Cursushandleiding Wiskunde Blok 2

CK1W0006

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Inhoudsopgave

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1 Literatuur

De bij het vak gebruikte literatuur is een syllabus waarvan er elke week een nieuw hoofdstuk op de site verschijnt.

2 Indeling

Hoorcolleges: Donderdag 11:00 - 12:45
Werkcolleges: Dinsdag 13:15 - 15:00 .

De indeling van de drie werkcollegegroepen is als volgt:
groep 1 (Max Knobbout - Ruppert 117): achternaam A-G.
groep 2 (Jelle Don- Ruppert 135): achternaam H-M.
groep 3 (Rik Jansen - Ruppert 136): achternaam N-Z.

3 Inleveropgaven

Elke week is er een inleveropgave die aan het begin van het werkcollege ingeleverd dient te worden bij de docent die jouw werkcollege verzorgt. Indien je het wekcollege niet bij kunt wonen, mail de opgave dan voor aanvang naar de docent die jouw werkcollege verzorgt. In totaal zijn er zeven inleveropgaven.

4 Tentamen en cijfer

Toets blok 2: Maandag 25 Januari 15.30 - 18.30 uur in Educatorium, Megaron
Er is in Februari een herkansing, waarbij beide toetsen apart of beide herkanst kunnen wroden. Datum en plaats worden nog bekend gemaakt.

De twee deelcijfers (blok 1 en blok 2) zijn beide als volgt samengesteld:

$$(\text{cijfer toets}) \cdot 0.9 + (\text{gemiddelde inleveropgaven}) \cdot 0.1.$$

De inleveropgaven worden alleen meegeteld indien er per blok voor minimaal 5 van de 7 inleveropgaven een voldoende is behaald. Het eindcijfer is het gemiddelde van de twee deelcijfers. Het vak is gehaald indien het eindcijfer, zowel als de beide deelcijfers minimaal een 6 zijn.

Bij de herkansing tellen de inleveropgaven op dezelfde wijze mee, mits aan bovenstaande voorwaarde is voldaan.

On sets, functions and relations

Chapter 1

Rosalie Iemhoff

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Introduction

Sets, functions and relations are some of the most fundamental objects in mathematics. They come in many disguises: the statement that 2+3=5 could be interpreted as saying that a set of two elements taken together with a set of three elements results in a set of five elements; it also means that the function +, when given the input 2 and 3, outputs the number 5; in saying that the probability of the number 2 is 1/6 when throwing a dice, one states that the set of outcomes of throwing a dice has six elements that occur with equal probability. In other settings the presence of sets, functions or relations is more evident: every polynomial is a function, in analysis one studies functions on the real numbers, in computer science functions play an essential role, as the notion of an algorithm is central in the field.

In these notes we will study some elementary properties of sets, functions and relations. Although this exposition will be mainly theoretical, it is always instructive to keep in mind that through the study of these basic notions one obtains knowledge about the subjects in which these notions play a role, as e.g. in the examples above.

1 Sets

In this section the properties of sets will be studied. We start with the informal but intuitive notion of what a set is and what it means for an element to belong to a set, without describing it formally. This is not to say that one cannot approach the subject more precisely, but such an approach is related to many deep and complex problems in mathematics and its foundations, and therefore falls outside the scope of this exposition.

Taken that one has an intuition about what these two undefined notions set and membership are, one can, surprisingly enough, build all of mathematics on these two notions. That is, all the mathematical objects and methods can, at least in principle, be cast in terms of sets and membership, not using any other notions.

What is the intuition behind sets and their elements? In general, a set consists of elements that share a certain property: the set of tulips, the set of people who were born in July 1969, the set of stars in the universe, the set of real numbers, the set of all sets of real numbers. A special set is the empty set, that is the set that does not contain any elements. In contrast, do you think a set containing everything exists?

1.1 Notation

Sets are denoted by capitals, often X, Y or A, B, the elements of sets by lower case letters. $x \in X$ means that x is an element of the set X. Sets can be given by listing their elements: $\{0, 1, 2, 3, 4\}$ is the set consisting of the five elements 0, 1, 2, 3 and 4; $\{a, 7, 000\}$ consists of the elements a, 7, and 000. The elements of a set do not have to have an order and do not occur more than once in it: thus $\{1, 2, 1\}$ is the same set as $\{2, 1\}$.

Sometimes we cannot list the elements of a set and have to describe the set in another way. For example: the set of natural numbers; the set of all children born on July 12, 1969. Of course, the elements of the latter could be listed in principle, but it is much easier to describe the set in the mentioned way. Even sets of one element can be difficult to list, such as the set consisting of the 2^{1000}th digit of π. It has one elements, but we cannot compute it fast enough to know the answer before the end of time. Sets given by descriptions are often denoted as follows:

$$\{n \in \mathbb{N} \mid \ n \text{ is an even natural number}\},$$

$$\{p \in \mathbb{N} \mid \ p \text{ is a prime number}\}.$$

Thus given a set A

$$\{x \in A \mid \varphi(x)\}$$

denotes the set of elements of A for which φ holds. Thus the symbol $\mid$ can be read as "for which". Here φ is a property, which in a formal setting is given by a predicate formula and in an informal setting by a sentence.

The set $\{x \in A \mid \varphi(x)\}$ can also be denoted as $\{x|x \in A, \varphi(x)\}$. I have a slight preference for the first option, but the second one is correct too.

Example 1 1. $\{n \in \mathbb{N} \mid\ n$ is odd $\}$ is the set of odd numbers, and whence is the same as the set $\{n \in \mathbb{N} \mid \exists m\,(n = 2m+1)\}$.

2. $\{w \mid w$ is a sequence of 0's and 1's which sum is 2 $\}$ is a set, the same set as $\{w \mid w$ is a sequence of 0's and 1's containing exactly two 1's $\}$.

3. $\{\varphi \mid \varphi$ is a propositional tautology $\}$ is the set of propositional formulas that are true.

Some specific sets that one should know:

$\mathbb{N}$	the set of natural numbers $\{0, 1, 2, \dots\}$ (de natuurlijke getallen)
$\mathbb{Z}$	the set of integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$ (de gehele getallen)
$\mathbb{Q}$	the set of rational numbers (de rationale getallen)
$\mathbb{R}$	the set of real numbers (de reële getallen)
$\mathbb{C}$	the set of complex numbers (de complexe getallen)
$\mathbb{R}\backslash\mathbb{Q}$	the set of irrational numbers (de irrationale getallen)
$\emptyset$	the empty set.

For any number n, $\mathbb{N}_{\geq n}$ denotes the set $\{n, n+1, \dots\}$, and $\mathbb{N}_{>n}$ denotes the set $\{n+1, \dots\}$, and similarly for the other sets in the list above. $\mathbb{N}_{\geq 1}$, or equivalently $\mathbb{N}_{>0}$, is sometimes denoted by $\mathbb{N}^+$. For a finite set X, $|\,X\,|$ denotes the number of elements of X. A set that consists of one element is called a *singleton*. Thus $\{0\}$ is a singleton, and so is $\{\emptyset\}$.

1.2 Careful

We have to be careful with the $\{\dots\}$-notation. Consider the *Russell set*

$$R = \{x \mid x \text{ is a set}, x \notin x\}.$$

Thus R consists of the sets that are not an element of itself. Does R belong to this set (itself) or not? If it does, thus if $R \in R$, then, by definition of R, also $R \notin R$. This cannot be, and thus we conclude that $R \notin R$. But then, by the definition of R, also $R \in R$. This cannot be either. Our only conclusion can be that R itself is not a set! Intriguing as this example might be, we will in the following always remain on safe ground and not consider pathological cases like this one. In mathematics, in the field called *set theory*, the problem can be dealt with in a precise and satisfactory way.

1.3 Operations on sets

These are three standard operations on sets that often occur:

$A \cap B$	$=$	$\{x \mid x \in A \text{ and } x \in B\}$	intersection (doorsnede)
$A \cup B$	$=$	$\{x \mid x \in A \text{ or } x \in B\}$	union (vereniging)
$A\backslash B$	$=$	$\{x \in A \mid x \notin B\}$	difference (verschil of complement)

Two sets A and B are *disjunct* if $A \cap B = \emptyset$. The following abbreviation is often used in the context of infinite sets:

$$\bigcup A = \{x \mid \exists a \in A(x \in a)\}.$$

Thus for a set $A = \{a, b\}$, $\bigcup A = a \cup b$. For an infinite set $A = \{a_1, a_2, a_3, \dots\}$, $\bigcup A = a_1 \cup a_2 \cup \dots$, which is also written as $\bigcup A = \bigcup_{i=1}^{\infty} a_i$.

1.4 Subsets

$X \subseteq Y$ means that X is a *subset* of Y, i.e. every element of X is an element of Y. Thus

$$X \subseteq Y \Leftrightarrow \forall x\,(x \in X \Rightarrow x \in Y).$$

We write $X \subset Y$ if $X \subseteq Y$ and $X \neq Y$, and $X \not\subseteq Y$ if X is not a subset of Y. Thus $\{1, 2\} \subset \{1, 2, 3, 4\}$, and $\mathbb{N} \subset \mathbb{Z}$. There is another important operation on sets, namely the set of all subsets of a set, the so-called *powerset (machtsverzameling)* of a set:

$$P(Y) = \{X \mid X \subseteq Y\}.$$

Example 2

1. $P(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.
2. $P(\mathbb{R})$ is the set of sets of real numbers. E.g. $\{1, \pi, -72\} \in P(\mathbb{R})$.
3. $P(\{a\}) = \{\emptyset, \{a\}\}$.

$$P(P(\{a\})) = \{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\emptyset, \{a\}\}\}.$$

4. What is $P(\emptyset)$?

Theorem 1 If $X \subseteq Y$, then $P(X) \subseteq P(Y)$.

Proof Suppose $X \subseteq Y$ and consider $x \in P(X)$. Thus $x \subseteq X$. Since $X \subseteq Y$, also $x \subseteq Y$. Hence $x \in P(Y)$. This proves that $P(X) \subseteq P(Y)$. ♡

Theorem 2 For finite sets X:

$$|\,P(X)\,| = 2^{|\,X\,|}.$$

Proof Consider a set X with n elements. Put the elements of X in a certain order, it does not matter which, say $X = \{x_1, \dots, x_n\}$. There is a correspondence between sequences of 0's and 1's of length n, and subsets of X. Given a sequence $i_1, \dots, i_n$ of 0's and 1's, let it correspond to the subset X consisting of exactly those x_{i_j} for which $i_j = 1$, for $1 \leq j \leq n$. Note that every sequence correponds to a unique subset of X and vice versa. There are 2^n such sequences, and thus as many subsets of X. ♡

1.5 The natural numbers

The set ω is the smallest set closed under the following operation:

$\emptyset \in \omega$,

if $x \in \omega$, then $(x \cup \{x\}) \in \omega$.

Thus the first four elements of ω are

$$\emptyset \quad \{\emptyset\} \quad \{\emptyset, \{\emptyset\}\} \quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}.$$

If we interpret $\emptyset$ as 0, and $x \cup \{x\}$ as $x + 1$, then we could view ω as a representation of $\mathbb{N}$ with $+$.

1.6 The Cantor set

Not all sets are easy to visualize. This is an example of an intriguing set: it is the result of the following process:

start with the interval [0,1] and delete the open middle third $(\frac{1}{3}, \frac{2}{3})$,

from the remaining line fragments delete the open middle third, and repeat this process indefinitely.

Thus after the first step remain $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. After the second step remain $[0, \frac{1}{9}]$ and $[\frac{2}{9}, \frac{3}{9}]$, $[\frac{6}{9}, \frac{7}{9}]$ and $[\frac{8}{9}, 1]$, and so on. The Cantor set has many special properties, which we will encounter in the next chapters. Which elements certainly belong to the set?

1.7 Exercises

1. Write in set-notation the set of numbers that are squares of natural numbers.
2. Write in set-notation the set of all vowel letters.
3. Give three set-notations for the set of non-negative integers divisable by 3.
4. Describe in words the set $\{x \in \mathbb{Q} \mid 0 < x < 1\}$.
5. Describe in words the set $\{x \in \mathbb{R} \mid \exists y \in \mathbb{Q}(x = y^2)\}$.
6. Describe in words the set $\{x \in \mathbb{R} \mid \exists y \in \mathbb{R}(x = y^2 \ \wedge \ y > 2)\}$.
7. How many elements has the set $\{x \in \mathbb{R} \mid x^2 = x\}$? Give a different set-notation for the set.
8. Give an example of a set for which it is (at present) difficult to decide if it is empty or not.

9. Does $\{0\} \in \mathbb{N}$ hold? How many elements has $\{\{\mathbb{N}\}\}$?
10. Which set is $\{n \in \mathbb{N} \mid n^2 > n\}$?
11. Does $\{\{x, y\} \mid x, y \in \mathbb{N}_{>0}, \frac{x}{y} = \frac{y}{x}\} = \{\{n\} \mid n \in \mathbb{N}\}$ hold? Prove your answer.
12. Does $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ hold? Prove your answer.
13. Does $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ hold? Prove your answer.
14. If you read $\vee$ for $\cup$ and $\wedge$ for $\cap$, and A, B, C are propositional formulas, are 10 and 11 true under this interpretation?
15. Does $C \backslash (A \cap B) = (C \backslash A) \cup (C \backslash B)$ hold? Prove your answer.
16. Does $C \backslash (A \cup B) = (C \backslash A) \cap (C \backslash B)$ hold? Prove your answer.
17. Is $X \backslash Y$ equal to $Y \backslash X$? Prove your answer.
18. Prove that $X \subseteq Y$ and $Y \subseteq Z$ implies $X \subseteq Z$.
19. Write down the subsets of $\{1, 2, 3, 4\}$.
20. Given any set A, is $\emptyset$ a subset of A?
21. What are the subsets of $\{x, y\}$? And of $\{x\}$?
22. Does $\{x \in \mathbb{R} \mid \exists y(x = y^2)\} = \mathbb{R}_{\geq 0}$ hold? And $\{x \in \mathbb{Q} \mid \exists y(x = y^2)\} = \mathbb{Q}_{\geq 0}$?
23. Are there sets $A_1, A_2, A_3, \ldots$ such that $A_1 \supset A_2 \supset A_3 \supset \ldots$? Here $A \supset B$ means that B is a subset of A and not equal to it.
24. Do there exist real numbers r, s such that $r < s$ and the interval $[r, s]$ is a subset of $\mathbb{R} \backslash \mathbb{Q}$? Prove your answer.
25. Do there exist real numbers r, s such that $r < s$ and the interval $[r, s]$ is a subset of $\mathbb{R} \backslash \{x \in \mathbb{R} \mid$ the decimal expansion of x does not contain the string $11\}$? Prove your answer.
26. Give an argument that in some sense shows that the sets $\{\{n\} \mid n \in \mathbb{N}\}$ and $\mathbb{N}$ have "the same size".
27. Describe $\bigcup \omega$.
28. Does there exist a subset of the natural numbers X for which $X \in X$? Prove your answer.
29. Does there exist sets of real numbers A and B such that $\{A\} \in B$ and $B \subseteq A$? Prove your answer.
30. If for sets $A_1, A_2, A_3, \ldots$ it is given that $A_i \in \{A_{i+1}\}$, what can you say about the sets?

31. If for sets $A_1, A_2, A_3, \ldots$ it is given that $A_1 = \{1\}$ and $A_{i+1} = \{A_1, \ldots, A_i\}$, and $A = \bigcup_{i=1}^{\infty} A_i$, does it hold that for every A_i that if $x \in y \in A_i$, then $x \in A_i$ or $x = 1$? Prove your answer.

32. Give $P(\emptyset)$.

33. Given a set X, is $\emptyset$ an element of $P(X)$? And is $X \in P(X)$?

34. Write down the subsets of $\{1, 2, 3, 4\}$.

35. How many subsets has $\{n \in \mathbb{N} \mid 0 \leq n \leq 5\}$? And $\mathbb{N}$?

36. Give $P(\{a, b\})$ and $P(P\{a, b\}))$.

37. What are the elements of $P(\mathbb{R} \backslash \mathbb{Q})$? What are the elements of $P(\mathbb{R}) \backslash \mathbb{Q}$?

38. Prove that if $X \neq Y$, then $P(X) \neq P(Y)$.

On sets, functions and relations

Chapter 2

Rosalie Iemhoff

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1 Relations

The elements of a set are not ordered. That is, $\{1, 2\}$ is the same set as $\{2, 1\}$. One sometimes calls such sets with two elements an *unordered pair*. In this section the notion of relation is introduced, which are sets with an order structure.

It is instructive to first consider the use of the word relation in daily speach. In the following sentences the word occurs explicitly: John has a relationship with Mary. There is a relation between mass and force. From these examples one can conclude that a relation, in many cases, is a "something" between two things. In these sentences the relation is implicit: John loves Mary, I have read Tolstoy's War and Peace. Here "to love" is a relation and so is "have read". These examples show that a relation is not necessarily symmetric: it might be that John loves Mary but she does not love him. You read these notes, but they do not read you.

On a more formal level we define an *ordered pair* to be a pair of two elements, denoted by $\langle a, b\rangle$. We want to cast it in terms of sets, those being our building blocks for the other mathematical notions. Therefore, we define

$$\langle a, b\rangle =_{def} \{\{a\}, \{a, b\}\}.$$

Note that this is a definition. Thus one has to verify that it has the properties ones wishes an ordered pair to have. It does: from $\{\{a\}, \{a, b\}\}$ we can read of which element is the first of the ordered pair, a, and which is the second, b. And

$$\langle a, b\rangle = \langle c, d\rangle \Leftrightarrow (a = c \text{ and } b = d).$$

A *binary relation* is a set of ordered pairs. Often we leave out the word binary. Note that a relation is a set which elements are of a special form. Clearly, $\{\langle 1, 2\rangle, \langle 3, 4\rangle\}$ is a relation, a relation consisting of two pairs. And so is $\{\langle a, b\rangle, \langle b, a\rangle\}$. The relation

$$\{\langle i, j\rangle \mid i, j \in \mathbb{R}, i < j\}$$

consists of all pairs of real numbers for which the second element is larger than the first element. And

$$\{\langle i, j\rangle \mid i, j \in \mathbb{N}, \exists n \in \mathbb{N}(i^j = 2n)\}$$

is the relation consisting of all pairs of natural numbers such that the first number to the power of the second number is even. Unwinding the definition of ordered pair one readily sees that e.g. $\{\langle 1, 2\rangle, \langle 3, 4\rangle\}$ is short for

$$\{\langle 1, 2\rangle, \langle 3, 4\rangle\} = \{\{\{1\}, \{1, 2\}\}, \{\{3\}, \{3, 4\}\}\}.$$

(So it is clear why we stick to the $\langle\rangle$ notation ...) Note that a subset of a relation is also a relation. We sometimes write xRy for Rxy, and $xRyRz$ for (xRy and yRz).

Here follow some definitions of important relations.

$$\leq_{\mathbb{N}} = \{\langle m, n\rangle \in \mathbb{N}^2 \mid m \leq n\} \qquad <_{\mathbb{N}} = \{\langle m, n\rangle \in \mathbb{N}^2 \mid m < n\}.$$

Similar notions we define for $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. When R is a relation on a set A, i.e. $R \subseteq A \times A$, we sometimes denote it as (A, R) to stress that it is a relation on A. Thus $\leq_{\mathbb{N}}$ denotes the same relation as $(\mathbb{N}, \leq)$, etc.

Example 1 The following are examples of relations.

1. $\leq_{\mathbb{N}}$.
2. The set $\{\langle a, b\rangle \mid$ a is the husband of b $\}$ is a relation on the set of human beings.
3. $\{\langle q, r\rangle \in \mathbb{Q}_{\geq 0} \times \mathbb{R} \mid \sqrt{q} = r\}$.
4. $\{\langle q, r\rangle \in \mathbb{Q} \times \mathbb{R} \mid (q \geq 0$ and $\sqrt{q} = r)$ or $(q < 0$ and $r = 0)\}$.
5. $\{\langle w, n\rangle \mid$ w is a sequence of n 0's and n 1's $\}$ is a relation.
6. $\{\langle \varphi, \psi\rangle \in \mathcal{P}^2 \mid \varphi \leftrightarrow \psi$ is a tautology $\}$ is a relation on the set of propositional formulas $\mathcal{P}$.

Since $\{x\} \subseteq \{x, y\}$, $\{x\} \in P(\{x, y\})$. Also, $\{x, y\} \in P(\{x, y\})$. Thus

$$\{\{x\}, \{x, y\}\} \subseteq P(\{x, y\}).$$

That is, $\langle x, y\rangle \subseteq P(\{x, y\})$. Whence

$$\langle x, y\rangle \in P(P(\{x, y\})).$$

1.1 Cartesian product

The *cartesian product* of two sets A and B is the set

$$A \times B =_{def} \{\langle a, b\rangle \mid a \in A, b \in B\}.$$

Note that the cartesian product of two sets is a relation. We often write A^2 for $A \times A$.

Example 2
1. $\{1, 2\} \times \{3, 4\} = \{\langle 1, 3\rangle, \langle 1, 4\rangle, \langle 2, 3\rangle, \langle 2, 4\rangle\}$.
2. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ can be interpreted as the set of coordinates of points in the plane.
3. $\{\langle x, y\rangle \mid x \in A, y \in A, x \in y\}$ is a relation on A^2.

1.2 Disjoint sum

The *disjoint sum* of two sets A and B is the set

$$A + B =_{def} \{\langle a, 0\rangle \mid a \in A\} \cup \{\langle b, 1\rangle \mid b \in B\}.$$

The disjoint sum is used when one wants to consider the elements of two sets, but without losing track of which element belongs to which set.

1.3 Relations of arbitrary arity

Above we saw relations between two elements. Examples of relations of arity greater than two are: "being the mother and the father of", i.e. the relation consisting of triples $\langle a, b, c\rangle$ such that a is the mother and b is the father of c; the relation $\{\langle n, m, k\rangle \in \mathbb{N} \mid n+m = k\}$; the relation R of five-tuples $\langle a, b, c, d, e\rangle$ of letters such that *abcde* is a word in the Dutch language. ($\langle r, a, d, i, o\rangle$ belongs to R, and so does $\langle h, a, l, l, o\rangle$, but $\langle h, e, r, f, s\rangle$ does not). There are various ways to define relations of arbitrary arity in terms of sets, e.g.

$$\langle a, b, c\rangle =_{def} \langle a, \langle b, c\rangle\rangle \qquad \langle a, b, c, d\rangle =_{def} \langle a, \langle b, c, d\rangle\rangle,$$

etc. We define relations of arity n inductively as follows

$$\langle a_1, \ldots, a_{n+1}\rangle = \langle a_1, \langle a_2, \ldots, a_{n+1}\rangle\rangle.$$

In the same way as above one can then show that

$$\langle a_1, \ldots, a_n\rangle = \langle b_1, \ldots, b_n\rangle \;\Leftrightarrow\; \forall i \leq n(a_i = b_i).$$

Expressions $\langle a_1, \ldots, a_n\rangle$ are called *n-tuples*. A set consisting of n-tuples is an *n-ary relation*. As mentioned above, a set of pairs we also call a *binary* relation. We define

$$A^n = \{\langle a_1, \ldots, a_n\rangle \mid \forall i \leq n(a_i \in A)\}.$$

Example 3

1. $\{\langle n, m, k\rangle \in \mathbb{N} \mid n \cdot m = k\}$ is a 3-ary relation.
2. $\{\langle a_1, \ldots, a_n\rangle \in \mathbb{R}^{n+1} \mid a_1 + a_2 + \ldots a_n = 0\}$ is a $(n + 1)$-ary relation.
3. $\{\langle \varphi, \psi, \chi\rangle \in \mathcal{P}^3 \mid \varphi \wedge \psi \to \chi$ is a tautology$\}$ is a 3-ary relation.

1.4 Pictures

There is an elegant way of depicting binary relations on a set, i.e. relations $R \subseteq A^2$. We draw Rxy as

$$x \longrightarrow y$$

If both xRy and yRx hold we draw in one of the three following ways:

$$\begin{tikzcd} x \arrow[r, bend left] & y \arrow[l, bend left] \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, leftrightarrow] & y \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, dash] & y \end{tikzcd}$$

Thus the picture that corresponds to the relation $\{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle\}$ is

$$\begin{tikzcd} 1 \arrow[rr] & & 2 \\ & 0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

And $\{\langle 0,0\rangle, \langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle, \langle 2,1\rangle\}$ corresponds to the picture

$$\begin{tikzcd} 1 \arrow[rr, leftrightarrow] & & 2 \\ & 0 \arrow[ul] \arrow[ur] \arrow[loop below] & \end{tikzcd}$$

Using suggestive dots we can also draw infinite relations, e.g.

$$\{\langle 0,1\rangle, \langle 0,2\rangle, \langle 0,3\rangle, \dots\} \cup \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,4\rangle, \dots\},$$

with picture

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & {} \arrow[r, dotted] & \cdots \\ & 0 \arrow[ul] \arrow[u] \arrow[ur] \arrow[urr, dotted] \arrow[urrr, dotted] & & & \end{tikzcd}$$

The relation $\{0,1,2\}^2 \backslash \{\langle x,x\rangle \mid x \in \{0,1,2\}\}$ corresponds to this picture:

$$\begin{tikzcd} 0 \arrow[rr, dash] & & 1 \\ & 2 \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

1.5 Properties of relations

Here follow the names and corresponding pictures of some important properties of relations. Dotted arows denote the arrows that have to exist given the solid arrows.

reflexive $\quad \forall w(wRw)$

$$\begin{tikzcd} w \arrow[loop above] \end{tikzcd}$$

transitive $\quad \forall w \forall v \forall u(wRv \wedge vRu \to wRu)$

$$\begin{tikzcd} w \arrow[r] \arrow[rr, dotted, bend right] & v \arrow[r] & u \end{tikzcd}$$

symmetric $\quad \forall w \forall v(wRv \to vRw)$

$$\begin{tikzcd} w \arrow[r, bend left] & v \arrow[l, dotted, bend left] \end{tikzcd}$$

euclidean $\quad \forall w \forall v \forall u(wRv \wedge wRu \to vRu)$

$$\begin{tikzcd} v \arrow[rr, dotted] & & u \\ & w \arrow[ul] \arrow[ur] & \end{tikzcd}$$

dense $\quad \forall w \forall v(wRv \to \exists u(wRu \wedge uRv))$

$$\begin{tikzcd} w \arrow[rr] \arrow[dr, dotted] & & v \\ & u \arrow[ur, dotted] & \end{tikzcd}$$

The following properties are a bit harder to draw. Therefore only their descriptions are given.

antisymmetric	$\forall x \forall y(xRy \wedge yRx \to x = y)$
weakly connected	$\forall x \forall y(xRy \vee yRx \vee x = y)$
partial order	reflexive, transitive and antisymmetric
total (linear) order	weakly connected partial order
equivalence relation	reflexive, transitive and symmetric
serial	$\forall x \exists y(xRy)$
completely disconnected	$\forall x \forall y \neg(xRy)$

A remark on notation. Although elements of (the pairs in) relations always range over sets, these sets are not mentioned in case they are clear from the context. For example, a relation R on a set A is reflexive if $\forall w \in A(wRw)$. A relation on the cartesian product $A \times B$ is serial if $\forall w \in A \exists v \in B(wRv)$, and so on.

Example 4 1. $\leq_{\mathbb{N}}$ is a reflexive, transitive, linear, antisymmetric, and dense relation: $n \leq n$ (reflexivity); $k \leq n \leq m$ implies $k \leq m$ (transitivity); $n \leq m$ or $m \leq n$ or $n = m$ (linearity); $n \leq m \wedge m \leq n$ implies $m = n$ (antisymmetry); if $n \leq m$, then $n \leq n \leq m$ (dense).

2. Note that $<_{\mathbb{N}}$ has the same properties as $\leq_{\mathbb{N}}$ except that it is not reflexive (since not $n < n$) and not dense, since $1 < 2$ but there is no $n \in \mathbb{N}$ such that $1 < n < 2$.

3. $<_{\mathbb{R}}$ is dense, and so is $<_{\mathbb{Q}}$.

4. The relation "eat" has none of the above mentioned properties.

5. The relation "to meet" between human beings is symmetric.

6. The relation $\{\langle r, s\rangle \in \mathbb{R}^2 \mid x^2 = y\}$ is not linear.

7. $P(A)$ with the relation $\subseteq$ is a partial order. You will be asked to prove this in the exercises.

8. $\in$ is transitive on ω. You will be asked to prove this in the exercises.

1.6 Equivalence relations

Equivalence relations are used in settings where we have a notion of "sameness" and want to treat all elements of a set that are the "same" as identical. The rationals numbers are an example: although given by different expressions, the numbers $\frac{2}{7}$ and $\frac{4}{14}$ are considered to be "the same", as are -1 and $\frac{-9}{9}$, and so on.

We first discuss equivalence relations in general, and then return to $\mathbb{Q}$. Equivalence relations give rise to a partition of a set in the following way. If R is an equivalence relation on a set A, then we define the *equivalence class* of an element a as the set $\{b \in A \mid Rab\}$, and denote it by $[a]_R$, or by $[a]$, when R is clear from the context. Note that because R is symmetric, $[a]$ is the same set as $\{b \in A \mid Rba\}$. Because R is reflexive, $a \in [a]$. The set $\{[a] \mid a \in A\}$ is denoted by A/R.

Example 5 1. $\{\langle a, b\rangle \mid$ persons a and b have the same birthday $\}$ is an equivalence relation.

2. $\{\langle w, v\rangle \mid w, v$ are sequences of 0's and 1's of the same length $\}$ is an equivalence relation.

3. $\{\langle \varphi, \psi\rangle \in \mathcal{P}^2 \mid \varphi \leftrightarrow \psi$ is a tautology $\}$ is an equivalence relation.

Theorem 1

$$\forall b \in [a] : \ [a] = [b].$$

Proof If $b \in [a]$ then Rab, and thus Rba since R is symmetric. We show that $[a] = [b]$. First, if $c \in [b]$, then Rbc. Since also Rab, Rac follows by transitivity, and thus $c \in [a]$. Second, if $c \in [a]$, then Rac. Since also Rba, Rbc follows by transitivity, and thus $c \in [b]$. This proves that $[a] = [b]$. ♡

Theorem 2

$$\forall b \notin [a] : \ [a] \cap [b] = \emptyset.$$

Proof If there would be a c in $[a] \cap [b]$, then Rac because $c \in [a]$ and Rcb because $c \in [b]$. Hence Rab by transitivity, and thus $b \in [a]$. ♡

From the two theorems above it follows that

Corollary 1 The set $\{[a] \mid a \in A\}$ is a partitioning of A into disjoint sets given by the equivalence classes.

Example 6 Given the equivalence relation

$$R = \{\langle \varphi, \psi \rangle \in \mathcal{P}^2 \mid \varphi \leftrightarrow \psi \text{ is a tautology } \},$$

$\mathcal{P}/R$ is a partitioning of the set of formulas into classes of equivalent formulas.

1.7 The rational numbers

The rational numbers $\mathbb{Q}$ can be represented in an elegant way using the following equivalence relation on $\mathbb{Z} \times \mathbb{N}_{>0}$ R:

$$\langle x, y \rangle R \langle a, b \rangle \ \Leftrightarrow \ xb = ya.$$

Thus, $\langle x, y \rangle$ represents $\frac{x}{y}$. In the exercises you will be asked to show that R is indeed an equivalence relation. An instance of the relation is for example $\langle 2, 7 \rangle R \langle 4, 14 \rangle$, and a non instance is that not $\langle -5, 8 \rangle R \langle 1, 1 \rangle$. In terms of equivalence classes this can be expressed as

$$[\langle 2, 7 \rangle] = [\langle 4, 14 \rangle] \qquad [\langle -5, 8 \rangle] \neq [\langle 1, 1 \rangle].$$

The rationals $\mathbb{Q}$ can now be represented as

$$\mathbb{Z} \times \mathbb{N}_{>0}/R.$$

1.8 Exercises

1. Give a set-notation for the relation of pairs of reals for which the second element is the square of the first.

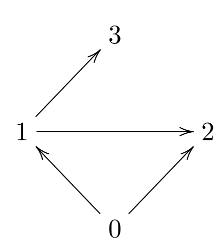
2. Write down the elements of $\{a, b\} \times \{a, c, d\}$.

3. Given two finite sets A and B, how many elements have $A \times B$ and $A + B$? What can you say about the number of elements in $A \cup B$?
4. Which subset of the plane $\mathbb{R}^2$ is the set $\{\langle x, y\rangle \in \mathbb{R}^2 \mid x = y\}$?
5. Is the relation $\{\langle r, s\rangle \in \mathbb{R}^2 \mid (r + s) \in \mathbb{Q}\}$ symmetric? Is it linear?
6. Is the relation $\{\langle n, m\rangle \in \mathbb{Z}^2 \mid n^2 = m\}$ dense?
7. Show that for euclidean relations $\forall x \forall y \forall z(Rxy \wedge Rxz \rightarrow Ryz \wedge Rzy)$ holds.
8. Write down in set notation the relation consisting of the 3-tuples $\langle x, y, z\rangle \in \mathbb{Z}^3$ such that $x^2 + y^2 = z^2$. Which arity has this relation? Give two elements of the relation.
9. Prove that the relation that is the cartesian product $A \times B$ of two sets is serial if and only if B is not empty or A is empty.
10. Prove that the relation that is the cartesian product $A \times B$ of two sets is symmetric if $A = B$.
11. In which cases is A equal to A^2?
12. Prove that the relation $\{\langle x, y\rangle \in \mathbb{R}^2 \mid x^2 = y\}$ is not total on $\mathbb{R}$.
13. Given a relation $R \subseteq A^2$, $R_{\upharpoonright B}$ denotes the restriction of R to B: $R_{\upharpoonright B} = \{\langle x, y\rangle \mid \langle x, y\rangle \in R, x \in B, y \in B\}$. A property is called *subset-hereditary* if whenever R has a property, then so does $R_{\upharpoonright B}$ for all subsets B of A. Which of the properties given in Section 1.5 are subset-hereditary, and which are not? In the last case, provide counter examples.
14. Prove that $\langle a, b\rangle = \langle c, d\rangle$ if and only if $a = c$ and $b = d$.
15. Why would $\{a, b\}$ not be a useful definition for an ordered pair $\langle a, b\rangle$? What about the definition $\{\{a\}, \{b\}\}$?
16. Is $\{a\} \in \{\langle a, b\rangle\}$? Is $\{b\} \in \{\langle a, b\rangle\}$?
17. Is $\{\langle 1, 2\rangle\} \subseteq \mathbb{N}$? Is $\{\langle 1, 2\rangle\} \subseteq P(\mathbb{N})$?
18. Show that the relation R on $\mathbb{Z} \times \mathbb{N}_{>0}$ given by

$$\langle x, y\rangle R \langle a, b\rangle \;\Leftrightarrow\; xb = ya,$$

that was used to construct $\mathbb{Q}$, is an equivalence relation.

19. Is the relation given by the picture euclidean?



Which arrows have to be added to make it a transitive relation?

20. Is the following relation dense? Serial?

$$
\begin{tikzcd}
a_1 \arrow[rr] \arrow[dr] & & a_2 \arrow[rr] \arrow[dr] & & a_3 \arrow[rr] \arrow[dr] & & a_4 & \cdots \\
& b_1 \arrow[ur] & & b_2 \arrow[ur] & & b_3 \arrow[ur] & \cdots &
\end{tikzcd}
$$

And the same questions for this relation:

$$
\begin{tikzcd}
a_1 \arrow[rr] \arrow[dr] & & a_2 \arrow[rr] \arrow[dr] & & a_3 \arrow[rr] \arrow[dr] & & a_4 & \cdots \\
& b_1 \arrow[ur] \arrow[loop below] & & b_2 \arrow[ur] \arrow[loop below] & & b_3 \arrow[ur] \arrow[loop below] & \cdots &
\end{tikzcd}
$$

21. Show that the relation $\leftrightarrow$ on the set of propositional formulas is an equivalence relation.

22. Draw a diagram of the relation $\subseteq$ on $P(\{0,1,2\})$.

23. Prove that $(P(A), \subseteq)$ is a partial order for every set A. What about the set $(P(A), \subset)$?

24. How many elements has a set A at least if $(P(A), \subseteq)$ is not a total ordering?

25. Show that given a reflexive relation R, the relation S defined by

$$Sxy \;\Leftrightarrow\; Rxy \vee Ryx$$

is a reflexive symmetric relation.

26. Prove that for finite sets X and Y, $|\, X \cup Y \,| \leq |\, X + Y \,|$.

Inleveropgave 3 - 3 December

1. (2pt) Geldt $X \in P(P(X))$? Als X eindig is, hoeveel elementen heeft $P(P(X))$?

2. (2pt) Hoeveel elementen van $P(\{a, b, c, d\})$ bevatten a?

3. (3pt) Als $X \cap Y = \emptyset$, wat zijn dan de elementen van $P(X) \cap P(Y)$? Bewijs je antwoord.

4. (3pt) Leg uit waarom voor geen enkele $r < s$ geldt dat het interval $[r, s]$ een deelverzameling van $\mathbb{R} \backslash \mathbb{Q}$ is.

On sets, functions and relations

Chapter 3

Rosalie Iemhoff

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1 Functions

In the previous chapter it has been explained how relations can be viewed as sets, namely as sets of ordered pairs. Functions can also be viewed as sets, or relations, but with certain additional properties.

A *function* $f : A \to B$ is a subset $f \subseteq A \times B$ such that for each $x \in A$ there exists exactly one $y \in B$ such that $\langle x, y\rangle \in f$. In formal notation: $f \subseteq A \times B$ is a function if

$$\forall x \in A \exists! y \in B\,(\langle x, y\rangle \in f).$$

($\exists! y$ means "there exists a unique y".) Functions are also called *maps* or *mappings*. When $A = B$ we say that f is a function *on* A. We write $f(x) = y$ for $\langle x, y\rangle \in f$. Using this notation the above formula becomes the well-known property of functions:

$$\forall x \forall y\,(f(x) = f(y) \to x = y).$$

This is not a computational view on functions, as f is not viewed as an operation or algorithm that on input x provides an outcome $f(x)$, like e.g. the function $\mathrm{ggd}(x, y)$ that outputs the greatest common divisor of numbers x and y. The intuitive notion of a function $f : A \to B$ is *intensional*: f is given by a rule or computation that associates an element in B with every element in A. Thus it might be that $f(x)$ and $g(x)$ are the same for all values x, but we still do not want to consider f and g as equal because they are given by a different computation.

These intuitions we lose in set theory, as in this setting functions are *extensional*: if $f(x) = g(x)$ for all x, then $f = g$, because the sets $f = \{\langle x, y\rangle \mid f(x) = y\}$ and $g = \{\langle x, y\rangle \mid g(x) = y\}$ are the same set of pairs, that is, they are equal. Thus there is no reference to the processes underlying f and g which might distinguish them. What we gain by this at first sight somewhat unnatural set-theoretic view is the insight that functions can be defined in terms of sets, thus again showing that basic notions of mathematics can be defined in terms of sets.

Example 1

1. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$ is the set $\{\langle x, y\rangle \in \mathbb{R}^2 \mid 2x = y\}$.
2. The identity function id_A on a set A is the set $\{\langle x, y\rangle \in A^2 \mid x = y\}$. Thus $id_A(x) = x$.

1.1 Domain and range

The *domain* of a function $f : A \to B$ is A. The *range* of the function is the set $\{y \in B \mid \exists x \in A\, f(x) = y\}$. The *codomain* of the function is B. The domain of f is denoted by $\mathrm{dom}(f)$, and its range by $\mathrm{rng}(f)$. If $f(x) = y$, then y is called the *image* of x under f. Given a set $X \subseteq A$, $f[X]$ denotes the set $\{f(x) \mid x \in X\}$ and is called the *image* of X under f. Thus $f[A]$ is the set of elements in B

that can be reached from A via f. For $Y \subseteq B$, the set $\{x \in A \mid f(x) \in Y\}$ is denoted by $f^{-1}[Y]$.
The set of *all* functions from A to B is denoted by B^A.

Example 2 1. The domain of the function $f : \mathbb{N} \to \mathbb{Z}$ given by $f(n) = -n$ is $\mathbb{N}$, and the range is $\mathbb{Z}_{\leq 0}$. $f[\{n \in \mathbb{N} \mid n < 7\}] = \{n \in \mathbb{Z}_{\leq 0} \mid n > -7\}$.

2. The domain of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is $\mathbb{R}$ and its range is $R_{\geq 0}$. $f^{-1}[\{4, 9\}] = \{2, -2, 3, -3\}$.

$$f^{-1}[\mathbb{R}_{\geq 25}] = \{x \in \mathbb{R} \mid x \geq 5\} \cup \{x \in \mathbb{R} | x \leq -5\}.$$

3. For the function sgn: $\{0, 1\} \to \{0, 1\}$ with sgn(0)=1 and sgn(1) $= 0$, domain and range are $\{0, 1\}$. $f(0) = \{1\}$ and $f^{-1}[\{0, 1\}] = \{0, 1\}$.

4. Let $\mathcal{P}$ be the set of propositional formulas. Then $f(\varphi) = \neg\varphi$ is the function on $\mathcal{P}$ that maps formulas to their negation. $f[\{\varphi, \varphi \vee \psi\}] = \{\neg\varphi, \neg(\varphi \vee \psi)\}$.

5. $\mathbb{R}^{\mathbb{R}}$ is the set of all the functions on the reals. $\{0, 1\}^{\mathbb{N}}$ is the set of all functions from the natural numbers to $\{0, 1\}$, which can also be viewed as the set of infinite sequences of zeros and ones.

6. Let $\mathcal{W}$ be the set of finite sequences of 0's and 1's, and $f, g \in \mathcal{W}^{\mathcal{W}}$ given by $f(w) = 0w0$ and $g(w) = ww$. The range of f is all words that start and end with a 0, and the range of g are all words of even length.

Theorem 1 For finite sets X and Y the number of functions from X to Y is $|Y|^{|X|}$.

Proof You will be asked to prove this in the exercises. ♡

1.2 Composition

The *composition* of functions stands for the consecutive application of them. For example, the composition of the function $f(x) = x^2$ with the function $g(x) = x - 5$, is the function $h(x) = g(f(x)) = x^2 - 5$, while the composition of g with f is $f(g(x)) = (x - 5)^2$.
More formally, given two functions $f : A \to B$ and $g : B \to C$, the *composition* of f with g is denoted by $g \circ f$ or gf, and defined by

$$(g \circ f)(x) = g\big(f(x)\big).$$

Of course, the notion $g \circ f$ only makes sense when the range of f is part of the domain of g: rng(f)⊆dom(g). E.g. for $f : \mathbb{N} \to \mathbb{Z}$ with $f(n) = -n$ and $g : \mathbb{N} \to \mathbb{N}$ with $g(n) = 2n$ the composition $g \circ f$ is not well-defined, since $f(2) = -2$, but $g(-2)$ is not defined. On the other hand, the composition $f \circ g$ is in this case defined, since rng(g)⊆dom(f). Note that this also shows that $f \circ g$ is in general different from $g \circ f$.

We can repeat this process and, given functions $f_1, \dots f_n$ where $f_i : A_i \to B_i$ and $B_i \subseteq A_{i+1}$, we can define $f_n \circ \dots \circ f_1$ as

$$f_n \circ \dots \circ f_1(x) = f_n(f_{n-1}(\dots(f_2(f_1(x))\dots).$$

Example 3 1. For $f, g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with $f(x) = x + 2$ and $g(x) = \sqrt{x}$, the composition $g \circ f$ maps x to $\sqrt{(x+2)}$, and $f \circ g$ maps x to $\sqrt{x} + 2$.

2. Let $\mathcal{P}$ be the set of propositional formulas, and $f, g \in \mathcal{P}^{\mathcal{P}}$ defined by $f(\varphi) = \neg\varphi$ and $g(\varphi) = \varphi \vee p$. Then $(g \circ f)(\varphi) = \neg\varphi \vee p$ and $(f \circ g)(\varphi) = \neg(\varphi \vee p)$.

3. Given $f, g, h : \mathbb{N} \to \mathbb{N}$ with $f(n) = (n+2)$, $g(n) = 2n$ and $h(n) = n^2$, then $h \circ g \circ f = (2(n+2))^2$.

4. Let $\mathcal{W}$ be the set of finite sequences of 0's and 1's, and $f, g \in \mathcal{W}^{\mathcal{W}}$ given by $f(w) = 0w0$ and $g(w) = ww$. Then $gf(w) = 0w00w0$.

5. Given $f, g, h : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with $f(x) = x^2$, $g(x) = \sqrt{x}$ and $h(x) = x^2$, then $h \circ g \circ f = (\sqrt{x^2})^2$, and thus $h \circ g \circ f = f$.

The proof of the following theorem is simple, but it provides a nice example of a formal proof.

Theorem 2 If f, g, h are functions such that rng(f)$\subseteq$dom(g) and rng(g)$\subseteq$dom(h), then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

This property says that composition is *associative*.

Proof Let $f : A \to B$, $g : B \to C$ and $h : C \to D$. We have to show that for all $x \in A$ we have $(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$. We prove this by applying the definition of composition:

$$\begin{array}{rcl}(h \circ (g \circ f))(x) & = & h\big((g \circ f)(x)\big) \\ & = & h\big(g\big(f(x)\big)\big) \\ & = & (h \circ g)\big(f(x)\big) \\ & = & \big((h \circ g) \circ f\big)(x)\end{array}$$

♡

1.3 Injections, surjections and bijections

There are function that do not different elements to the same element. For example, the function $f : \mathbb{N} \to \mathbb{N}$ with $f(n) = n + 1$. Such functions are called *injections*. A function $f : A \to B$ is *injective* if

$$\forall x \in A \forall y \in A\ (x \neq y \to f(x) \neq f(y)).$$

Note that this is equivalent to

$$\forall x \in A \forall y \in A (f(x) = f(y) \to x = y).$$

Example 4 1. The identity function is injective.

2. $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$ is not injective: $f(-2) = 4 = f(2)$. If we consider the same function, but now as a function on $R_{\geq 0}$, then it is injective.

You will be asked to prove the following observation in the exercises.

Theorem 3 The composition of two injective functions $f : A \to B$ and $g : B \to C$ is injective.

There are functions $f : A \to B$ that do not reach all elements in B, that is, the range of f is a real subset of B. The function $f : \mathbb{N} \to \mathbb{N}$ that maps all numbers to 0, $f(n) = 0$, is an example of this since rng(f)=$\{0\} \subset \mathbb{N}$. Functions that do reach *all* of B are called *surjections*. A function $f : A \to B$ is *surjective* if $f[A] = B$, in other words if

$$\forall y \in B \exists x \in A \ f(x) = y.$$

Example 5 1. The identity function is surjective.

2. $f : \mathbb{Z} \to \mathbb{Q}$ with $f(n) = 1/n$ is not surjective, as $2 \notin f[\mathbb{Z}]$. It is injective.

3. $f : \{\{a, b\}, \{c\}, \{d\}\} \to \{0, 1, 2\}$ given by $f(\{a, b\}) = 0$ and $f(\{c\}) = f(\{d\}) = 2$, is not injective since $\{c\}$ and $\{d\}$ are mapped to the same element. Neither is it surjective, as there is no x such that $f(x) = 1$.

4. Let R be an equivalence relation on a set A, and consider the map $f : A \to A/R$ which maps elements to their equivalence class, $f(a) = [a]$. f is surjective, but it is only injective if no equivalence class contains more than one element.

5. Let $\mathcal{P}$ be the set of propositional formulas, and let R be the equivalence relation on formulas given by (see Chapter 2):

$$\varphi R \psi \Leftrightarrow \varphi \leftrightarrow \psi \text{ is a tautology.}$$

Consider $f \in \mathcal{P}^{\mathcal{P}}$ and $g : \mathcal{P} \to \mathcal{P}/R$ given by $f(\varphi) = \neg\varphi$ and $g(\varphi) = [\neg\varphi]$. Then f is injective, as for no two different formulas φ and ψ, $\neg\varphi$ equals $\neg\psi$. g is not injective: $g(\varphi) = g(\neg\neg\varphi)$. f is not surjective, but g is.

Observe that the surjectivity of a function depends on the choice of the codomain. For example, the function $f : \mathbb{N} \to \{0\}$ given by $f(n) = 0$ is surjective, but the same function, $f(n) = 0$, considered as a function $f : \mathbb{N} \to \mathbb{N}$ is not.

A function is *bijective* if it is both injective and surjective. Sometimes, bijections are called *1-1 functions*.

Example 6 1. The function $f(x) = x - 1$ on the reals is bijective.

2. The function $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n + 7$ is not bijective, as it is not surjective.

3. Let $\{a_1, \ldots, a_n\}$ be a finite set, let $p_1, \ldots, p_n$ be n propositional variables, and let $\mathcal{P}$ be the set of all propositional formulas in $p_1, \ldots, p_n$ and $\bot$. The function $f : P(A) \to \mathcal{P}$ is given by:

$$f(X) = \bigwedge_{a_i \in X} p_i \quad f(\emptyset) = \bot.$$

   Then f is a injection, but not a surjection, and thus not a bijection. You will be asked to prove this in the exercises.

4. Let $\mathcal{W}_n$ be the set of finite sequences of 0's and 1's of length n, and let $\{a_1, \ldots, a_n\}$ be a finite set. The function $f : \mathcal{W} \to P(A)$ given by

$$f(w) = \{a_i \in A \mid w_i = 1\},$$

   where w_i is the i-th element of w, is a bijection. You will be asked to prove this in the exercises.

Intuitively, a bijection between sets A and B associates with every element in A a unique element in B and vice versa. Thus it can be seen as a correspondence between the sets A and B. Because of this, bijections have a natural inverse, which is the function "turned around". That is, if $f : A \to B$ is a bijection, then we can define the function $g : B \to A$ by

$$g(y) = x \Leftrightarrow f(x) = y.$$

Note that then

$$(g \circ f)(x) = g\big(f(x)\big) = x.$$

For example, for the bijection $f(x) = x+2$ on the reals, g would be $g(y) = y-2$. And indeed, $g\big(f(x)\big) = g(x + 2) = (x + 2) - 2 = x$. This is the content of the following theorem.

Theorem 4 A function $f : A \to B$ is bijective if and only if there exists a function $g : B \to A$ such that $(f \circ g) = id_B$ and $(g \circ f) = id_A$. g is called the *inverse* of f and denoted by f^{-1}.

Proof An *if and only if* statement has to directions: from left to right and from right to left, which we denote by $\Rightarrow$ and $\Leftarrow$.
$\Rightarrow$: in the direction from left to right we have to show that if $f : A \to B$ is bijective, then such a function g as in the theorem exists. Thus suppose that f is bijective. Recall that functions are sets of pairs, that is, $f \subseteq A \times B$. Therefore we can define g according to the intuition of "turning f around":

$$g = \{\langle y, x\rangle \mid \langle x, y\rangle \in f\} = \{\langle y, x\rangle \mid f(x) = y\}.$$

We have to show that g is a function from B to A and that $(f \circ g) = id_B$ and $(g \circ f) = id_A$. You will be asked to prove this in the exercises.

$\Leftarrow$: in the direction from right to left we have to show that if there is a $g : B \to A$ such that $(f \circ g) = id_B$ and $(g \circ f) = id_A$, then f is a bijection. Thus we have to show that f is injective and surjective.

First, we show that f is injective by showing that if $f(x) = f(y)$, then $x = y$. So suppose $f(x) = f(y)$ for two elements x and y. Since $(g \circ f) = id_A$ it follows that $g(f(x)) = (g \circ f)(x) = id_A(x) = x$. Similarly, $g(f(y)) = (g \circ f)(y) = id_A(y) = y$. But since $f(x) = f(y)$, also $g(f(x)) = g(f(y))$, and thus $x = y$.

Second, we show that f is surjective. Consider an $y \in B$. We have to find an $x \in A$ such that $f(x) = y$. Now take $x = g(y)$. Indeed, $x \in A$. Also, since $(f \circ g) = id_B$ it follows that $f(x) = f(g(y)) = y$, and we are done.

Note that in $\Rightarrow$ we used that $(g \circ f) = id_A$, and in $\Leftarrow$ we used $(f \circ g) = id_B$. $\heartsuit$

Example 7

1. $id_A^{-1} = id_A$.
2. For the function $f : \mathbb{N} \to \{n \in \mathbb{N} \mid \exists m \in \mathbb{N}\,(2m = n)\}$ given by $f(x) = 2x$, $f^{-1} : \{n \in \mathbb{N} \mid \exists m \in \mathbb{N}\,(2m = n)\} \to \mathbb{N}$ is given by $f^{-1}(n) = n/2$.
3. The inverse of the function $\sqrt{x}$ on the positive reals $\mathbb{R}_{\geq 0}$ is the function x^2.
4. The only bijections on $\{0, 1\}$ are $\{\langle 0, 0\rangle, \langle 1, 1\rangle\}$ and $\{\langle 0, 1\rangle, \langle 1, 0\rangle\}$. Note that there are no injections on this set except bijections.
5. The unique function from $\{a\}$ to $\{b\}$ is a bijection.

1.4 Fixed points

The identity function maps every element to itself. There are functions that only map some elements to themselves, like the real-valued function $f(x) = x^2$ that is the identity on 0 and 1 but not on any of the other elements in $\mathbb{R}$. Clearly, there are functions that map no element to itself, for example the function $f(n) = n + 1$.

Given a function $f : A \to B$, an element $x \in A$ is called a *fixed point of f* if $f(x) = x$. Thus 0 and 1 are fixed points of the function $f(x) = x^2$.

The famous Dutch mathematician L.E.J. Brouwer (1881 - 1966) has proved the *Fixed Point Theorem*: every continuous function on the unit ball has a fixed point.

1.5 Notation

The definition of a function can be given in many ways. In words, in set-notation, or by a formula, like this:

$$f \text{ is the function on the integers that multiplies a number by } 7$$

$$f = \{\langle n, m\rangle \in \mathbb{Z}^2 \mid m = 7n\}$$

$$f : \mathbb{Z} \to \mathbb{Z} \quad f(n) = 7n.$$

Sometimes more complex notation is needed: $F : \mathbb{R} \to \mathbb{R}$ and

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}$$

describes the function that maps positive reals to their square root and negative reals to 1. We call such a definition a *definition by case distinction* or a *definition by cases*. Such definitions are often used in programming languages.

1.6 Exercises

1. Give a set-notation for the function that maps real numbers m different from 0 to their inverse, and 0 to 0. What are the domain and range of this function?

2. What is the domain and what is the range of the function $f(n) = 7n$ on the natural numbers?

3. Given the function $f(x) = \sqrt{x}$ on the positive reals, write down its set-notation. What is $f[\mathbb{R}_{\geq 4}]$? And what is $f^{-1}[\mathbb{R}_{\leq 4}]$?

4. List the elements of the set $\{0, 1, 2\}^{\{0\}}$.

5. Show that the number of functions from $\{0, 1\}$ to $\{0, 1\}$, i.e. the size of $\{0, 1\}^{\{0,1\}}$, is 2^2, by giving all the functions explicitly, as sets.

6. Given $f, g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with $f(x) = \sqrt{x}$, $g(x) = x^3$, which function is $f \circ g$? And which $g \circ f$?

7. Let $\mathcal{W}$ be the set of finite sequences of 0's and 1's, and $f, g \in \mathcal{W}^{\mathcal{W}}$ given by $f(w) = www$ and $g(w) = ww^R$, where w is the reverse of w. Give domain and range of these functions, and gf and fg.

8. Let $\mathcal{W}$ be the set of finite sequences of 0's and 1's, and $f, g \in \mathcal{W}^{\mathcal{W}}$ given by $f(w) = ww$ and $g(w) = w^R$, does $fg = gf$ hold? Prove your answer.

9. Let $\mathcal{P}$ be the set of propositional formulas, and consider the function $f(\varphi) = \varphi \to p$ and $g(\varphi) = p \to \varphi$. Give $f \circ g$ and $g \circ f$. Are there φ for which $(f \circ g)(\varphi) \leftrightarrow (g \circ f)(\varphi)$? Prove your answer.

10. Prove for the f and g in the first part ($\Rightarrow$) of the proof of Theorem 4 that g is a function from B to A, and that $(f \circ g) = id_B$ and $(g \circ f) = id_A$.

11. Prove Theorem 1.

12. Is the function $f : \mathbb{Z} \to \mathbb{Z}$ with $f(n) = n + 1$ surjective? Is it surjective when considered as a function on the natural numbers? Explain your answer.

13. Is the exponentation function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2^x$ injective? And surjective? Describe the set $f[\{x \in \mathbb{R} \mid -2 \leq x \leq 2\}]$ and the set $f^{-1}[\{x \in \mathbb{R} \mid 4 \leq x \leq 16\}]$.

14. Prove that the composition of two injective functions is injective.

15. Let $\mathcal{W}_n$ be the set of finite sequences of 0's and 1's of length n, and let $\{a_1, \ldots, a_n\}$ be a finite set. The function $f : \mathcal{W} \to P(A)$ is given by

$$f(w) = \{a_i \in A \mid w_i = 1\},$$

where w_i is the i-th element of w. Prove that f is a bijection.

16. Let $\{a_1, \ldots, a_n\}$ be a finite set, let $p_1, \ldots, p_n$ be n propositional variables, and let $\mathcal{P}$ be the set of all propositional formulas in $p_1, \ldots, p_n$ and $\perp$. The function $f : P(A) \to \mathcal{P}$ is given by:

$$f(X) = \bigwedge_{a_i \in X} p_i \quad f(\emptyset) = \perp.$$

$\bigwedge_{a_i \in X} p_i$ denotes the conjunction of those p_i for which $a_i \in X$. For example, $f(\{a_1, a_7\}) = p_1 \wedge p_7$. Prove that f is an injection, but not a bijection.

17. The same notation as in the previous exercise. $g : P(A) \to \mathcal{P}$ is given by: $g(X) = p_i$, where i is the smallest number $\leq n$ with $a_i \in X$, and $f(\emptyset) = \perp$. Is g an injection or a surjection?

18. Are the sinus and cosinus functions on the real numbers injective? And surjective?

19. Prove that all functions from a nonempty set to a singleton (a set with one element) are surjective, i.e. all $f : A \to \{a\}$, with $A \neq \emptyset$, are surjective. In which cases are they also injective?

20. Given finite sets A and B, give a condition under which there are no injections from A to B.

21. Show that for any injective function $f : A \to B$, the function f as considered from A to $f[A]$ is a bijection.

22. Prove that all injections on a finite set A are bijections.

23. Prove that all injections from a finite set to a finite set with the same number of elements, are bijections.

24. Let $\mathcal{P}$ be the set of propositional formulas, and consider the function $f(\varphi) = \neg\neg\varphi$. Does f have a fixed point? If $f(\psi) \leftrightarrow \psi$ is a tautology, does this imply that ψ is a fixed point of f?

    Let R be the equivalence relation on formulas given by (see Chapter 2):

    $$\varphi R \psi \Leftrightarrow \varphi \leftrightarrow \psi \text{ is a tautology.}$$

    If we consider the function $g : \mathcal{P} \to \mathcal{P}/R$ given by $g(\varphi) = [\neg\neg\varphi]$ (thus $g(\varphi) = [f(\varphi)]$), what are the fixed points of g?

25. Show that if $f : A \to A$ has a fixed point x, then also $f^n(x)$ (f composed with itself n times) is a fixed point of f and equal to x.

26. Give a definition by cases of the function $f : \mathbb{R} \to \mathbb{R}$ that maps all reals between 0 and 1 to 0, that maps 0 and 1 to 1, and that maps all other reals to -1.

27. Given two finite sets A and B, how many injections are there from A to B?

28. Show that for a function f, $\{\langle x, y\rangle \mid f(x) = f(y)\}$ is an equivalence relation.

29. Show that for a function $f : A \to B$, for $R = \{\langle x, y\rangle \mid f(x) = f(y)\}$, there is an injection from A/R to B.

30. Show that for finite A there is no surjection from A to $P(A)$.

Inleveropgave 4 - 8 December

1. (3pt) Zij X een verzameling. Bewijs dat de relatie $\subseteq$ op $P(X)$ transitief en seriëel (serial) is. Bewijs dat de relatie $\subset$ op $P(X)$ niet seriëel en niet dicht is.

2. (3pt) Zij $\mathcal{P}$ de verzameling van propsitionele formules. Is de relatie

$$\{\langle\varphi, \psi\rangle \in \mathcal{P}^2 \mid \varphi \to \psi \text{ is een tautologie}\}$$

   een equivalentierelatie? Licht je antwoord toe.

3. (4pt) Bewijs dat als $f : A \to B$ en $g : B \to C$ twee injecties zijn, dat dan gf ook een injectie is. Geldt hetzelfde voor surjecties? Licht je antwoord toe.