Semantics – CKI – Utrecht, Spring 2011

Quiz in Set Theory

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Instructions

1. The quiz contains 5 questions. Please answer all of them on the exam sheet.
2. You are not permitted to use any help from people or pre-prepared material.
3. Exam duration: 30 minutes.
4. Please write your student number here: _____________.

Good luck!

Question 1 (15 points)
Define all the functions in the set $\{a, b\}^{\{1,2\}\times\{3\}}$.

Question 2 (12 points)
Write down all the members in the set $\wp(\{a, b\}) \times \{c\}$.

Question 3 (12 points)
Write down all the members in the set $\wp(\{a, b\} \times \{c\})$.

Question 4 (12 points)

Complete the equations below:

a. $\{a\} \cap \wp(\{a\})$ = ___________

b. $\{\{a\}\} \cap \wp(\{a, b\})$ = ___________

c. $\wp(\{a, b\}) \cap \wp(\{b, c\})$ = ___________

d. $(\{a, b\} \times \{c\}) \cap (\{a\} \times \{b, c\})$ = ___________

Question 5 (25 points)

For a given set $E \neq \emptyset$ we define the function $f : \wp(E) \to \wp(E)$ as follows:

for every set $A \subseteq E$, $f(A) = E - A$.

Further, we define the function $g : \wp(\{a, b\}) \to \wp(\{a, b, c\})$ as follows:

for every set $A \subseteq \{a, b\}$, $g(A) = A \cup \{c\}$.

a. The function f is an injection/surgection/ingection (erase the incorrect possibility/ies).

b. Does f has an inverse function f^{-1}? yes/no

If it does, define it: ___________

c. The function g is an injection/surgection/ingection (erase the incorrect possibility/ies).

d. Does g has an inverse function g^{-1}? yes/no

If it does, define it: ___________

e. Write down the value of the function $f \circ g$, for each member of its domain:

Question 6 (24 points)

Let R be a binary relation over a set E. Thus we have $R \subseteq E \times E$. Now we define:

R^{-1}, the *inverse* relation of R, is the relation that satisfies for all $x, y \in E$: $\langle x, y\rangle \in R^{-1}$ iff $\langle y, x\rangle \in R$.

$\overline{R}$, the *complement* relation of R, is the relation $(E \times E) - R$.

Encircle the **correct** statements among the following ones:

a. If R is symmetric then R^{-1} is symmetric.

b. If R is symmetric then $\overline{R}$ is symmetric.

c. If R is reflexive then R^{-1} is reflexive.

d. If R is reflexive then $\overline{R}$ is reflexive.

e. If R is transitive then R^{-1} is transitive.

f. If R is transitive then $\overline{R}$ is transitive.

For the cases that you did **not** encircle, please give a simple counter-example for R and E that shows that the claim is incorrect.

Quiz 2011 – remark

In question 4, the function $g : \wp(\{a,b\}) \to \wp(\{a,b,c\})$ is defined as follows: for every set $A \subseteq \{a,b\}$, $g(A) = A \cup \{c\}$.
g **is an injection** because for every $A_1, A_2 \in \wp(\{a,b\})$ s.t. $A_1 \neq A_2$, since $c \notin A_1$ and $c \notin A_2$, we have $A_1 \cup \{c\} \neq A_2 \cup \{c\}$. Hence $g(A_1) \neq g(A_2)$.
g **is <u>not</u> a surjection** because for any $B \in \wp(\{a,b,c\})$ s.t. $c \notin B$ (i.e. B is $\emptyset$, $\{a\}$, $\{b\}$ or $\{a,b\}$), there is no $A \in \wp(\{a,b\})$ s.t. $g(A) = B$. This is because $c \in g(A)$ for every $A \in \wp(\{a,b\})$, but $c \notin B$.
Thus, g is not a bijection.
By definition, a function $G : Y \to X$ is the *inverse function* of $F : X \to Y$ if and only if F is a bijection and $G(F(x)) = x$ for every $x \in X$.
Therefore: the function g above **does not have an inverse function**, since it is not a bijection.

However, for any injection $F : X \to Y$, we can define a *bijection* $F' : X \to Y'$, where $Y' \subseteq Y$ is the set $F[X] = \{F(x) \in Y : x \in X\}$, and for every $x \in X$: $F'(x) = F(x)$. By definition of Y', the function F' is a surjection (onto Y'), and like F it is an injection, hence F' is a bijection.
In the case of the injection g above, the function $g' : \wp(\{a,b\}) \to \{\{c\},\{a,c\},\{b,c\},\{a,b,c\}\}$ defined by $g'(A) = g(A)$ for every $A \in \wp(\{a,b\})$ is a bijection, and its inverse function $g'^{-1} : \{\{c\},\{a,c\},\{b,c\},\{a,b,c\}\} \to \wp(\{a,b\})$ is defined by $g'^{-1}(A) = A - \{c\}$.

Let the function $h : \wp(\{a,b,c\}) \to \wp(\{a,b\})$ be defined by $h(A) = A - \{c\}$. The function h extends g'^{-1} in a natural way, but it is **not** the inverse function of g.