BETH'S NONCLASSICAL VALUATIONS

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This paper is about E.W. Beth's use of nonclassical valuations to various purposes. In Beth's own terminolgy these nonclassical valuations are called *pseudo-valuations*. One can actually distinguish at least three periods for his use of the pseudo-valuations. In the first period he has a grandiose idea, which unfortunately does not turn out to work as he would have liked it, the second period consists of some simple but elegant applications of the idea, and the third period is the application of the idea in a direction in which one might say that present day logic still uses them. This little history is intertwined with the birth of the concept of semantic tableaux. We will touch on the latter subject in so far as this is necessary for our considerations but the reader can find considerably more detail in Guillaume's contribution to this volume.

1. The first period

The first mention of pseudo-valuation we can find is in a letter to Alfred Tarski of June 30, 1954, but let us first explain the idea of pseudo-valuations in the context of pure implicational logic, also Beth's favorite system to demonstrate his ideas on.

A *valuation* is a function v from, in this case propositional, formulas to 0 and 1 such that:

$v(A\rightarrow B)=1$ iff $v(A)=0$ or $v(B)=1$

or in other words:

$v(A \to B)=0$ iff $v(A)=1$ and $v(B)=0$

(Actually Beth usually used 0 and 2 instead of 0 and 1. According to a personal oral communication to D. de Jongh of around 1962, the purpose of that was to make room for a third truth value, like undefined, in between, but we will follow the more standard way of writing 0 and 1.)
This, of course, gives you standard classical logic and Beth wanted to apply it more generally. He was thinking of subsystems of classical logic axiomatized by some axiom schemes **Ax** with the rule of modus ponens:

$A, A \to B/B$.

This means in terms of the valuations that

if $v(A)=1$ and $v(A \to B)=1$, also $v(B)=1$, or more perspicuously,

if $v(A)=1$ and $v(B)=0$, then $v(A \to B)=0$

just half of the equivalence above.

Beth realized that this was precisely enough to describe the situation of arbitrary subschemes of classical logic and even more generally:
If **Ax** is a set of axiom schemes of implicational logic one can give the value 1 to all the formulas C that are derivable from **Ax** by modus ponens, $v(C)=1$, and value 0 to all other formulas, and one indeed has obtained such a pseudo-valuation. This pseudo-valuation simply makes everything true that is derivable from **Ax** and makes everything else false.

Of course, this gives one a kind of general *completeness theorem* for propositional logics. Beth's grandiose idea, that actually did not work out, was that this could be made to work in such a way that not only could one get completeness, but also *decidability*, which would solve a problem that Tarski stated in a lecture in Princeton in 1946:

"To be able to decide when a set of formulas is an *adequate* axiom system, a system from which all tautologies are derivable."

Namely, to decide whether an axiom system **Ax** is adequate, one would only need to ascertain whether one of the well-known axiom systems would be derivable from **Ax** (and this would be decidable), besides checking of course that **Ax** consists of only of tautologies.

Of course, Beth consulted his friend Tarski on this. Beth included in his letter to Tarski some manuscript. It is not 100% sure which unpublished paper of his this is, but we may assume that it is a paper he intended to be in honour of Feys, a copy of which can be found in the archives. There also exists a copy of an abstract for the ASL meeting in Amsterdam that same year with this content. Beth himself organized this conference and he was in addition to organizing the conference planning to talk about this subject, but later decided not to. Also the paper in honour of Feys was replaced by a different one, the reasons for this we will come to in a minute.

How incredibly active Beth was in this period can be demonstrated by his time table in August-September. The conference of the ASL that he organized started on September 1. On August 31 he gave a lecture on Nieuwentyt, a Dutch philosopher of Science, for the Nederlandse vereniging voor Logica. On September 11, just a few days after the logic conference, he gave a lecture in the Hague on the philosophy of Henri Poincaré.

That the papers remained unpublished is because Tarski answered him on July 13: "I haven't had time to study your paper, but there is one remark which I have to make at once. Your Corollary 3 (which gives an affirmative answer to a problem formulated in my Princeton talk) is in direct contradiction to a result stated by Lineal and Post in the Bulletin of the Amer. Math. Soc vol. 55, 1949, p. 50.": an abstract of Post and Lineal that showed that that there are finite sets of tautologies for which it is undecidable what is derivable from it. Actually Beth's answer to Tarski on July 22 shows that this did not really surprise Beth very much, because in the meantime he had realized, when he tried to write down a full proof of his ideas that seemingly small gaps were unexpectedely difficult to fill, in fact a letter to Tarski of July 14 crossing Tarski's letter to him already mentions these problems: "I found several gaps …. It seems that I will have to resort to several other tricks besides the introduction of pseudo-valuations." On July 22 he writes "… it explains why I could gradually improve my argument but not finish it." So, he accepted this setback immediately (from the same letter: "… there is no reason to doubt its truth …") and tried to contact Post about the proof, because only an abstract without proof had appeared.

Just for the sake of history, the story does not give much intrinsic insight: Beth got an answer from Post's wife that Post had just died. A little while later he got a letter from Post's coauthor with the name Samuel Lineal that turned out to be a pseudonym of Samuel Gulden. The story does not tell unfortunately why somebody writes logic under a pseudonym, was he ashamed to have written a logic paper? We can only find the name—and this time it is Gulden—10-15 years later in some work in topology, he wasn't ashamed of that, appparently. He didn't do any more logic in as far as we can verify. Even the writing of the proofs of his and Post's theorem he left to other people. But even though the proofs were not available at the time Beth was convinced and didn't publish his papers.

It is also good to stand still for a moment and consider how Beth intended to prove decidability here. Just this year he was not only concerned with pseudo-valuations and all these other activities, but also one his most lasting contributions to logic, his semantic tableaux, were getting their shape in his mind. We find a kind of *prototableaux*, both in the lectures on "L'existence en mathématiques" in March-April in Paris, and in the the unpublished paper in honor of Feys that I mentioned above. I call them prototableaux because as Beth says in a letter to Hasenjaeger of February of the next year in these "es fehlte jedoch noch etwas wesentliches" (something essential was yet missing: he had not invented the splittings of the tableaux, those were added in his mind in December of this very fruitful year. In his "L'existence en mathématiques" lecture he for example obtained B on the true left side of the tableau from A and A□ B on the left side (modus ponens) directly without splitting the tableau.

It is very interesting that from the very start his tableaux, even his protoableaux, were used for radically distinct purposes. In "L'existence en mathématiques" it was used to construct a what we now call *closed* tableau for a valid sequence to show its validity. In his unpublished subformula theorem paper in honour of Feys he uses a tableau to show that a certain tautology, p□ (q□ (p□ q)), can be falsified only by a pseudo-valuation.

???In fact he says (somewhere) that in his mind tableaux and pseudo-valuations are very much connected.???

His idea was to use these tableau-like methods to show that only subformulas of the schemes to be investigated needed to be substituted in the schemes, i.e., a finite number of formulas so that decidability follows. This is of course always the advantage that tableaux give. A final word on this first period. Beth was not directly successful in applying his pseudo-valuations in the manner he envisaged, but we are reasonably sure that something can be done with his ideas even nowadays, they have not been fully exploited.

2. The second period.

In the second period Beth used the idea of pseudo-valuations to prove independence of axiom systems in propsitional, and even predicate logic. Of course, an adequate set of axiom schemes **Ax** is independent, if, for all schemes S in **Ax**, **Ax**–S is not adequate. And naturally, to determine whether **Ax**–S is adequate, it suffices to check whether S is derivable from **Ax**–S. To show that it is not, it suffices to give a pseudo-valuation that makes **Ax**–S true and S false. Even though this is in general apparently not decidable, in practice it can of course be successful, and it turns out to be. We have always found this technique highly original and elegant. The normal method would be to give some complicated matrix and this works much better. Let me simply give some example; you do not find this in the regular logic text books.

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Beth consulted with with his friend Alonzo Church on this matter.
For Peirce rather difficult.

The third period.

We now get to the third period, and we will switch to the word nonclassical valuations here and use the word pseudo-valuation only for the very special

valuations of the second (and first) period.
As an introduction let us recur to the semantic tableaux. The next year, not only had he completed his concept of semantic tableau, he had within that year published the definitive article "Semantic entailment and formal derivability" on it. But not even that is all, in September of that year, 1955, he had already lectured in Paris on the form of semantic tableaux for *intuitionistic* logic, another proof of the fact that from the very start semantic tableaux were not taken to be the expression of an orthodox classical view on truth values. Curiously enough however these tableaux were not by Beth in the ensuing papers connected to nonclassical valuations, but to choice sequences. His justly famous paper "A semantic construction of intuitionistic logic" appeared already in 1956. Of course choice sequences in that paper are much more than pseudo-valuations connected to the intended meaning of the intuitionistic connectives; Beth was one of the very few people who succeeded in working as a nonintuitionist, but nevertheless from inside the intuitionistic point of view, and the highly technical paper was the start of some very complicated research by Kreisel, Troelstra, Veldman and Friedman on the completeness of intuitionistic logic from an intuitionistic point of view. His completeness result was heavily criticized as inadequate by Kreisel. D. de Jongh remembers that Beth in one of his later conversations with me expressed the opinion that the difficulties that Kreisel pointed out would be overcome. Actually, the germ of the ideas that Veldman used later is already contained in the Foundations of Mathematics. He was not able to do that himself, but he was vindicated later by the work of Wim Veldman and Harvey Friedman. In another sense however, the point of view of choice blocked the way for him to the discovery of the models later known as Kripke-models. At least from the present day view Kripke-models for intuitionistic logic are just the combination of Beth's models with the idea of pseudo-valuations. Beth had both ideas, but with pseudo-valuations one looks at intuitionistic logic more as an outsider, and he wasn't prepared to do that at that moment.

That story continues in January, February of 1957, the year after the main publication on intuitionism. Beth's friend Curry who later took his chair for a number of years, wrote to him about a brilliant 16-year old student of his, Saul Kripke, whom he was tutoring. He asked Beth to contact Kripke and send him material on semantic tableaus etc. Beth did so immediately, and sent Kripke his two main papers on semantic tableaux for classical and intuitionistic logic, the Semantic entailment and the Semantic construction papers. It is pretty obvious that this must have greatly influenced Kripke. His two famous papers of the period after this, in 1959 and 1963, containing his new models with possible worlds, first for modal logic and then for intuitionistic logic, had a backbone of semantic tableaux.

Kripke's semantic tableaux for intuitionistic logic were slightly different from the ones Beth used in his original semantic construction of intuitionistic logic. The difference centers exactly on the fact that Kripke in his semantic tableaux is inspired directly by what one may call nonclassical valuations whereas Beth thinks as I said of choice-sequences.

... ...

When the first author arrived as a student in 1961, Beth was, in the few years that remained to him, taking an approach like Kripke's by using nonclassical valuations. Beth introduced these valuations (I-valuations) to him, and it was his task in Beth's European Atomic Community research project to work on these models. His first paper, of 1962, ("Recherches sur les I-valuations") was devoted to these studies, and, he kept working on problems connected with the work on I-valuations on and off throughout his carreer (e.g the Journal of Symbolic Logic, in 1995, together with Lilia Chagrova, on "The decidability of dependency in intuitionistic propositional logic which is directly in line with the researches started in 1961).

There is a second line of research intimitely connected with the previous one that still extends into the present day. In 1963/64 Hans Kamp, my co-student with Beth, and I worked on the semantic tableaux for intuitionistic logic that I had worked on with Beth himself to convert them into computer programs that computed whether formulas are intuitionistic tautologies. Obviously in doing this we were stimulated by Beth's ideas on logical or thought machines. We were successful in the project, but even rather simple formulas did send the computers of those years into calculatations that lasted for minutes. Now, throughout the years I have been coming back also to this reasearch program. I didn't do any programming myself, it was not really my favorite business, but I have found students to do the programming for me. The last one is Lex Hendriks, who after a PhD on the subject in 1996 continues in a postdoc postion subsidized by the Dutch Research Council NWO. The backbone of the whole project that extends to many other logics now still is formed by the semantic tableaux developed by Beth in the early sixties. A difference is that present day computers do no longer take more than two minutes for a formula but grind through thousands of them each minute.

Acknowledgements.

I am very happy to be able to talk at this conference in honour of my teacher E.W. Beth, and in fact to talk about the origin of ideas I have been working on throughout my own carreer, and we thank the Stichting Beth for giving me this opportunity.
Visser.
Let me end this talk by expressing my thanks to Beth himself for introducing me to the beautiful subjects that have kept me busy ever since.

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