Using the $\breve{\Delta}$ representation of multiset, we define a natural transformation μ^o from $\breve{\Delta}^2$ to $\breve{\Delta}$ as follows. Suppose $\delta \in \breve{\Delta}^2(A)$, that is, $\delta : \breve{\Delta}(A) \to k$. Then if $a \in A$ we define $\mu^o_A(\delta)(a) := \mathsf{sup}\{\alpha(a)\delta(\alpha) : \alpha \in \breve{\Delta}(A)\}$. One can then prove that the triple $\langle \breve{\Delta}, \nu, \mu^o \rangle$ forms a monad (and since $\breve{\Delta}$ is naturally isomorphic to Δ we obtain a corresponding monad on the functor Δ).

Lemma 6.3 *The triple $\langle \breve{\Delta}, \nu, \mu^o \rangle$ forms a monad.*

6.2 Kleisli Categories for Multisets

Given a monad $\langle F, \nu, \mu \rangle$ on a category $\mathcal{C}$, the Kleisli category $\mathcal{C}_F$ of $\langle F, \nu, \mu \rangle$ has the same objects as $\mathcal{C}$, while a Kleisli-arrow from A to B is a $\mathcal{C}$-arrow from A to $F(B)$. The monad structure allows to define the composition $\star$ of Kleisli arrows as follows: if $f : A \to B$ and $g : B \to C$ are arrows in the Kleisli category, then $g \star f : A \to C$ is defined as $g \star f := \mu_C \circ F(g) \circ f$. In particular, if f is a Kleisli arrow from A to B then $f \star \nu_A = f = \nu_B \star f$, and ν_A serves as the identity arrow in the Kleisli category. Notice that F-coalgebras can be identified with looping arrows. Hence, if the functor F can be extended to a monad, then the monad structure allow us to define the composition of F-coalgebras. This composition is particularly interesting from a coalgebraic point of view, because we can prove that it is preserved under bisimulation.

Lemma 6.4 *Suppose R is a bisimulation between $e_1 : A \to F(A)$ and $e'_1 : A \to F(A)$, say via $r_1 : R \to F(R)$ and that* the same *R is a bisimulation between $e_2 : A \to F(A)$ and $e'_2 : A \to F(A)$, say via $r_2 : R \to F(R)$. Then R is a bisimulation between $e_2 \star e_1$ and $e'_2 \star e'_1$, viz. via $r_2 \star r_1$.*

Proof.
Since $r_2 \star r_1 = \mu_R \circ F(r_2) \circ r_1$, one can see that R is a bisimulation between $e_2 \star e_1$ and $e'_2 \star e'_1$ by composing the commutative diagrams below.

$$\begin{tikzcd}
A \arrow[r, "e_1"] & F(A) \arrow[r, "F(e_2)"] & F^2(A) \arrow[r, "\mu_A"] & F(A) \\
R \arrow[u, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[r, "r_1"] & F(R) \arrow[u, "F(\pi_1)"'] \arrow[d, "F(\pi_2)"'] \arrow[r, "F(r_2)"] & F^2(R) \arrow[u, "F^2(\pi_1)"'] \arrow[d, "F^2(\pi_2)"'] \arrow[r, "\mu_R"] & F(A) \arrow[u, "F(\pi_1)"'] \arrow[d, "F(\pi_2)"'] \\
A \arrow[r, "e'_1"] & F(A) \arrow[r, "F(e'_2)"] & F^2(A) \arrow[r, "\mu_A"] & F(A)
\end{tikzcd}$$

$\square$

Let us consider our multiset monads. The Kleisli composition is a very familiar object in the case of Γ, because it can be identified with *matrix multiplications*. To see this, notice first that a multigraph (or a Γ-coalgebra) $\mathcal{A} = \langle A, \rho \rangle$ can be seen as a matrix having A-rows and A-columns, with entry $\langle a, b \rangle$ equal to $\rho(a, b)$. If $\mathcal{A}' = \langle A, \rho' \rangle$ is another coalgebra with the same domain, then the reader can check that the matrix corresponding to $\mathcal{A} \star \mathcal{A}'$ is the rows by columns product of the matrix corresponding to $\mathcal{A}'$ and $\mathcal{A}$.

7 Afterword

What have we accomplished in this paper? We gained a better understanding of the fact that there are two salient notions of multiset. It was shown how the Scott universe can be fitted into the coalgebraic framework. The relationship between multisets and trees was elaborated. Some insight was provided on why unraveling fails to be a functor. Finally we briefly considered how the dynamics of multirelations can be added on top of the coalgebraic framework.

The present work can be viewed as a case study in coalgebraic theory. We looked in detail at particular functors. It turned out that reasonable uniform versions could be found for both types of functors considered. One may wonder precisely which endofunctors of the category of sets and classes of the AFA-universe do have uniform naturally isomorphic variants. A striking phenomenon is the fact that the uniform versions seem to be philosophically superior. They seem to be closer to an 'explanatory modeling' than their non-uniform brethren. Note however that there are many uniform variants of a given functor modulo natural isomorphism. Some of them could be utterly philosophically unenlightening. However that may be, we could be moved to consider the following hypothesis. Whenever we have a sufficiently clear intuitive concept that lends itself to coalgebraic analysis at all, then there is a uniform functor that models the intuitive concept better than any naturally isomorphic non-uniform functor.

What have we not accomplished? First, we did not develop axiomatizations of the two universes of multisets. We are not sure how interesting this question is. Secondly, we feel that there are two closely related defects to the coalgebraic framework that we have employed. (i) It is not abstract enough and (ii) it is not rich enough. The lack of abstraction shows itself where it is not fully perspicuous which specific properties of sets and classes are employed in the proofs. The poverty shows itself where we construct a universe of multisets, but e.g. the question about what the appropriate multiset morphisms are is left undecided by the framework. Of course we know what the morphisms should be, but this insight is not fully reflected in the framework. It seems that a more full understanding of the universes of multisets from the coalgebraic point of view would require a reworking of the coalgebraic framework. Thus we end our paper with a challenge for the future, the challenge to *generalize and enrich* the coalgebraic framework.

References

[1] P. Aczel. *Non-Wellfounded Sets.* CSLI, Lecture Notes 14, 1988.

[2] P. Aczel and N. Mendler. A Final Coalgebra Theorem. In *Category Theory and Computer Science,* Lecture Notes in Computer Science 389, ed Pitt et al. 1989.

[3] J. Barwise and L. Moss. *Vicious Circles.* Volume 60 of *CSLI Lecture Notes*, Stanford California, 1996.

[4] S. Mac Lane. *Categories for the Working Mathematician.* Volume 5 of the *Graduate Text in Mathematics*, Springer 1991.

[5] L. Moss. Coalgebraic Logic. Preprint

[6] J.J.M.M. Rutten. Universal Coalgebra: a Theory of Systems. Technical Report of the Centrum voor Wiskunde en Informatica CS-R9652 1996.

[7] D. Scott. A Different Kind of Model for Set Theory. Unpublished paper, given at the *1960 Stanford Congress of Logic, Methodology and Philosophy of Science.*

[8] A.S. Troelstra and H. Schwichtenberg. *Basic Proof Theory.* Cambridge Tracts in Theoretical Computer Science 43, CUP, 1996.

[9] J.J.M.M. Rutten and D. Turi. On the Foundation of Final Semantics: Non-Standard Sets, Metric Spaces, Partial Orders. Technical Report of the Centrum voor Wiskunde en Informatica CS-R9241 1992.