# Monotonicity and Relative Scope Entailments[1]
### first version, comments welcome!

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### Abstract

This paper explores the hypothesis that simple monotonicity properties of quantifiers in natural language determine to a large extent the entailment relations between their wide/narrow scope readings. We prove that the *disjunctive normal form* of upward monotone quantifiers using principal ultrafilters correlates with whether an object narrow scope reading entails an object wide scope reading. This result naturally extends the familiar entailment relations between $\exists\forall$ and $\forall\exists$ quantification in first order logic into arbitrary "finitely based" upward monotone determiners (over possibly infinite models), which are precisely defined.

Given a simple sentence of the form *Subject-Verb-Object*, we are interested in the logical relations between the *object narrow scope* (ONS) and the *object wide scope* (OWS) readings of the sentence. In [4], Zimmermann (1993) fully characterizes the class of "scopeless" object quantifiers – those for which the ONS and OWS readings are equivalent for any subject. Zimmermann shows that this class is closely related to the class of (principal) ultrafilters (names). In [3], Westerståhl (1996) fully characterizes the class of "self-commuting" quantifiers, i.e. the quantifiers Q for which ONS and OWS readings are equivalent when Q is substituted for both subject and object. However, as far as we know, the more general problem of characterizing (possibly one-way) entailment relations between ONS and OWS readings has not been given serious attention.

*Global determiners* (see [2]) are functors that map any non-empty domain $E$ to a binary relation over $\wp(E)$. Any set $Q_E \subseteq \wp(E)$ is called a *generalized quantifier* (GQ) over $E$. Thus, a determiner $D_E$ over $E$ maps any $A \subseteq E$ to the generalized quantifier $D_E(A)$ over $E$.

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Let $Q_1$ and $Q_2$ be the GQs over $E$ that the subject and object respectively denote in a given model. The ONS and OWS readings of the sentence in this model with respect to a binary relation $R \subseteq E \times E$, are defined using the following polyadic GQs over $E \times E$:

$$
(1) \quad
\begin{aligned}
Q_1\text{-}Q_2(R) &\stackrel{def}{=} Q_1(\{x \in E : Q_2(\{y \in E : R(y)(x)\})\}) \quad \text{(ONS reading)} \\
Q_1{\sim}Q_2(R) &\stackrel{def}{=} Q_2(\{y \in E : Q_1(\{x \in E : R(y)(x)\})\}) \quad \text{(OWS reading)}
\end{aligned}
$$

Let $D_1$ and $D_2$ be global determiners that correspond to the subject and object determiners respectively. The polyadic determiners $D_1\text{-}D_2$ and $D_1{\sim}D_2$, which give rise to ONS and OWS readings respectively, are defined as ternary relations between $A, B \subseteq E$ and $R \subseteq E \times E$.

$$
(2) \quad
\begin{aligned}
D_1\text{-}D_2(A)(B)(R) &\stackrel{def}{=} ((D_1(A))\text{-}(D_2(B)))(R) \\
D_1{\sim}D_2(A)(B)(R) &\stackrel{def}{=} ((D_1(A)){\sim}(D_2(B)))(R)
\end{aligned}
$$

A quantifier $Q_E$ is called *upward* (*downward*) *monotone* if it is closed under supersets (subsets). A global <u>determiner</u> $D$ is called *upward* (*downward*) <u>*right monotone*</u> if for every $A \subseteq E$, the quantifier $D_E(A)$ is upward (downward) monotone. Symmetrically, $D$ is called *upward* (*downward*) <u>*left*</u> <u>*monotone*</u> if for every $A \subseteq E$, the quantifier $\{B \subseteq E : D(B)(A)\}$ is upward (downward) monotone. We use the following abbreviations:

$Q_E \in \text{MON}{\uparrow}/\text{MON}{\downarrow}$:    $Q_E$ is upward/downward monotone

$D \;\; \in \text{MON}{\uparrow}/\text{MON}{\downarrow}$:    $D$ is upward/downward right monotone

$D \;\; \in {\uparrow}\text{MON}/{\downarrow}\text{MON}$:    $D$ is upward/downward left monotone

We would like to characterize whether the relation $D_1\text{-}D_2$ is contained in the relation $D_1{\sim}D_2$. When $D_1$ is *every* (*some*) and $D_2$ is *some* (*every*), it is well-known that the answer is negative (positive) respectively. For instance, the ONS reading of the sentence *some student saw every teacher* entails, but is not entailed by, its OWS reading. We show that in fact, in sentences with upward monotone subjects and objects, the *existential* determiner is the basis for the class of *subject* determiners that guarantee entailment from the ONS reading to the OWS reading. Symmetrically, for upward monotone subjects and objects, the *universal* determiner is the basis for the class of *object* determiners that guarantee entailment from the ONS reading to the OWS reading.

We use the fact (cf. [1]) that any upward monotone quantifier $Q_E$ can be represented as a union of intersections of principal ultrafilters.

**Fact 1** *Let $Q_E$ be an upward monotone GQ over $E$. Then $Q_E = \cup_{N \in M} \cap_{x \in N}$*

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$I_x$, for some subset $M$ of $\wp(E)$, where $I_y$ is the principal ultrafilter $\{A \subseteq E : y \in A\}$ generated by $y \in E$.

**Proof** Let $M$ be $Q_E$. If $A \in Q_E$, then clearly $A \in \cap_{x \in A} I_x$, thus $Q_E \subseteq \cup_{N \in Q_E} \cap_{x \in N} I_x$. In the other direction, if $A \in \cup_{N \in Q_E} \cap_{x \in N} I_x$, then there is $N \in Q_E$ s.t. $A \in \cap_{x \in N} I_x$. Thus, there is $N \in Q_E$ s.t. $N \subseteq A$. Due to upward monotonicity of $Q_E$, we have: $A \in Q_E$. □

We call $M$ the *signature* of a *disjunctive normal form* (DNF) of an upward monotone quantifier. We define a hierarchy of the upward monotone quantifiers by requiring a DNF for a quantifier $Q_E \in \text{MON}\uparrow$, with a signature $M$ that satisfies certain conditions. The classes in the hierarchy, with the respective conditions on $M$ that define them, are listed below.

TRIV$_0$:    $M = \emptyset$: $Q_E$ is empty

TRIV$_1$:    $\emptyset \in M$: $Q_E$ is equal to $\wp(E)$

PUF:      $M = \{\{a\}\}$ for some $a \in E$:
          $Q_E$ is the *principal ultrafilter $I_a$ generated by $a$*;

PUF$_\cap$:    $M = \{A\}$ for some (possibly empty) $A \subseteq E$:
          $Q_E$ is an intersection of PUFs: the *principal filter $F_A$ generated by $A$*

PUF$_\cup$:    $M$ is a (possibly empty) collection of singletons in $\wp(E)$:
          $Q_E$ is a *union of PUFs*.

Obviously, the following relations hold between these classes of GQs: PUF $\subset$ PUF$_\cap$ $\subset$ MON $\uparrow$; PUF $\subset$ PUF$_\cup$ $\subset$ MON $\uparrow$; TRIV$_0$ $\subset$ PUF$_\cup$; TRIV$_1$ $\subset$ PUF$_\cap$.

Further, observe the following simple facts.

**Fact 2** *A quantifier $Q$ is in* PUF$_\cup$ *iff* $Q = \cup_{\{x\} \in Q} I_x$.

**Fact 3** *A quantifier $Q$ is in* PUF$_\cap$ *iff* $Q = \cap_{x \in \cap Q} I_x$ $(= F_{\cap Q})$.

Consider now the following simple relation between the above hierarchy and scope entailments.

**Lemma 4** *Let $Q_1, Q_2 \subseteq \wp(E)$ be upward monotone GQs over $E$. If $Q_1 \in$* PUF$_\cup$ *or $Q_2 \in$* PUF$_\cap$ *then $Q_1$-$Q_2 \subseteq Q_1 \sim Q_2$.*

**Proof** Assume first that $Q_1 \in$ PUF$_\cup$.
Assume that $R \in Q_1$-$Q_2$. That is: $Q_1(\{x \in E : Q_2(\{y \in E : R(y)(x)\})\})$.
We conclude that $Q_1 \neq \emptyset$. Because $Q_1$ is in PUF$_\cup$, we have by fact 2:

$Q_1 = \cup_{\{t\}\in Q_1} I_t$. Thus, there exists $\{t\} \in Q_1$ s.t. $\{x \in E : Q_2(\{y \in E : R(y)(x)\})\} \in I_t$. Let $t_0 \in E$ satisfy $\{t_0\} \in Q_1$ and $Q_2(\{y \in E : R(y)(t_0)\})$. But $\{y \in E : R(y)(t_0)\} \subseteq \{y \in E : \exists\{t\} \in Q_1[R(y)(t)]\}$. Hence, from $Q_2 \in \mathrm{MON}\!\uparrow$ we conclude: $Q_2(\{y \in E : \exists\{t\} \in Q_1[R(y)(t)]\})$. (i)

From $Q_1 \in \mathrm{MON}\!\uparrow$ we conclude $\{y \in E : \exists\{t\} \in Q_1[R(y)(t)]\} \subseteq \{y \in E : Q_1(\{x \in E : R(y)(x)\})\}$.

Hence, from $Q_2 \in \mathrm{MON}\!\uparrow$ and (i) we conclude: $Q_2(\{y \in E : Q_1(\{x \in E : R(y)(x)\})\})$.

Therefore, $R \in Q_1 {\sim} Q_2$.

We have shown that if $Q_1 \in \mathrm{PUF}_\cup$ then $Q_1$-$Q_2 \subseteq Q_1{\sim}Q_2$. The proof for $Q_2 \in \mathrm{PUF}_\cap$ is analogous. $\square$

We use our classification of $\mathrm{MON}\!\uparrow$ local quantifiers in order to classify $\mathrm{MON}\!\uparrow$ global determiners as follows. For any global determiner $D$:

$D$ is $\mathrm{PUF}_\cup^1$ iff for all $A \subseteq E$: $D_E(A)$ is in $\mathrm{PUF}_\cup \cup \mathrm{TRIV}_1$.

$D$ is $\mathrm{PUF}_\cap^0$ iff for all $A \subseteq E$: $D_E(A)$ is in $\mathrm{PUF}_\cap \cup \mathrm{TRIV}_0$.

$D$ is $\mathrm{TRIV}_0$ ($\mathrm{TRIV}_1$) iff for all $A \subseteq E$: $D_E(A)$ is in $\mathrm{TRIV}_0$ ($\mathrm{TRIV}_1$). When $D$ is $\mathrm{TRIV}_0$ or $\mathrm{TRIV}_1$ we say that $D$ is *trivial*.

$D$ is $\mathrm{TRIV}_0^\exists$ ($\mathrm{TRIV}_1^\exists$) iff there exist $A \subseteq E$ s.t. $D_E(A)$ is in $\mathrm{TRIV}_0$ ($\mathrm{TRIV}_1$).

$D$ is $\mathrm{PUF}_0$ ($\mathrm{PUF}_1$) iff for all $x \in E$: $D_E(\{x\}) = I_x$, while for all $A \subseteq E$ s.t. $|A| \neq 1$: $D_E(A)$ is in $\mathrm{TRIV}_0$ ($\mathrm{TRIV}_1$). When $D$ is $\mathrm{PUF}_0$ or $\mathrm{PUF}_1$ we say that $D$ is PUF.

Thus, a determiner is called $\mathrm{PUF}_\cup^1$ ($\mathrm{PUF}_\cap^0$) when it generates only $\mathrm{PUF}_\cup$ ($\mathrm{PUF}_\cap$) and trivial quantifiers. Note that a determiner is classified as $\mathrm{PUF}_\cup^1$ ($\mathrm{PUF}_\cap^0$) or $\mathrm{TRIV}_0$ ($\mathrm{TRIV}_1$) according to its behavior on *all* domains and arguments. By contrast, for classifying a determiner as $\mathrm{TRIV}_0^\exists$ ($\mathrm{TRIV}_1^\exists$), it is sufficient to find *one* domain and one argument for which it is $\mathrm{TRIV}_0$ ($\mathrm{TRIV}_1$). The usefulness of both "universal" and "existential" classifications of determiners will be clarified as we go along.

Meanwhile, to illustrate these definitions, consider the following simple facts.

- The determiner *some* is $\mathrm{TRIV}_0^\exists$ and $\mathrm{PUF}_\cup^1$, but neither $\mathrm{TRIV}_1^\exists$ nor $\mathrm{PUF}_\cap^0$;

- The determiner *every* is $\mathrm{TRIV}_1^\exists$ and $\mathrm{PUF}_\cap^0$, but neither $\mathrm{TRIV}_0^\exists$ nor $\mathrm{PUF}_\cup^1$;

- The determiner *some and (in fact) every* is $\mathrm{TRIV}_0^\exists$ and $\mathrm{PUF}_\cap^0$, but neither $\mathrm{TRIV}_1^\exists$ nor $\mathrm{PUF}_\cup^1$;

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- The determiner *some or (perhaps even) every* is $\text{TRIV}_1^\exists$ and $\text{PUF}_\cup^1$, but neither $\text{TRIV}_0^\exists$ nor $\text{PUF}_\cap^0$.

Of course, the determiners *some and/or every* are respectively the intersection/union of the standard relations for the existential and universal determiners.

Our main claim is that this typology of determiners allows us to determine in which cases of $\text{MON}\uparrow$ determiners $D_1$ and $D_2$, the ONS reading $D_1$-$D_2$ entails (or is entailed by) the OWS reading $D_1{\sim}D_2$. Before proving that, there is one qualification concerning this result that we should explain. We will assume that both $D_1$ and $D_2$ are *finitely based*, in a sense that is defined below. This restriction is needed because $\text{MON}\uparrow$ determiners such as *infinitely many* behave with respect to re lative scope entailments differently than $\text{MON}\uparrow$ determiners such as *at least three*. Consider the following examples.

(3)    a. Infinitely many students saw John or Mary.

    b. At least three students saw John or Mary.

(4)    a. Infinitely many students saw at least one of the two students.

    b. At least three students saw at least one of the two students.

In (3a), the ONS reading entails the OWS reading: if there are infinitely many students that have the property *saw John or saw Mary*, then either John or Mary has the property *was seen by infinitely many students*. But this is obviously not the case in (3b). A similar contrast is observed between (4a) and (4b), under a Russellian treatment of the definite article. For instance:

(5) **at_least_one_of_the_n$'$**$(A)(B) = 1 \iff |A| = n \wedge A \cap B \neq \emptyset$

(6) **each_of_the_n$'$**$(A)(B) = 1 \iff |A| = n \wedge A \subseteq B$

We have seen that $\text{MON}\uparrow$ determiners such as *infinitely many* show scope entailments that are different than those of similar "finite" determiners. Such "infinite" determiners, which are common in the mathematical jargon, are much less common – and have a much less defined meaning – in everyday speech. This is in contrast to more ordinary determiners such *at least three* or *every*, which English speakers use by and large with the same meaning as logicians do. The formal distinction between determiners that is held responsible for this difference is defined as follows.

**Definition 1** (FB **quantifiers**) *Let $E$ be a denumerable non-empty domain. A sequence $A_i|_{i=1}^{\infty}$ of subsets of $E$ is called* properly monotone *if $A_i \subset A_{i+1}$ for every $i \geq 1$, or $A_i \supset A_{i+1}$ for every $i \geq 1$.*
*Two properly monotone sequences $A_i|_{i=1}^{\infty}$ and $B_j|_{j=1}^{\infty}$ are called* mutually monotone *if $A_i \subset B_j$ for all $i, j \geq 1$, or $A_i \supset B_j$ for all $i, j \geq 1$.*
*A quantifier $Q_E$ over $E$ is called* finitely based (FB) *iff for any two mutually monotone sequences $A_i|_{i=1}^{\infty}$ and $B_j|_{j=1}^{\infty}$ s.t. $Q_E$ is constant on both sequences, $Q_E$ sends both sequences to the same value.*

By "constancy" of a quantifier $Q_E$ on a set $\mathcal{X} \subseteq \wp(E)$, we of course mean: $\mathcal{X} \subseteq Q_E$ or $Q_E \cap \mathcal{X} = \emptyset$. In the first case say we say that $Q_E$ sends $\mathcal{X}$ to 1. In the second case say we say that $Q_E$ sends $\mathcal{X}$ to 0.

The definition of FB determiners is derived from the definition of FB quantifiers.

**Definition 2** (FB **determiners**) *A global determiner $D$ is* FB *iff for any denumerable non-empty domain $E$, $D_E(E)$ is an* FB *quantifier.*

Note that this definition pays attention only to the behavior of $D_E$ on the whole $E$ domain, and does not take into account proper subsets of $E$. Thus, a determiner such as *all* is provably FB, even though on the domain of natural numbers, the quantifier *all odd natural numbers* is *not* FB. This is in accordance with the intuition that the determiner *all* does not inherently pertain to infinite sets. By contrast, the determiner *all but finitely many* provably maps any infinite domain to a non-FB quantifier, hence it is not FB itself.

Let us consider an example for a pair of FB/non-FB determiners that belong in the same class of the above hierarchy. Consider first the determiner *infinitely many*. Let $\mathbf{N}$ by the set of natural numbers, with $\mathbf{N}_O \subset \mathbf{N}$ the set of odd natural numbers. Consider two sequences $(\mathbf{N}_O \cap [1..2i])|_{i=1}^{\infty}$ – the increasing sequence of sets of odd numbers; and $(\mathbf{N}_O \cup [1..2i])|_{i=1}^{\infty}$ – the unions of the odd numbers with elements in the increasing sequence of sets of even numbers. These two sequences are mutually monotone, but the denotation of *infinitely many natural numbers* on the domain $E = \mathbf{N}$ is constantly false on the first sequence but constantly true on the second sequence. Consequently, the determiner *infinitely many* is *not* FB. It is impossible to find two such sequences for the determiner *at least three*: trivially, for any domain $E$, the quantifier $\mathbf{at\_least\_3}'_E(E)$ cannot be false over an infinite properly monotone sequence. Consequently, the determiner *at least three* is FB. Note however that, for each of the determiners *infinitely many* and *at least three*, there are quantifiers that the determiner forms that belong in the class MON $\uparrow$

$\setminus(\mathrm{PUF}_\cup\cup\mathrm{PUF}_\cap)$. Hence, both determiners are in the class $\mathrm{MON}\!\uparrow\,\setminus(\mathrm{PUF}_\cup^1\cup\mathrm{PUF}_\cap^0)$. Some more examples for FB and non-FB determiners are given in table 1. We note without proof that the class of FB determiners is closed under complements and finite intersections and unions.

| FB | non-FB |
|---|---|
| at least three | finitely many |
| at most three | infinitely many |
| exactly three | |
| all | |
| all but (at least) three | all but finitely many |
| all but at most three | all but infinitely many |

Table 1: FB and non-FB determiners

We observe the following fact about upward monotone FB quantifiers.

**Lemma 5** *Let $Q$ be an FB upward monotone quantifier over a denumerable domain $E$. If $C_1 \supset C_2 \supset \ldots$ is a properly decreasing infinite sequence of sets in $Q$, then there is a finite set $A \subseteq C_1$ in $Q$.*

**Proof** Assume for contradiction that every $A \subseteq C_1$ in $Q$ is infinite. We will show that $Q$ is not FB.

Assume first that there is $A \subseteq C_1$ in $Q$ s.t. $E \setminus A$ is infinite. Let us denote $A = \{a_1, a_2, \ldots\}$, $E \setminus A = \{e_1, e_2, \ldots\}$. Consider the following two sequences:

$$X_1 = A; \quad X_{i+1} = X_i \cup \{e_i\} \text{ for every } i \geq 1.$$

$$Y_1 = \{a_1\}; \quad Y_{j+1} = Y_j \cup \{a_{j+1}\} \text{ for every } j \geq 1.$$

These two infinite sequences are mutually monotone.

For every $i \geq 1$ we have: $X_i \in Q$ (by monotonicity of $Q$ and $X_1 = A \in Q$).

For every $j \geq 1$ we have: $Y_j \notin Q$ (by finiteness of $Y_j$ and $Y_j \subseteq A \subseteq C_1$).

Hence $Q$ is not FB, in contradiction to the assumption that it is.

Assume now that for every $A \subseteq C_1$ in $Q$: $E \setminus A$ is finite. Consider the following two sequences:

$$X_1 = C_1; \quad X_{i+1} = X_i \setminus (C_{2i} \setminus C_{2i+1}).$$

$$Y_1 = C_1 \setminus C_2; \quad Y_{j+1} = Y_j \cup (C_{2j+1} \setminus C_{2j+2}).$$

These two infinite sequences are mutually monotone.

For every $i \geq 1$ we have: $X_i \in Q$, because $C_{2i-1} \in Q$, $C_{2i-1} \subseteq X_i$, and $Q \in$ MON↑.

For every $j \geq 1$ we have: $Y_j \subseteq \cup_{k=1}^{\infty}(C_{2k-1} \setminus C_{2k}) \stackrel{def}{=} D$. But $E \setminus D \supseteq \cup_{n=1}^{\infty}(C_{2k} \setminus C_{2k+1})$ is infinite and $D \subseteq C_1$, hence by our assumption: $D \notin Q$. We conclude $Y_j \notin Q$ by monotonicity of $Q$.

Therefore we proved again that $Q$ is not FB, in contradiction to the assumption that it is. $\square$

For the statement of our main claim, recall the following definitions, which are standard in GQ theory ([2]). For any global determiner $D$:

$D$ satisfies *extension* (EXT)   iff   for all $A, B \subseteq E \subseteq E'$: $D_E(A)(B) = D_{E'}(A)(B)$.

$D$ is *isomorphism invariant* (ISOM)   iff   for all bijections $\pi : E \to E'$, for all $A, B \subseteq E$: $D_{E'}(\{\pi(x) : x \in A\})(\{\pi(y) : y \in B\}) = D_E(A)(B)$.

$D$ is *conservative* (CONS) iff for all $A, B \subseteq E$: $D_E(A)(B) = D_E(A)(A \cap B)$.

As in other works on GQ theory, we restrict our attention to determiners in natural language that are EXT, ISOM and CONS.

It is now possible to move on to our main claim.

**Theorem 6** *Let $D_1$ and $D_2$ be two global* MON↑ *determiners that satisfy* FB, EXT *and* CONS. *Then $D_1$-$D_2 \subseteq D_1 \sim D_2$ for any domain denumerable non-empty $E$   iff   both following conditions hold: (1) $D_1$ is* PUF$_\cup^1$ *or $D_2$ is* PUF$_\cap^0$; *and (2) $D_1$ is not* TRIV$_1^\exists$ *or $D_2$ is not* TRIV$_0^\exists$.

**Proof (if)**

We prove that if $D_1$ is PUF$_\cup^1$ and condition 2 holds then $D_1$-$D_2 \subseteq D_1 \sim D_2$. The proof in case that $D_2$ is PUF$_\cap^0$ is analogous.

If $D_1$ is not TRIV$_1^\exists$ then for all $A \subseteq E$: $D_{1E}(A) \in$ PUF$_\cup$. Hence, by lemma 4, for all $B \subseteq E$: $(D_1(A))$-$(D_2(B)) \subseteq (D_1(A)) \sim (D_2(B))$. In other words: $D_1$-$D_2 \subseteq D_1 \sim D_2$ for any domain $E$.

Otherwise, $D_1$ is TRIV$_1^\exists$ and by condition 2: $D_2$ is not TRIV$_0^\exists$. (i)

For all $A \subseteq E$: $D_{1E}(A) \in$ PUF$_\cup \cup$ TRIV$_1$. If $D_{1E}(A) \in$ PUF$_\cup$, then again by lemma 4, for all $B \subseteq E$: $(D_1(A))$-$(D_2(B)) \subseteq (D_1(A)) \sim (D_2(B))$.

If $D_{1E}(A) \in$ TRIV$_1$, then $\{y \in E : D_1(A)(\{x \in E : R(y)(x)\})\} = E$ for any $R \in E \times E$. But for all $B \subseteq E$: $E \in D_{2E}(B)$, because $D_2(B)$ is not TRIV$_0$ (by (i)) and upward monotone. We conclude that for all $B \subseteq E$: $(D_1(A))$-$(D_2(B)) \subseteq (D_1(A)) \sim (D_2(B))$.

We have shown that if $D_1 \in \mathrm{PUF}_\cup^1$ and condition 2 holds, then $D_1\text{-}D_2 \subseteq D_1 {\sim} D_2$. The proof for $D_2 \in \mathrm{PUF}_\cap^0$ is analogous. $\square$

To prove the "only if" direction of theorem 6, we will first prove the following two lemmas, which rely on the FB property.

**Lemma 7** *Let $D$ be an* FB *determiner in* $\mathrm{MON}{\uparrow}\,\backslash\mathrm{PUF}_\cup^1$ *that satisfies* EXT *and* CONS*. Then there are $A \subseteq E$, for which there is $B \in D_E(A)$ s.t. $|B| \geq 2$ and for every $X \subset B$: $X \notin D_E(A)$.*

**Proof** We first show that there is a domain $A$ s.t. $D_A(A) \in \mathrm{MON}{\uparrow}\,\backslash(\mathrm{PUF}_\cup\cup \mathrm{TRIV}_1)$. The proof is routine in usages of CONS and EXT. By the assumption that $D$ is not $\mathrm{PUF}_\cup^1$, there are $A \subseteq E$ s.t. $D_E(A) \notin \mathrm{PUF}_\cup \cup \mathrm{TRIV}_1$. By fact 2 and upward monotonicity of $D_E(A)$, there is $B \in D_E(A)$ s.t. for every $x \in B$: $\{x\} \notin D_E(A)$. By conservativity of $D$: $A \cap B \in D_E(A)$. Because $D_E(A) \in \mathrm{MON}{\uparrow}\,\backslash\mathrm{TRIV}_1$ we conclude $A \cap B \neq \emptyset$. Because $D$ satisfies EXT: $A\cap B \in D_A(A)$. Hence by assumption on $B$ and extension, for all $x \in A\cap B$: $\{x\} \notin D_A(A)$. By fact 2 and upward monotonicity of $D_A(A)$: $D_A(A)$ is not $\mathrm{PUF}_\cup$. Because $A \cap B \neq \emptyset$, $D_A(A)$ is not $\mathrm{TRIV}_1$ either.
Let us denote $Q = D_A(A)$.
Let us further denote $Q' \overset{def}{=} \{X \in Q : \text{for every } x \in X: \{x\} \notin Q\}$.
Because $Q \in \mathrm{MON}{\uparrow}\,\backslash\mathrm{PUF}_\cup$ and fact 2: $Q' \neq \emptyset$.
Because $Q \in \mathrm{MON}{\uparrow}\,\backslash\mathrm{TRIV}_1$: $\emptyset \notin Q'$.
Assume that there is no properly decreasing infinite sequence in $Q'$. Then for every $C \in Q'$, there is $B \subseteq C$ s.t. $B \in Q'$ and for every $X \subset B$: $X \notin Q'$. Any such $B \in Q'$ is non-empty and by definition of $Q'$: $|B| \geq 2$. Because $Q' \neq \emptyset$, we proved existence of $B$ as required.
Assume now that there is a properly decreasing infinite sequence $C_1 \supset C_2 \supset \ldots$ in $Q' \subseteq Q$. By lemma 5, there is a finite set $C \subseteq C_1$ in $Q$. From definition of $Q'$ we conclude $C \in Q'$. Hence, by finiteness of $C$, there is $B \subseteq C$ s.t. $B \in Q'$ and for every $X \subset B$: $X \notin Q'$. Thus, by $\emptyset \notin Q'$ and definition of $Q'$ we again conclude $|B| \geq 2$. $\square$

**Lemma 8** *Let $D$ be an* FB *determiner in* $\mathrm{MON}\uparrow\,\backslash\mathrm{PUF}_\cap^0$ *that satisfies* EXT *and* CONS*. Then there is a domain $E$ and $A \subseteq E$, for which there are $B_1, B_2 \in D_E(A)$ s.t. $B_1 \cap B_2 \notin D_E(A)$.*

**Proof** For similar considerations as in the proof of lemma 7, we have $D_A(A) \in \mathrm{MON}{\uparrow}\,\backslash(\mathrm{PUF}_\cap \cup \mathrm{TRIV}_0)$ for some $A$. Let us denote $Q = D_A(A)$. Assume that there is no properly decreasing infinite sequence in $Q$. Let us

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denote $Q_{min} \stackrel{def}{=} \{A \in Q : \text{for every } B \subset A: B \notin Q\}$. By assumption on $Q$, we get that for every $A \in Q$ there is $B \subseteq A$ s.t. $B \in Q_{min}$. Therefore, by monotonicity of $Q$: $Q = \cup_{A \in Q_{min}} F_A$. Because $Q \notin \text{TRIV}_0$: $Q_{min} \neq \emptyset$. Because $Q \notin \text{PUF}_\cap$: $|Q_{min}| \neq 1$. Hence, any two sets $B_1, B_2 \in Q_{min}$ are as required.

Assume now that there is a properly decreasing infinite sequence in $Q$. Because $Q$ is MON$\uparrow$ and FB, it follows by lemma 5 that there is a finite set $A_0 \in Q$. Assume for contradiction that for all $B_1, B_2 \in Q$: $B_1 \cap B_2 \in Q$ (assumption (i)). The set $Q_0 \stackrel{def}{=} \{A \cap A_0 : A \in Q\} \subseteq Q$ is finite and non-empty, thus by assumption (i): $\cap Q_0 \in Q$. But $\cap Q_0 = \cap Q$, and therefore $\cap Q \in Q$. By monotonicity of $Q$ and fact 3: $Q$ is in $\text{PUF}_\cap$, in contradiction to our assumption that it is not. We conclude that there are $B_1, B_2 \in Q$ s.t. $B_1 \cap B_2 \notin Q$. $\square$

We can now finally prove the "only if" direction of theorem 6.

**Proof of theorem 6 (only if)**

Let $D_1$ and $D_2$ be two global MON$\uparrow$ determiners that satisfy FB, EXT and CONS.

We assume that at least one of the two conditions 1 and 2 does not hold, and will show that $D_1$-$D_2 \not\subseteq D_1 \sim D_2$.

Assume first that condition 1 does not hold: $D_1$ is not $\text{PUF}_\cup^1$ and $D_2$ is not $\text{PUF}_\cap^0$. According to lemma 7, there are $A_1, B_0 \subseteq E$ s.t. $|B_0| \geq 2$ and $Q_1 \stackrel{def}{=} D_1(A_1)$ on $E$ satisfies: $B_0 \in Q_1$, and for all $X \subset B_0$: $X \notin Q_1$. Let us denote $a_1, a_2 \in B_0$, for arbitrary $a_1 \neq a_2$.

According to lemma 8, there are $A_2, B_1, B_2 \subseteq E'$ s.t. $B_1, B_2 \in Q_2 \stackrel{def}{=} D_2(A_2)$ on $E'$, but $B_1 \cap B_2 \notin Q_2$.

Because $D_1$ and $D_2$ satisfy EXT, we can assume without loss of generality that $E = E'$ (otherwise, choose $E'' = E \cup E'$).

Let us define $R \subseteq E \times E$ as follows:

$$R(y)(x) \Leftrightarrow (x = a_1 \wedge y \in B_1) \vee (x \in B_0 \setminus \{a_1\} \wedge y \in B_2).$$

We shall now show that $(A_1, A_2, R) \in D_1$-$D_2$: From definition of $R$, $\forall x \in B_0 : \{y \in E : R(y)(x)\} \in \{B_1, B_2\} \subseteq Q_2$. Hence: $B_0 \subseteq \{x \in E : \{y \in E : R(y)(x)\} \in Q_2\}$. From monotonicity of $Q_1$ and $B_0 \in Q_1$ we get that $(A_1, A_2, R) \in D_1$-$D_2$.

To prove that $(A_1, A_2, R) \notin D_1 \sim D_2$, we will show that $\{y \in E : \{x \in E : R(y)(x)\} \in Q_1\} = B_1 \cap B_2$, which is sufficient because $B_1 \cap B_2 \notin Q_2$.

Direction "$\supseteq$": We assume $b \in B_1 \cap B_2$. We have for every $x \in E$:

$R(b)(x) \Leftrightarrow x = a_1 \vee x \in B_0 \setminus \{a_1\} \Leftrightarrow x \in B_0$. Hence $\{x : R(b)(x)\} = B_0$.
But $B_0 \in Q_1$, hence we conclude $b \in \{y \in E : \{x \in E : R(y)(x)\} \in Q_1\}$.
Direction "$\subseteq$": We assume $b \in \{y \in E : \{x \in E : R(y)(x)\} \in Q_1\}$.
Hence, $\{x \in E : R(b)(x)\} \in Q_1$. From definition of $R$, we conclude
$\{x \in E : R(b)(x)\} \subseteq B_0$. From our assumption about minimality of $B_0$ in
$Q_1$, it follows that $\{x : R(b)(x)\} = B_0$. Especially: $R(b)(a_1)$ and $R(b)(a_2)$.
Hence, by definition of $R$: $b \in B_1 \cap B_2$.
From the assumption that condition 1 does not hold, we have shown $\langle A_1, A_2, R \rangle \in$
$D_1\text{-}D_2 \setminus D_1 {\sim} D_2$, which means that $D_1\text{-}D_2 \nsubseteq D_1 {\sim} D_2$.

Assume now that condition 2 does not hold: $D_1$ is $\mathrm{TRIV}_1^\exists$ and $D_2$ is
$\mathrm{TRIV}_0^\exists$. Let $A \subseteq E'$ be s.t. $D_1(A)$ on $E'$ is $\wp(E')$ and let $B \subseteq E''$ be s.t.
$D_2(B)$ on $E''$ is empty. On $E \stackrel{def}{=} E' \cup E''$ we have:
$D_1(A) = \wp(E)$, by MON$\uparrow$ and EXT of $D_1$.
$D_2(B) = \emptyset$, by CONS and EXT of $D_2$.
Thus, for every $R \in E \times E$:
$((D_1(A))\text{-}(D_2(B)))(R)$ trivially holds; but
$((D_1(A)){\sim}(D_2(B)))(R)$ trivially does not hold.
From the assumption that condition 2 does not hold, we have again shown
that $D_1\text{-}D_2 \nsubseteq D_1 {\sim} D_2$. $\square$

Theorem 6 characterizes all the FB logical cases of upward right-monotone
subject and object determiners that make the ONS reading entail (or be en-
tailed by) the OWS reading. Simple cases like that are when the subject
determiner is $\mathrm{PUF}_\cup^1 \setminus \mathrm{TRIV}_1^\exists$ or when the object determiner is $\mathrm{PUF}_\cap^0 \setminus \mathrm{TRIV}_0^\exists$.
That is: when the subject always denotes a $\mathrm{PUF}_\cup$ quantifier or the object
always denotes a $\mathrm{PUF}_\cap$ quantifier. This is the case in the following sen-
tences.

(7)    a. Some student saw every/most/at least two teachers.

        b. Every/most/at least two student(s) saw every teacher.

However, to characterize completely the cases of MON$\uparrow$ logical deter-
miners for which the ONS reading entails the OWS reading, we have also
considered some more complex cases of global determiners. An example for
a member in $\mathrm{PUF}_\cup^1 \cap \mathrm{TRIV}_1^\exists$ is the determiner *some or every*. Examples
for members in $\mathrm{PUF}_\cap^0 \cap \mathrm{TRIV}_0^\exists$ are the determiner *some and every* and the
determiner *each of the five* (cf. definition (6)). These determiners show en-
tailments from the ONS reading to the OWS reading in sentences such as
the following.

(8) Some or (perhaps even) every student saw some or (perhaps even)
    every teacher.

(9)  a. At least two teachers saw some and (in fact) every student.

b. At least two teachers saw each of the five students.

The complete characterization of scope entailments with MON↑ determiners explains why there is no entailment from the ONS reading to the OWS reading in simple cases such as the following.

(10) Every/most/at least two student(s) saw some/most/at least two teacher(s).

Also in more complex cases such as the following, there is no entailment from the ONS reading to the OWS reading, as theorem 6 expects.

(11) Some or (perhaps even) every student saw some teacher.

(12) Every student saw some and (in fact) every teacher.

In both cases, when there are no students and no teachers, the ONS reading is true but the OWS reading is false.

Another result concerns the following fact that is mentioned in [3] about *local* quantifiers. Westerståhl calls two quantifiers $Q_1$ and $Q_2$ over $E$ *independent* when $Q_1$-$Q_2 = Q_1{\sim}Q_2$. Then he makes the following claim.

**Proposition 9 (Westerståhl)** *Let* $Q_1$ *and* $Q_2$ *be two quantifiers over* $E$ *that are* MON↑, *non-trivial and* ISOM. *Then* $Q_1$ *and* $Q_2$ *are independent iff* $Q_1 = Q_2 = \mathbf{every}'_E(E)$ $(= \{E\})$ *or* $Q_1 = Q_2 = \mathbf{some}'_E(E)$ $(= \wp(E)\setminus\emptyset)$.

When we consider global determiners, we call $D_1$ and $D_2$ *independent* if $D_1$-$D_2 = D_1 \sim D_2$ for any domain $E$. Theorem 6 entails the following fact about independent determiners. Note the ISOM requirement (as in Westerståhl's proposition), in addition to the requirements in theorem 6.

**Corollary 10** *Let* $D_1$ *and* $D_2$ *be two global* MON↑ *determiners that satisfy* FB, EXT, ISOM *and* CONS.
*Then* $D_1$ *and* $D_2$ *are independent iff both of the following conditions hold:*

1. *At least one of the following holds: (a)* $D_1$ *and* $D_2$ *are both* $\mathrm{PUF}^1_\cup$; *or (b)* $D_1$ *and* $D_2$ *are both* $\mathrm{PUF}^0_\cap$; *or (c)* $D_1$ *is trivial; or (d)* $D_2$ *is trivial; or (e)* $D_1$ *is* PUF; *or (f)* $D_2$ *is* PUF.

2. *At least one of the following holds: (a) Neither* $D_1$ *nor* $D_2$ *are* $\mathrm{TRIV}^\exists_0$; *or (b) Neither* $D_1$ *nor* $D_2$ *are* $\mathrm{TRIV}^\exists_1$.

**Proof** In the "if" direction, if neither $D_1$ nor $D_2$ are trivial or PUF, the claim follows directly from theorem 6. We first assume without loss of generality that $D_1$ is trivial. If $D_1$ is $\mathrm{TRIV}_0$, we conclude that $D_1\text{-}D_2 = \emptyset$. From condition 2 above we conclude that $D_2$ is not $\mathrm{TRIV}_1^\exists$. From monotonicity of $D_2$, it follows that for all $A \subseteq E$: $\emptyset \notin D_{2E}(A)$. From triviality of $D_1$ we know that for all $A \subseteq E$, $R \subseteq E \times E$: $\{y \in E : D_{1E}(A)(\{x \in E : R(y)(x)\})\} = \emptyset$. Thus, $D_1 \sim D_2 = \emptyset$. The proof for the case that $D_1$ is $\mathrm{TRIV}_1$ is analogous.

We now assume without loss of generality that $D_1$ is PUF. Let $E$ an arbitrary domain. We shall look at an arbitrary $\langle A, B, R \rangle \in D_{1E}\text{-}D_{2E}$ and prove that $\langle A, B, R \rangle \in D_{1E} \sim D_{2E}$. If $|A| \neq 1$, then from definition of PUF, $D_1(A)$ is trivial. From PUF we also know that if $D_1(A)$ is $\mathrm{TRIV}_1$ then $D_1$ is not $\mathrm{TRIV}_1^\exists$, and if $D_1(A)$ is $\mathrm{TRIV}_0$ then $D_1$ is not $\mathrm{TRIV}_0^\exists$. As above, we can now see that $\langle A, B, R \rangle \in D_1 \sim D_2$. For $A = \{a\}$, we assumed $D_1(\{a\})(\{x : D_2(B)(\{y : R(y)(x)\})\})$. From PUF, $a \in \{x : D_2(B)(\{y : R(y)(x)\})\}$, therefore $D_2(B)(\{y : R(y)(a)\})$, now from PUF, $D_2(B)(\{y : D_1(\{a\})(\{x : R(y)(x)\})\})$, therefore $\langle \{a\}, B, R \rangle \in D_1 \sim D_2$. The other direction is analogous, thus proving $D_1\text{-}D_2 = D_1 \sim D_2$.

In the "only-if" direction, we assume that $D_1\text{-}D_2 = D_1 \sim D_2$ and prove that both conditions (1) and (2) above hold. From theorem 6, we conclude that the following four propositions hold:

(i) $D_1$ is $\mathrm{PUF}_\cup^1$ or $D_2$ is $\mathrm{PUF}_\cap^0$ (from $D_1\text{-}D_2 \subseteq D_1 \sim D_2$).

(ii) $D_2$ is $\mathrm{PUF}_\cup^1$ or $D_1$ is $\mathrm{PUF}_\cap^0$ (from $D_1\text{-}D_2 \supseteq D_1 \sim D_2$).

(iii) $D_1$ is not $\mathrm{TRIV}_1^\exists$ or $D_2$ is not $\mathrm{TRIV}_0^\exists$ (from $D_1\text{-}D_2 \subseteq D_1 \sim D_2$).

(iv) $D_2$ is not $\mathrm{TRIV}_1^\exists$ or $D_1$ is not $\mathrm{TRIV}_0^\exists$ (from $D_1\text{-}D_2 \supseteq D_1 \sim D_2$).

Assume first for contradiction that condition 2 above does not hold. Thus, $D_1$ or $D_2$ are $\mathrm{TRIV}_0^\exists$, and $D_1$ or $D_2$ are $\mathrm{TRIV}_1^\exists$. From propositions (iii) and (iv) above, we conclude that either $D_1$ or $D_2$ are neither $\mathrm{TRIV}_0^\exists$ nor $\mathrm{TRIV}_1^\exists$. But, both $D_1$ and $D_2$ are CONS, hence for any $E \neq \emptyset$: $D_{1E}(\emptyset), D_{2E}(\emptyset) \in \{\emptyset, \wp(E)\}$, which contradicts our assumption, hence condition 2 holds.

Assume now for contradiction that condition 1 above does not hold. Thus, either $D_1$ or $D_2$ are not $\mathrm{PUF}_\cup^1$, either $D_1$ or $D_2$ are not $\mathrm{PUF}_\cap^0$, and $D_1$ and $D_2$ are both neither trivial nor PUF.

From propositions (i) and (ii) above, we conclude that either $D_1$ or $D_2$ must be both $\mathrm{PUF}_\cup^1$ and $\mathrm{PUF}_\cap^0$. Without loss of generality, assume that $D_1$ is both $\mathrm{PUF}_\cup^1$ and $\mathrm{PUF}_\cap^0$. By definition of $\mathrm{PUF}_\cup^1$ and $\mathrm{PUF}_\cap^0$, for every $A \subseteq E : D_{1E}(A) \in \{\emptyset, I_x, \wp(E)\}$ for some $x \in E$.

First, we shall prove that for all $A \subseteq E \subseteq E'$ and for all $y \in E$, if $D_{1E}(A) = I_y^E$, then also $D_{1E'}(A) = I_y^{\overline{E'}}$. For any $B \in \wp(E')$, if $y \in B$ then also $y \in B \cap E$, therefore $D_{1E}(A)(B \cap E) = 1$, and thus from MON↑ and EXT, $D_{1E'}(A)(B)$. We have shown $I_y^E \subseteq D_{1E'}(A)$. But, from EXT, $D_{1E'}(A)(\emptyset) = D_{1E}(A)(\emptyset) = 0$, thus $D_{1E'}(A)$ is not trivial, therefore $D_{1E'}(A)$ must be $I_y^{E'}$.

If $A = \emptyset$, then from CONS, $D_{1E}(A) \in \{\emptyset, \wp(E)\}$.

If $A = \{x\} \subseteq E$, from CONS, $D_{1E}(\{x\}) \in \{\emptyset, \wp(E), I_x, \overline{I_x}\}$. From MON↑, $D_{1E}(\{x\}) \in \{\emptyset, \wp(E), I_x\}$.

If $|A| \geq 2$, assume $D_{1E}(A) = I_x$. From CONS, $x \in A$. From $|A| \geq 2$, there is $y \in A$ s.t. $y \neq x$. Let $\pi$ be a permutation of $E$ that swaps $x$ and $y$ and maps any other element of $E$ to itself. From ISOM, $D_{1E}(\pi(A)) = I_y$. But, $D_{1E}(\pi(A)) = D_{1E}(A) = I_x$, hence a contradiction.

From condition 2 proven above, we know that either $D_1$ is not TRIV$_0^\exists$ or $D_1$ is not TRIV$_1^\exists$. Therefore, $D_1$ must be trivial with the same value for all $A \subseteq E$ for which it is trivial. From nontriviality, we know that there is $A \subseteq E$ s.t. $D_{1E}(A) = I_x$ for some $x$. We have shown that this can occur only for $A = \{x\}$. From EXT, $D_{\{x\}}(\{x\}) = I_x^{\{x\}}$. From the claim above, we know that $D_{E'}(\{x\}) = I_x^{E'}$ for all $E'$ s.t. $x \in E'$. We will now show that $D_E(\{y\}) = I_y^E$ for all $E$ and for all $y \in E$. Let $E$ be some domain and let $y \in E$ be some element. Let $E'$ be a domain s.t. $|E'| = |E|$ and $x \in E'$. Let $\pi$ be a bijection $E' \to E$ that maps $x$ to $y$. From ISOM, $D_{1E}(\{y\})(B) \leftrightarrow D_{1E}(\pi(\{x\}))(\pi^{-1}(B)) \leftrightarrow D_{1E'}(\{x\})(\pi^{-1}(B)) \leftrightarrow \pi^{-1}(B) \in I_x \leftrightarrow x \in \pi^{-1}(B) \leftrightarrow y \in B$. Thus, $D_{1E}(\{y\}) = I_y$ for all $y \in E$. We have already shown that for all $A \subseteq E$ s.t. $|A| \neq 1$, $D_{1E}(\{x\})$ is trivial with the same value, thus $D_1$ must be PUF, contradicting our assumption.

We have shown that both conditions hold, thus proving the corollary. □

Examples for identical $D_1$ and $D_2$ that are independent are the following cases: $D_1 = D_2 = $ *some, every, some-or-every, some-and-every*. However, independent determiners do not have to be identical. For instance: *each of the two* and *each of the five* are independent determiners, since according to the Russellian definition in (6), they are both in PUF$_\cap^0 \setminus$ TRIV$_1^\exists$.

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