Independent choices and the interpretation of IF logic

Theo M.V. Janssen, (theo@science.uva.nl)
Institute for Logic, Language and Computation, University of Amsterdam

Abstract. In this paper it is argued that Hintikka's game theoretical semantics for Independence Friendly logic does not formalize the intuitions about independent choices; it rather is a formalization of imperfect information. Furthermore it is shown that the logic has several strange properties (e.g. renaming of bound variables is not allowed). An alternative semantics is proposed which formalizes intuitions about independence.

Keywords: IF logic, game theoretical semantics, imperfect information, independent choices, Henkin quantifier, branching quantifier

1. Introduction

This paper deals with the definition of independent choices in logic. The notion arises in game theoretical semantics, where the truth of a formula is determined by a game between two players, one who tries to check the formula, and one who tries to refute it. A version of such games, introduced by J. Hintikka, is IF logic: *independence friendly logic*. This logic uses the quantifier $\exists y_{/x}$, which means that a value for y has to be chosen independent of the value for x, and the disjunction $\psi \vee_{/x} \theta$, which means that a subformula has to be chosen independent of x. IF logic is described in a number of publications e.g. Hintikka (1974), Sandu (1993), Hintikka (1996), Hintikka and Sandu (1997).

Hintikka calls the players Me and Nature, or Verifier and Falsifier, but we will follow Hodges (1997a) and call them ∃loise (female) and ∀belard (male); this has the advantage that pronouns can be used without the danger of confusion. Furthermore, the names reflect the choices the players make (in the situations arising in this paper): ∃loise makes the choice for $\exists y, \exists y_{/x}, \vee$, and $\vee_{/x}$, and ∀belard for $\forall$ and $\wedge$. A formula is true if ∃loise has a winning strategy and false if ∀belard has one.

Below, some examples are given which illustrate the aims of IF logic. These, and all later examples, are interpreted on the natural numbers ($\mathbf{N}$). For variables bound by universal quantifiers preferably x and y will be used, for variables bound by existential quantifiers u and v.

(1) $\forall x\, \exists u_{/x}[x = u]$

When ∀belard has chosen, ∃loise has to choose u independently of x, and therefore it may happen that she selects a different number. Hence

∃loise has no winning strategy, the formula is not true. Also ∀belard has no winning strategy, so the formula is neither true nor false. The same holds for

(2) $\forall x\, \exists u_{/x}[u \neq x]$

The formalization of a strategy for a choice is a function which takes as arguments the choices previously made and yields the choice that has to be made. Such a strategy is called independent of a variable if this variable is not among its arguments. A strategy for a formula consists of strategies for each of the choices that has to be made. In this paper we will (with one exception) only consider winning strategies for ∃loise, so all examples are only investigated for being true or not true, and not for falsehood.

The next example is true on the natural numbers; ∃loise follows the strategy $u := 0$ (read as: 'u becomes 0' or 'for u is chosen the value 0').

(3) $\forall x\, \exists u_{/x}[u \leq x]$

The next formula is not true because without information on x, a guaranteed correct choice between the disjuncts is not possible.

(4) $\forall x[x = 7 \vee_{/x} x \neq 7]$

The following example is somewhat more complex:

(5) $\forall x \exists u \forall y\, \exists v_{/x}[x < u \wedge y < v]$

The choice of v is independent of x, but may depend on y. A more familiar representation of this example is by means of a branching quantifier or Henkin quantifier:

(6) $\begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} [x < u \wedge y < v]$

The formula is true (on $\mathbf{N}$).

As one may have noticed in the examples, the playing of the game and the application of the strategies proceeds from outside to inside. For this (and other) reasons Hintikka's game theoretical interpretation of IF logic is not compositional. He claimed that a compositional semantics would not be possible. However, Hodges has given an equivalent compositional interpretation: trump semantics (Hodges, 1997a, Hodges, 1997b). By doing so, he clarified several aspects of the logic. A related formalization is given by Caicedo and Krynicki (1999).

The main aim of this paper is to show that the existing interpretations for IF logic (game theoretical semantics and trump semantics) do not capture the intuitions about independence. In Sections 2–7 many examples will be given where the intuitive judgements about the truth of formulas involving independent choices differs from the results in these interpretations. An alternative semantics, called subgame semantics, will be proposed in Section 10.

2. The context of a formula

Consider:

(7) $\forall x[x \neq 2 \vee \exists u_{/x}[x = u]]$

In the formula it is mentioned that it is possible that x differs from 2. This is no news, x runs over all numbers. Furthermore, as $\exists u_{/x}$ indicates, information concerning x should play no role when u is selected. We have seen in example (1) that such an independent choice is not possible. So one expects the formula not to be true. But in game theoretical semantics the context in which this subformula occurs makes a difference. For $\exists u_{/x}$ $\exists$loise chooses the strategy $u := 2$. Her strategy for the disjunction is *if* $x \neq 2$ *then* L *else* R. So if x is different from 2, she chooses the left hand side of the conjunction, which then is true, and otherwise the right hand side, which then becomes true by her choice for u. So (7) is true in game theoretical semantics.

In (1) $\exists$loise could not find a u independent of x such that $x = u$. According to my intuition that also means that she cannot find a u in situations where she can deduce from the context that $x = 2$, because she is supposed *not* to use the value of x. Using this information is cheating. This is a first illustration that game theoretical semantics semantics does not formalize independence.

If one has to find a value for u that for sure equals the value of x, then one would expect this task to be as difficult as finding a value that for sure is different: if one knows the value of x, both tasks can be done easily, but if they have to be performed independently of the value of x, both tasks are unsolvable. However, in game theoretical semantics changing $\neq$ into $=$ (or vice versa) may cause a difference. Such a change turns the true sentence (7) into (8) which is not true.

(8) $\forall x[x = 2 \vee \exists u_{/x}[x = u]]$

On the other hand (9) and (10) are true:

(9) $\forall x[x = 2 \vee \exists u_{/x}[x \neq u]]$

(10) $\forall x[x \neq 2 \vee \exists u_{/x}[x \neq u]]$

In (9) ∃loise plays the strategies *if* $x = 2$ *then* L *else* R and $u := 2$, and in (10) the strategies *if* $x = 3$ *then* L *else* R and $u := 3$. Also these variants illustrate that the game theoretical interpretation does not correspond with intuitions on independence.

3. Repetition of subformulas

It is no printing error that in (11) the same subformula occurs twice.

(11) $\forall x[\exists u_{/x}[x \neq u] \vee \exists u_{/x}[x \neq u]]$

According to my intuition, if one cannot find a u independent of x, one cannot find that if one may do so, at ones choice, in the left hand side subformula, or in the right hand side one: it remains the same choice, and is as difficult on the left as on the right.

In game theoretical semantics, however, (11) is true. A possible strategy is as follows. If $x = 3$ then ∃loise chooses the left subformula, and follows there the strategy always to choose $u := 4$. Otherwise, she chooses right, where her strategy is to choose $u := 3$. Whatever the value of x is, she always ends in a situation such that $x \neq u$ is true. So this is a winning strategy: the formula is true.

The above example shows a strange property of game theoretical semantics. Consider the subformula $\exists u_{/x}[x \neq u] \vee \exists u_{/x}[x \neq u]$. If we replace this by $\exists u_{/x}[x \neq u]$, so changing (11) into (2), the interpretation of the whole formula changes from 'true' into 'not true'. This phenomenon of IF logic is discovered by Janssen (1997, 1999): *$\phi \vee \phi$ is not in all contexts equivalent with ϕ.*

Again, one might expect that finding a value u that equals x will be as difficult as finding one different from x. However changing $\neq$ into $=$ in (11) makes a difference. Consider:

(12) $\forall x[\exists u_{/x}[x = u] \vee \exists u_{/x}[x = u]]$

A strategy for the left disjunct has to be a constant function, say $u := n$, and the same for the right disjunct, say $u := m$. This means that ∃loise can only design a strategy that works for two values of x, whereas ∀belard has many more choices. So she has no winning strategy, the formula is not true. But when we have as many disjuncts as there are elements in the domain, the formula becomes true again. For instance, the infinite disjunction is true on **N**:

(13) $\forall x[\exists u_{/x}[x = u] \vee \exists u_{/x}[x = u] \ldots]$

For conjunction a related phenomenon arises. Notice that ∀belard has no strategy which guarantees him that (14) becomes false:

(14) $\exists u \forall x_{/u}[x = u]$

One might expect that he also has no winning strategy if he may choose whether he will falsify the formula on the left hand side of a conjunction or on the right hand side:

(15) $\exists u[\forall x_{/u}[x = u] \wedge \forall x_{/u}[x = u]]$

That is, however, not the case. ∀belard's strategy for this game is analogous to the one of ∃loise for (11). For $\wedge$ he could choose the left disjunct if $u = 3$, and the right disjunct otherwise. On the left he follows the strategy $x := 4$, which makes the conjunct false, and on the right $x := 3$ with the same effect. So he has a strategy which makes the formula always false, hence (15) is false. So here again a strange property of the game theoretical interpretation: *$\phi \wedge \phi$ is not in all contexts equivalent with ϕ.*

The point of this example, and the one concerning $\phi \vee \phi$, is not that the laws of traditional logic are broken (although that is not attractive), but that intuitions concerning independence are violated.

4. Existential quantifiers

Consider

(16) $\forall x \exists u \, \exists v_{/x}[x = v]$

This examples resembles example (1), the difference is that a vacuous quantifier u is inserted. The information given by $\exists u$ that there exists a number tells us nothing new, so one might expect that this change makes no difference for the task to find a v independent of x. In the semantics for IF logic as given in Hodges (1997a), the surprising result -Hodges warns the reader- holds that (16) is true. The strategy ∃loise follows for $\exists v_{/x}$ is to play $v := u$. This strategy does not mention the value for x, and therefore is allowed. The strategy for $\exists u$ is to play $u := x$; this dependence is allowed because $\exists u$ has no slash. So $u = x$, and, since $v = u$, it follows that $x = v$. By choosing these strategies, ∃loise always wins, whereas the formula without the empty quantification is not true.

This result is not accordance with intuitions about independence. The value of v is carefully chosen in such a way that it equals x, hence is very dependent on x.

The issue is clarified in Hintikka (1996), which appeared after Hodges (1997a) was written. He says (Hintikka, 1996, p. 63):

> the small extra specification that is needed is that moves connected with existential quantifiers are always independent of earlier moves with existential quantifiers

This means that in case an existential quantifier $\exists v$ occurs within the scope of $\exists u$, it should be interpreted as if it was written as $\exists v_{/u}$. So in (16) the quantifier $\exists v_{/x}$ should be interpreted as $\exists v_{/x,u}$. In the appendix to Hintikka's book, Sandu presents a formal interpretation of IF logic in which this independence of existential variables on other ones is formalized (Hintikka, 1996, p.256).

That such an independence was always intended, can be seen for instance in the examples given in one of the earlier papers on game theoretical semantics: Hintikka, 1974. Furthermore, Hintikka (1996, p.64) gives a formula which only is true in infinite models and in the empty model, but loses this property (which is not expressible in first order logic) when the extra specification mentioned above is dropped.

We will refer in the sequel to this independence of existential quantifier as the 'slashing convention' and with 'GTS' to the semantics with this convention. It is the 'official' interpretation of IF logic and described by Sandu in the appendix Hintikka (1996). Notice that, although in the language of IF logic the only variables which may occur after a slash are the variables bound by a universal quantifier, also variables bound by existential quantifiers may, due to the convention, implicitly arise after a slash.

The slashing convention means that Hodges (1997a) does not give a compositional version of GTS, but for a closely related interpretation. In Hodges (1997b) it is indicated how he would design a compositional semantics obeying the slashing convention.

The slashing convention has consequences which are intuitively very strange. Consider:

(17) $\exists u \exists v[u = v]$

Due to the convention the second quantifier is implicitly slashed for the first. This means that the formula is equivalent with:

(18) $\exists u\, \exists v_{/u}[u = v]$

According to my intuition, it is impossible to find a v independent of u such that the two are equal, so (18) is not true. This reaction on the slashing convention is found on several places in the literature. In their review of Hintikka, 1996, Cook and Shapiro (1998, p. 311) comment on Hintikka's extra specification as follows: 'However [...] then a sentence like $\exists x \exists y[x = y]$ would not be true over the natural numbers, contradicting Hintikka's claim that game theoretical semantics agrees with ordinary semantics on first-order sentences (Hintikka, 1996, p. 65)'. Related remarks are found in the review by de Swart et al. (1997) and in the unpublished PhD-dissertation of Pietarinen (2002, p. 31).

However, these intuitions about (17) and (18) are not reflected in their game theoretical interpretation: (18) is in game theoretical semantics a true formula. ∃loise's strategy for the first quantifier is, for instance, $u := 7$, and for the second quantifier $v := 7$. This second strategy does not mention u, and therefore it allowed in GTS. So, although in (18) the strategies for u and v must be the same and a change in the strategy for u has immediate consequences for the strategy for u, they are nevertheless formally independent in GTS.

Note the strange fact that, due to the convention, in (17) the obvious strategy $u := v$ is not allowed, whereas in the comparable formula (19) the strategy $u := x$ is allowed:

(19) $\forall x \exists u[u = x]$

Also in the next example the slashing convention prohibits the obvious strategy:

(20) $\forall x \exists u[u > x \wedge \exists v[v > u]]$

We would like to follow a strategy like $v := u + 2$: how could we otherwise achieve that $v > u$? This is, however, not allowed. The trick is that the strategy for v incorporates the strategy for u. If we have chosen the strategy $u := x+1$, then we now follow the strategy $v := (x+1)+2$. So again, although u and v are formally independent, their strategies have a close relationship.

The examples given in this section show again that there is a difference between intuitions on independence and the formalization of independence in game theoretical semantics. Furthermore, it was shown that there is no direct connection between strategies in GTS and the classical interpretation.

5. Disjunction

As was the case with existential quantifiers in example (16), also in a disjunction a vacuous variable u can be used to transfer information concerning x towards the choice for $\vee$ which must be independent of x:

(21) $\forall x \exists u[x = 4 \vee_{/x} x \neq 4]$

Example (21) becomes true if one chooses $u := x$ and then decides on $\vee_{/x}$ using the value of u. To avoid this signaling the same solution can be used as for (16): state that a disjunction is implicitly slashed for the existential variables which have scope over it. Although Hintikka does not mention this variant of the slashing convention, it is incorporated in the appendix by Sandu (see Hintikka, 1996, p.256). So (21) is not true in game theoretical semantics.

Also the slashing convention for disjunctions has strange consequences. We will consider three examples. The first is (22) which is, due to the convention, equivalent with (23):

(22) $\exists u[u = 4 \vee u \neq 4]$

(23) $\exists u[u = 4 \vee_{/u} u \neq 4]$

Of course, (22) should be true, but according to my intuition of independence, (23) is not true because it requires to make independent of u a choice between $u = 4$ and $u \neq 4$. Nevertheless, both (22) and (23) are true in GTS. The trick is to follow a constant strategy for the existential quantifier, e.g. $u := 2$, and for the disjunction always to choose R. So instead of letting the value of u determine the choice (which is not allowed), the strategy for $\exists u$ determines the choice.

The second example has no slashes and is classically true.

(24) $\forall x \exists u[u = x \wedge [u = 4 \vee u \neq 4]]$

Since $u = x$, a constant strategy for $\vee$ cannot be used, and in GTS the obvious strategy for the disjunction *if* $u = 4$ *then* L *else* R is not available because of implicit slashing. But (24) is nevertheless true because x can be used: follow the strategy *if* $x = 4$ *then* L *else* R. This is an unnatural strategy because x does not occur in $u = 4 \vee u \neq 4$.

In the third example a slash is added to (24):

(25) $\forall x \exists u[u = x \wedge [u = 4 \vee_{/x} u \neq 4]]$

This formula is intuitively true because the choice between $u = 4$ and $u \neq 4$ has nothing to do with x. In GTS neither this strategy, nor one using x is allowed. The only strategy $\exists$loise might play is a constant strategy, but then she will lose for at least one x. Hence (25) is not true in GTS. It is remarkable that the following variants of (25) are true in GTS:

(26) $\forall x \exists u[u \leq x \wedge [u = 4 \vee_{/x} u \neq 4]]$

(27) $\forall x \exists u[u > x \wedge [u = 4 \vee_{/x} u \neq 4]]$

The strategies for the first example are $u := 0$ and for the disjunction choose always R, and for the second example $u := x + 5$ and always R.

Example (25) is a remarkable example. In all other cases where intuition differs from GTS results, it was the case that a formula which intuitively is not true, was nevertheless true in GTS. Here the situation is reverse.

6. Other forms of signaling

In Sections (4) and (5), we have seen how ∃loise could use her own choices to signal a value to herself that she is assumed not to know, and that the slashing convention is should prohibit this. But there is a form of signaling that is not prohibited: she may use a variable chosen by ∀belard to get information she should not use. Consider:

(28) $\forall x \forall y[x = y \vee \exists u_{/x}[u \neq x]]$

My intuition says that (28) should not be true because the left disjunct is not always true, and the right disjunct is not true at all. However, ∃loise can use y to signal the value of x: she follows the strategy $u := y$, and the formula becomes true.

7. Signaling blocked

In previous sections we have seen examples of formulas which were true because information was used that was derived from a variable which did not occur in the subformula. We will show that this mechanism can be blocked by an intervening quantifier.

An example we have discussed before is:

(29) $\forall x \exists u[u = x \wedge [u = 4 \vee u \neq 4]]$

The winning strategy for the disjunction had to be, due to implicit slashing, *if* $x = 4$ *then* L *else* R. An enriched variant is:

(30) $\forall x \forall y \exists u[u = x \wedge \exists v[v = y \wedge [u = 4 \vee u \neq 4]]]$

Here the same strategy for the disjunction wins. But if we change the name of the bound variable v to x (classically allowed), this strategy is not winning any more:

(31) $\forall x \forall y \exists u[u = x \wedge \exists x[x = y \wedge [u = 4 \vee u \neq 4]]]$

To be precise, in the appendix of Hintikka (1996) the interpretation is not defined for formulas in which a variable is bound within the scope of a quantifier that binds the same variable (as in (31)), so the truth value of (31) changes from 'true' into 'undefined'. If the definition is corrected in the obvious way, the truth value changes from 'true' to 'not true'. This example shows that in IF logic *renaming of bound variables is not allowed* because that may change the truth value.

Note that (31) is a formula of predicate logic which is classically true, but which is not true in GTS. So it is a *counterexample* to Hintikka's claim that IF logic is a *conservative extension* of classical logic (Hintikka, 1996, p.65).

The problem with renaming of bound variables does not only arise in connection with implicit slashing . Compare the following two extended forms of (28):

(32) $\forall x \forall y \forall z[x = y \vee \exists v\, \exists u_{/x}[u \neq x \wedge v = z]]$

(33) $\forall x \forall y \forall z[x = y \vee \exists y\, \exists u_{/x}[u \neq x \wedge y = z]]$

In the first example the strategy $u := y$ is winning, but in the second example it is not. This shows that also in case implicit slashing is not incorporated, as in Hodges, 1997b and Caicedo and Krynicki, 1999, change of bound variables is not allowed (so theorem 3.1.a in the latter paper is not correct in its present formulation).

8. Discussion

The examples given in the previous sections show that strategies in GTS are on several points in conflict with intuitions on independence. Most of these conflicts also arise in case implicit slashing is not incorporated. We have seen that, although the strategies for $\vee_{/x}$ and $\exists u_{/x}$ do not have x as argument, any information which can be deduced from other sources is used. Examples of such sources are:

1. The information whether the slashed variable is universally quantified or existentially (examples (17) and (19)).
2. The context in which the formula is used (examples (9), (11) and (15)).
3. The strategies used elsewhere in the formula (examples (18), (20) and (24))
4. Even the value of the slashed variable is used if this value can be deduced from other information (examples (7), (22) and (28))

These types of information are related, and most types can probably be conceived of as information obtained from strategies used elsewhere in the formula. Halpern (1996) discusses related phenomena in game trees of imperfect information games.

Because several types of partial information can be used, I conclude that game theoretical semantics is not a formalization of 'informational independence', but of 'imperfect information'. It is not impossible that Hodges would agree with this opinion, because the title of his paper Hodges (1997a) is 'Compositional semantics for a language of imperfect information', and *not* '... for a language of informational independence'.

9. Towards an alternative interpretation

9.1. The idea

The alternative which will be proposed, is based upon the following three ideas:

1. There is no implicit slashing.
2. Independence is a local phenomenon.
3. Independence is a kind of uniformity.

The first idea is implemented of course by not having implicit slashing, but also by enlarging the possibilities of explicit slashing. Not only variables bound by universal quantifiers, but also variables bound by existential quantifiers may occur after a slash.

The third idea is realized by uniformity restrictions on the strategies for independent choices. A strategy for $\exists u_{/x}$ is of course a function which yields the same value for any x. The discussion in the previous sections show that this strategy should not be tailored on special (imperfect) information about x but should uniformly be good. Therefore the restriction is added that for other values of x the same choice wins if there is a winning choice at all.

The second idea is obeyed by considering subformulas (including formulas with free variables) as games on their own. This explains the name of the alternative: 'subgame semantics', henceforth SGS. The strategy for a subgame may depend on the values of some or all variables occurring in the subgame, but not on the any other information. A winning strategy must be winning for that subgame for the current situation, but irrespective of the context. One might describe this approach by means of the following metaphor. Associated with a subgame there is a shelf of winning strategies. When a subgame is played, a strategy is taken from the corresponding shelf, and this strategy would also be good when the subgame is played in another context.

It might be useful to emphasize the distinction with other metaphors of 'independence' one finds in the literature. It is not assumed that there are players who forget a value for a variable and may remember it later (see e.g. Pietarinen, 2001), nor is it assumed that there are teams of players in which for each new variable a new player is introduced who gets only partial information.

The present approach differs, in a mathematical sense, from other approaches in the aspect that it is not based on equivalence classes of information sets within its game tree. One might say that equivalence classes are introduced among the information sets in *all* games in which the subgame arises. We will, however, not define it in that way.

9.2. Examples

As explained above, there are three conditions on a winning strategy for $\exists u_{/x}\,\phi$:

1. its arguments are the free variables in ϕ, except, of course, for x,
2. it yields a value for u which is winning,
3. if the value of x is changed, the same value for u is winning, if there is a winning choice at all.

Below we will consider examples which illustrate the conditions 1 and 3. The latter condition is split into two parts: the aspect 'if the value of x is changed', and 'if there is a winning choice at all'.

9.2.1. *Condition 1: arguments are the free variables in ϕ*

A first consequence is that there is no implicit slashing and the obvious strategies involving u can be used in examples below (35) and (36 (these were suggested by referees as counterexample to a previous version):

(34) $\exists u \exists v[u = v]$

(35) $\forall x \exists u[u = 0 \vee_{/x} u \neq 0]$

(36) $\forall x \exists v\, \exists u_{/x}[u = v]$

Also the example which intuitively was true, but was not true in GTS, now has a straightforward winning strategy (*if* $u = 4$ *then* L *else* R):

(37) $\forall x \exists u[u = x \wedge [u = 4 \vee_{/x} u \neq 4]]$

A second consequence is that certain formulas which were true only due to the use of non occurring variables, are not true anymore. An example is:

(38) $\forall x \forall y[x = y \vee \exists u_{/x}[u \neq x]]$

9.2.2. *Condition 3a: if the value of x is changed ...*

Consider the game

(39) $\forall x\, \exists u_{/x}[u = x]$

$\forall$belard starts the game by choosing a value for x, and next $\exists$loise has to make a choice in the subgame

(40) $\exists u_{/x}[u = x]$

Does she have a winning strategy for this game? Because of the first condition on winning strategies, the strategy can only be a constant function, say $u := n$. Now the third condition says that this strategy must be winning for all values of x for which there is a winning choice. Suppose $x = m$, where $n \neq m$. Then there certainly is a winning choice (viz. $u := m$), but the original strategy $u := n$ is not winning. Hence there is no strategy which satisfies the third condition, what means that there is no winning strategy for $\exists$loise in (40), so (39) is not true.

The observation that there is no winning strategy for (40) has consequences for many related examples. In

(41) $\forall x[x \neq 2 \vee \exists u_{/x}[x = u]]$

the right disjunct has only to be true in case $x = 2$. So one might try the strategy $u := 2$. However, this strategy does not satisfy the third condition: for other values of x there indeed is a winning choice, but that is not the choice given by the strategy $u := 2$. So there is no winning choice for the right disjunct and the formula is not true. Analogously, there are no winning strategies in (42) and (43).

(42) $\forall x[\exists u_{/x}[x \neq u] \vee \exists u_{/x}[x \neq u]]$

(43) $\forall x \exists v\, \exists u_{/x}[x \neq u]$

Special attention deserves (44) because it seems to provide a method to circumvent the condition that a strategy only depends on variables which occur in the subformula under consideration: insert dummy occurrences.

(44) $\forall x \exists v[\exists u_{/x}[x = u \wedge v = v]]$

Here, after $v := x$, a possible strategy (according to the first two conditions) is to play $u := v$. But does this satisfy the third condition? Consider the subgame:

(45) $\exists u_{/x}[x = u \wedge v = v]$

Even if $v = x$, the strategy $u := v$ is not a winning one, because if the value of x is changed (but not of other variables), the strategy $u := v$ does not give a guaranteed win. Related examples could be given for dummy occurrences after a slash. This all illustrates that, due to the third condition, adding dummy occurrences for the purpose of signaling is of no use.

9.2.3. *Condition 3b: ... if there is a winning choice*
Consider the branching quantifier formula

(46) $\forall x \exists u \forall y \, \exists v_{/x}[x < u \wedge y < v]$

The strategies $u := x+1$ and $v := y+2$ clearly win. But does $v := y+2$ satisfy the third condition? What happens when in (47) the value of x is changed?

(47) $x < u \wedge y < v$

If x is decreased, or increased with 1, then $v := y + 2$ still wins, but if x increased by 2 or more, then no choice wins (47) because $x < u$ is not true. So indeed, the strategy $v := y + 2$ wins for other values of x, if the subgame can be won at all. Therefore (46) is true.

Since in SGS a strategy may depend on existentially quantified variables, strategy $v := y + u + 2$ makes (48) true:

(48) $\forall x \exists u \forall y \, \exists v_{/x}[x < u \ \wedge \ (y + u) < v]$

Dependency on u can be excluded by expressing this explicitly (as is done in the formalizations of Hodges (1997a) and Caicedo and Krynicki (1999)). For instance (49) is not true.

(49) $\forall x \exists u \forall y \, \exists v_{/x,u}[x < u \ \wedge \ (y + u) < v]$

The proviso 'if there is a winning choice' is also important for:

(50) $\forall x[x \neq 1 \vee \exists u_{/x}[x = 1 \wedge u = x]]$

A winning choice for $\exists u_{/x}$ is only possible in positions where $x = 1$; then the strategy $u := 1$ wins. Due to the subformula $x = 1$ no information from context is needed to design this strategy. If the value of x is changed, the strategy does not win, but other moves do not win either. So the strategy $u := 1$ is a winning strategy which makes the right disjunct true in case that $x = 1$. In other cases the left disjunct is true, so (50) is true.

The last example (due to Väänänen) was a counterexample to a previous version. It is one you would not immediately think of:

(51) $\exists u \, \exists v_{/u}[u = 0]$

For v only constant strategies are possible. It does not matter which one is followed, if there is a winning strategy (so in case $u = 0$), then any strategy is winning, and otherwise no one is winning.

10. Definitions

10.1. Introduction

The following choices are made about the definition of the logic:

1. We will only define winning by $\exists$loise ; so the 'truth' of a sentence will be defined, but not its 'falsehood'.
2. As is the case in the version by Hintikka, negation only occurs in front of atomic formulas.
3. All kinds of variables may occur after a slash.
4. Hintikka's version allows for indexed disjunctions, conjunctions and relations (as in $\bigwedge_{i\in\{1,2\}}[\exists x P_i(x)]$), but these are not incorporated.

10.2. The language

The range of variables is denoted with A; in the examples that is $\mathbf{N}$: the natural numbers $0, 1, 2, \ldots$. The **relation symbols** are $R_1, R_2, \ldots$; each with a fixed arity. In the examples the binary relation symbols $=, \neq, <$, and $\leq$ are used.

Formulas are defined as follows:

1. If $v_1, \ldots v_n$ are variables, and n is the arity of R, then $R(v_1, \ldots v_n)$ and $\neg R(v_1, \ldots v_n)$ are formulas.
2. If ψ and θ are formulas, z is a variable, and W a set of variables, then also the following expressions are formulas: $\psi \wedge \theta$, $\psi \vee \theta$, $\psi \vee_{/W} \theta$, $\forall z \psi$, $\exists z \psi$. If $z \notin W$ then $\exists z_{/W}\, \psi$ is a formula. After a slash will omit the brackets from W; so we write $\exists u_{/x}$ and $\exists u_{/x,v}$.

$FV\phi$ is the set of **free variables** in ϕ. It consists of those variables in ϕ which do not occur in ϕ as first variable after a $\exists$ or $\forall$ symbol. By the variables occurring in ϕ are also understood the variables in W's occurring in $\vee_{/W}$ and $\exists u_{/W}$. So if ϕ is $\exists u_{/x}\, u = v$, then $FV\phi$ is $\{x, v\}$.

10.3. Positions

A position in a game describes the values of the free variables in ϕ. Therefore a **position** is defined as a finite partial function with domain the variables, and range the possible values for those variables. The name 'position' is chosen in analogy with a position in a chess game which gives the information where the pieces stand.

The following notations concern positions:

p_ϕ	the restriction of p to the free variables in ϕ
$\mathbf{P}_\phi$	the set of all positions p_ϕ
$p_{\phi\setminus W}$	the restriction of p to the free variables in ϕ except for those in W
$p * \left[\begin{smallmatrix}a\\u\end{smallmatrix}\right]$	the function obtained by extending p with the value a for argument u, or, if p was already defined for u, by adapting it to yield a for u
$p \sim_u q$	the position p differs from q only with respect to the value of u
$p \sim_W q$	the position p differs from q only with respect to the values of the variables in the set W

The notation is also used in combinations, e.g. $[p * \left[\begin{smallmatrix}a\\u\end{smallmatrix}\right]]_\psi$ denotes the restriction to the free variables in ψ obtained from position $p * \left[\begin{smallmatrix}a\\u\end{smallmatrix}\right]$.

10.4. Playing

Below it will be defined how the game proceeds: which player has to move and what are his/her possible moves. A **move** is a transition from a position p in a game ϕ to a position in some of its subgames (i.e. a game based upon a subformula of ϕ). The possible moves are defined with induction on ϕ. Then the occurrence of $\vee_{/W}$ or $\exists u_{/x}$ in ϕ indicates that the choice of the move has to be made independent of the variables in W. So it is a restriction on the motivation for the choice, but not on the choice itself: it a restriction on possible strategies. Therefore in the definition of possible moves it makes no difference whether $_{/W}$ occurs as subscript or not. The role of $_{/W}$ will be defined when we consider winning strategies in Section 10.6.

The initial position for a formula without free variables is the empty position: no values for free variables. Formulas with free variables arise as subgame of a larger game and then their initial position is inherited. If p in the larger game is defined for more variables than the free variables in the subgame, then it is restricted to those free ones.

The possible moves in position p in game ϕ and the player who has to move are defined by induction on the structure of ϕ:

- Case $\phi \equiv R(v_1, \dots v_n)$ or $\phi \equiv \neg R(v_1, \dots v_n)$
  Here no moves are possible, the game ends.
- Case $\phi \equiv \psi \wedge \theta$
  $\forall$belard chooses L or R. If he chooses L, then game ψ is played from position p_ψ (i.e. the restriction of p to $FV\psi$). Otherwise game θ is played from position p_θ.

* Case $\phi \equiv \forall x \psi$
  $\forall$belard chooses a value for x, say a, and the game proceeds by playing ψ from position $[p * \left[{a \atop x}\right]]_\psi$.
* Case $\phi \equiv \psi \vee \theta$ or $\phi \equiv \psi \vee_{/W} \theta$
  $\exists$loise chooses L or R. If she chooses L, game ψ is played from position p_ψ. If she chooses R, game θ is played from position p_θ.
* Case $\phi \equiv \exists u \psi$ or $\phi \equiv \exists u_{/W}\, \psi$
  $\exists$loise chooses a value for u, say b. Then game ψ is played from position $[p * \left[{b \atop u}\right]]_\psi$.

10.5. Strategies

A strategy for a game (a formula) is a function which for all positions describes which choice $\exists$loise will make. The requirement that C_ϕ does not depend on variables in a set W will be formalized in Section 10.6 by requiring that C_ϕ yields the same choice for all values of variables in W. Therefore there are two types of strategies:

Case $\phi \equiv \psi \vee \theta$ or $\phi \equiv \psi \vee_{/W} \theta$ $\quad C_\phi\colon \mathbf{P}_\phi \to \{L, R\}$

Case $\phi \equiv \exists u \psi$ or $\phi \equiv \exists u_{/W}\, \psi$ $\quad C_\phi\colon \mathbf{P}_\phi \to A$

C_ϕ is undefined for other ϕ 's.

10.6. Winning positions and winning strategies

A winning strategy for a choice is, informally said, a strategy which obeys the conditions discussed before and brings $\exists$loise in a winning position for some subgame. And a winning position is a position where she can follow a winning strategy. As you have notice, the one notion is needed in order define the other. Therefore the notions **winning position** and **winning strategy** (for $\exists$loise) in game ϕ are defined together by induction on the construction of ϕ as follows.

* Case $\phi \equiv R(w_1, w_2, \ldots w_n)$
  Let a_i be the value of w_i in p. Position p is a winning position if the tuple $\langle a_1, \ldots a_n \rangle$ belongs to the interpretation of R.
* Case $\phi \equiv \neg R(w_1, w_2, \ldots w_n)$
  Let a_i be the value of w_i in p. Position p is a winning position if the tuple $\langle a_1, \ldots a_n \rangle$ does not belong to the interpretation of R.
* Case $\phi \equiv \psi \vee \theta$
  The choice is a step towards winning.
  Position p is a winning if there is a strategy C_ϕ with the following property: if $C_\phi(p) = L$ then p_ψ is a winning position in game ψ and if $C_\phi(p) = R$, then p_θ winning position in game θ. A strategy with this property is called a winning strategy.

- Case $\phi \equiv \psi \vee_{/W} \theta$.
  Position p is a winning position if there is a strategy C_ϕ which satisfies the three requirements mentioned below; such a strategy is called a winning strategy.
  1. *The strategy does not have variables in W as argument.*
     If $q \sim_W p$, then $C_\phi(q) = C_\phi(p)$.
  2. *The choice is a step towards winning.*
     If $C_\phi(p) = L$ then position p_ψ is a winning position in game ψ, and if $C_\phi(p) = R$ then p_θ is a winning position in game θ.
  3. *If the values of variables in W are changed, and there is a winning choice, then the same choice is a step towards winning*
     Let $q \sim_W p$ where $q \neq p$. Then at least one of the following cases holds:
     1. Neither q_ψ is a winning position in for game ψ, nor q_θ is a winning position in for game θ
     2. $C_\phi(q) = L(= C_\phi(p))$ and q_ψ is a winning position for game ψ
     3. $C_\phi(q) = R(= C_\phi(p))$ and q_θ is a winning position for game θ.
- Case $\phi \equiv \exists u \psi$
  The choice is a step towards winning.
  Position p is winning if there is b such that $[p * [^b_u]]_\psi$ is a winning position in the game ψ. A strategy is a winning strategy in position p if it yields such a value.
- Case $\phi \equiv \exists u_{/W}\, \psi$
  Position p is winning if there exists a strategy C_ϕ which satisfies the three requirements mentioned below; such strategy is a winning strategy. The value $C_\phi(p)$ is denoted by b.
  1. *The strategy does not have variables in W as argument.*
     If $q \sim_W p$ then also $C_\phi(q) = b$.
  2. *The choice is a step towards winning.*
     $[p * [^b_u]]_\psi$ is a winning position in game ψ.
  3. *If the values of variables in W are changed, and there is a winning choice, then the same choice is a step towards winning.*
     If $q \sim_W p$, where $p \neq q$, and there is a c such that $[q * [^c_u]]_\psi$ is a winning position in game ψ, then in any case $[q * [^b_u]]_\psi$ is a winning position in game ψ.

A sentence ϕ is true if $\exists$loise has a winning strategy for this game starting in the empty position. This means that she has winning strategies for each of the choices she will encounter. If she plays according to the strategy, she will win, whatever ∀belard plays.

11. An example of the formalism

In order to illustrate the definitions from the previous section, we will consider one example in detail: the branching quantifier sentence we have met in (46):

(52) $\forall x \exists u \forall y\, \exists v_{/x,u}[x < u \wedge y < v]$

It will be shown that $u := x + 1$ and $v := y + 2$ form a winning strategy for this game.

Since the definitions work inside-out, we start with the subgame $x < u \wedge y < v$. The winning positions for this game are the positions in $\{\langle x\colon a, y\colon b, u\colon c, v\colon d\rangle \mid a, b, c, d \in \mathbf{N}$ and $a < c$ and $b < d\}$.

For the subgame $\exists v_{/x,u}[x < u \wedge y < v]$ candidates for winning positions are the positions in $\{\langle x\colon a, y\colon b, u\colon c\rangle \mid a, b, c \in \mathbf{N}$ and $a < c\}$; other positions are certainly not winning because no winning moves are possible if $c \leq a$. In all candidate positions the strategy $v := y + 2$ wins (other strategies are possible as well). But is it a winning strategy, i.e. does the strategy also bring ∃loise in a winning position if x is changed? Is it also winning in positions in $\{\langle x\colon a', y\colon b, u\colon c\rangle \mid a', b, c \in \mathbf{N}\}$ where a' is arbitrary? If $a' < c$ the strategy $v := y + 2$ wins indeed, and if $a' \geq c$ no choice at all wins. Hence strategy $v := y + 2$ is winning for all values of x if a winning choice is possible. So it is a winning strategy. This means that all positions in $\{\langle x\colon a', y\colon b, u\colon c\rangle \mid a', b, c \in \mathbf{N}\}$ are winning positions.

Next we consider the subgame $\forall y\, \exists v_{/x,u}[x < u \wedge y < v]$. All positions in $\{\langle x\colon a, u\colon c\rangle \mid a, c \in \mathbf{N}$ and $c > a\}$ are winning because whatever ∀belard plays, ∃loise comes in a winning position.

Finally the subgame $\exists u \forall y\, \exists v_{/x,u}[x < u \wedge y < v]$. Candidates for winning positions are $\{x\colon a \mid a \in \mathbf{N}\}$. Here the strategy $u := x + 1$ is a winning strategy because it brings ∃loise in a winning position, and no further conditions have to be satisfied. So the empty position is a winning position in the original game (52). This means that (52) is true : the game starts in a winning position, and ∃loise has winning strategies for $\exists u$ and $\exists v_{/x,u}$.

12. An application to natural language semantics

One of the earliest applications of game theoretical semantics is the analysis of natural language sentences with branching quantifiers (see Hintikka, 1974). In this section we will compare such an analysis with subgame semantics.

A typical example of a branching quantifier sentence (from Hintikka and Sandu, 1997) is:

(53) Some friend of each townsman and some neighbor of each villager envy each other

Hintikka (1974, p. 167) explains that, in order such sentences to be true, the choice of a neighbor of each villager must not depend on the townsman. This implies, in my opinion, that the choice of the neighbor must be independent of the choice of the friend as well; otherwise there would, through that choice, be an indirect dependency on the townsman.

The IF representation of this sentence, using selfexplaining abbreviations is:

(54) $\forall x \exists u \forall y\, \exists v_{/x}[T(x) \wedge V(y) \rightarrow [F(u,x) \wedge N(v,y) \wedge E(u,v)]]$

As we know, in GTS the slashed quantifier $\exists v_{/x}$ is implicitly slashed for u. In this way the requirement is formalized that the choice of the neighbor depends only on the townsman. In (55), the Skolem form of (54), this is made explicit: the Skolem function g yielding v has only x as argument.

(55) $\exists f \exists g \forall x \forall y[T(x) \wedge V(y) \rightarrow$
$[F(f(x),x) \wedge N(g(y),y) \wedge E(f(x),g(y))]]$

Consider now the following situation. Among the friends of the townsmen two groups are distinguished, viz. male and female ones, and the same among the neighbors of the villagers. Assume now that envying is a relation between all pairs of male friends and male neighbors, and also between female friends and female neighbors, but not between friends and neighbors of different sexes. In this situation the choices of friends for townsmen and neighbors for villagers have to correspond: in both cases male ones, or female ones. In this situation, the choice for the second quantifier ($\exists v_{/x}$) cannot be made independently of the choice for the first ($\exists u$). So in this model sentence (53) should not be true. However, (55) comes out true: take for e.g. f and g functions yielding male friends and neighbors respectively. This shows that the required independence is not captured by the formalizations (55) and (54). It is related to the dependency between existential quantifiers we have seen before on the level of individuals in $\exists u \exists v[u = v]$ and $\exists u\, \exists v_{/u}[u = v]$.

Let us now consider the subgame analysis of (53). In its representation v's independence of u is made explicit:

(56) $\forall x \exists u \forall y\, \exists v_{/x,u}[T(x) \wedge V(y) \rightarrow [F(u, x) \wedge N(v, y) \wedge E(u, v)]]$

Suppose now that in the model under discussion a male friend has been chosen. Then ∃loise must choose a male neighbor as well, say Jacob. And if a female friend had been chosen only the choice for a female neighbor would be winning. But the third condition on winning strategies for $\exists v_{/x,u}$ requires that for a female friend the original choice Jacob would be winning as well ('for other values of x and u the same choice is winning if there is a winning choice'). That is not the case in the given model, so there is no winning strategy for $\exists v_{/x,u}$, and therefore the formula is not true. So for this example subgame semantics gives the desired result, whereas that is not the case for game theoretical semantics.

13. Conclusions

We have investigated Hintikka's game theoretical semantics for IF logic. It turned out that this is not a formalization of 'informational independence', but rather of 'imperfect information'. The logical properties of IF logic are remarkable: $\phi \vee \phi$, $\phi \wedge \phi$, and ϕ are not equivalent, bound variables cannot be renamed, and formulas may get an interpretation different from the classical interpretation. An alternative interpretation for IF logic is proposed which agrees with intuitions about independent choices in logic.

Acknowledgments

The aim of my research was to find a definition of independence which is in agreement with intuitions. This was a sort of empirical enterprise: based upon examples, a definition was designed that dealt with the examples, and seemed plausible. But there were (and will always be) examples I never thought of. I am indebted to colleagues who studied my proposals and came with counterexamples: their reactions made it possible to find improvements of the proposal. I thank in this respect Jouko Väänänen, Wilfrid Hodges, Francien Dechesne, Yde Venema, Dick de Jongh, and two referees who, ipse facto, remain unnamed. If the proposal still is not as is should be, that is of course only my fault.

I especially thank Wilfrid Hodges for many stimulating discussions about trump semantics; without this interaction the ideas in this paper would not have emerged. I thank Johan van Benthem and Gabriël Sandu for organizing a visit to Helsinki; it was a useful push to work again on the subject. Furthermore, I thank Alexandru Baltag, Francien Dechesne, Kees Doets, Valentin Goranko, Joop Niekus, Marc Pauly, Gabriël Sandu, Allard Tamminga, and three referees for their useful remarks.

References

Caicedo, X. and M. Krynicki: 1999, 'Quantifiers for reasoning with imperfect information and Σ_1^1 logic'. In: W. A. Carnielli and I. M. d'Ottaviano (eds.): *Advances in Contemporary Logic and Computer Science. Proc. 11th Brazilian Conf. on Math.Logic 1996*, Vol. 235 of *Contemporary Mathematics*. Americal Mathematical Society, pp. 17–31.

Cook, R. and S. Shapiro: 1998, 'Hintikka's revolution "The Principles of Mathematics Revisited"'. *British Journal for the Philosophy of Science* **49**, 309–316.

Halpern, J.: 1996, 'On ambiguities in the interpretation of game trees'. *Games and Economics Behavior* **20**, 66–96.

Hintikka, J.: 1974, 'Quantifiers vs. Quantification theory'. *Linguistic Inquiry* **5**, 153–177.

Hintikka, J.: 1996, *The principles of mathematics revisited.* Cambridge: Cambridge University Press.

Hintikka, J. and G. Sandu: 1997, 'Game-theoretical semantics'. In: J. van Benthem and A. ter Meulen (eds.): *Handbook of logic and language.* Elsevier, Amsterdam and The MIT Press, Cambridge, Mass., Chapt. 6, pp. 361–410.

Hodges, W.: 1997a, 'Compositional semantics for a language of imperfect information'. *Journal of the IGPL* **5**(4), 539–563.

Hodges, W.: 1997b, 'Some Strange Quantifiers'. In: J. Mycielski et al. (ed.): *Structures in Logic and Computer Science*, No. 1261 in Lecture notes in computer science. Berlin: Springer, pp. 51–65.

Janssen, T.: 1997, 'A compositional semantics for the game-theoretical interpretation of logic'. In: P. Dekker, M. Stokhof, and Y. Venema (eds.): *Proceedings of the Eleventh Amsterdam Colloquium, dec. 1997.* pp. 181–185, ILLC/ Dept of Philosophy, University of Amsterdam.

Janssen, T.: 1999, 'IF logic and informational independence'. In: P. Dekker (ed.): *Proceedings of the Twelfth Amsterdam Colloquium, dec. 1999.* pp. 139–144, ILLC/ Dept of Philosophy, University of Amsterdam.

Pietarinen, A.: 2001, 'Varieties of IFing'. In: *Proceedings of the ESSLLI'01 Workshop on Logic and Games.* Dept. of Philosophy. University of Helsinki.

Pietarinen, A.: 2002, 'Games logic plays. Informational independence in game theoretical semantics'. Ph.D. thesis, School of cognitive and computing sciences, University of Sussex. Preliminary version.

Sandu, G.: 1993, 'On the logic of informational independence and its applications'. *Journal of Philosophical Logic* **22**, 26–60.

de Swart, H., T. Verhoeff, and R. Brands: 1997, 'Hintikka's "The principles of mathematics revisited"'. *Logique et Analyse* **159**, 281–289.