Consequence relations extending modal logic S4.3; an application of projective unification

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Solution: - using the fact that all logics extending S4.3 enjoy projective unification (D-W 2009).

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∘ $\vdash$ is characterized by a class of algebras of the form of the direct products $\mathcal{A} \times \mathcal{H}_n$, where $\mathcal{A} \in \mathcal{K}$ and $\mathcal{H}_n$ is so called *Henle algebra* with n-atoms, i.e. $\vdash$ has Strongly Finite Model Property (*SFMP*).

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• The lattice of all consequence relations extending S4.3 is countable and distributive.

$Var = \{p_1, p_2, \dots\}$ all propositional variables
Fm - modal formulas built up with $\{\wedge, \neg, \Box, \top\}$; Fm_n $\{p_i : i \leq n\}$
$\to, \vee, \leftrightarrow, \Diamond, \bot$ as usual;
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$$MP : \frac{\alpha \to \beta, \alpha}{\beta} \qquad \text{and} \qquad RN : \frac{\alpha}{\Box\alpha}.$$

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Here $\vdash$ denotes a structural global conseq. rel. extending $\vdash_{S4.3}$

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A conseq. rel. $\vdash$ has the Strongly Finite Model Property (*SFMP*) if there is a strongly adequate family $\mathbb{L}$ of finite algebras for $\vdash$.

If $\mathcal{A} = \mathcal{B} \times \mathcal{C}$, then $X \models_{\mathcal{A}} \alpha$ iff $X \models_{\mathcal{B}} \alpha$ and $X \models_{\mathcal{C}} \alpha$,
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If $\mathbb{K}$ is a class of topological BA *TBA* and $\mathcal{A}$ is a finite subdirectly irreducible *TBA*, then $\mathrm{Log}(\mathbb{K}) \subseteq \mathrm{Log}(\mathcal{A})$ iff $\mathcal{A} \in$ *SH*$(\mathbb{K})$.

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Note: $\mathfrak{1}^+ = \mathbf{2} =^{def} (\{\bot, \top\}, min, \neg, \Box)$, with $\Box a = a$.

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Projective unifier (formula) - S.Ghilardi (1999 Unification in INT)

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Rules: Admissible, Derivable, Passive, SC

r $:\ \alpha/\beta$ is *admissible* in *L*, if adding *r* does not change (the theorems of) *L*: $\tau(\alpha) \in L \ \Rightarrow\ \tau(\beta) \in L$, for every substitution τ.

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Every $\vdash$ has the SC extension $\vdash^0$ staisfying $(\bullet)$

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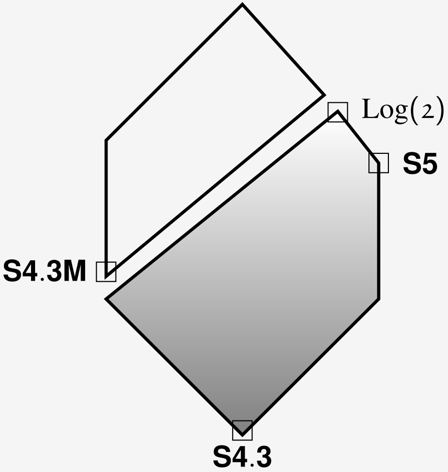
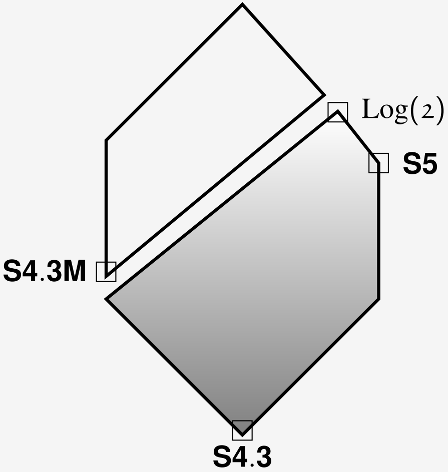
Non-axiomatic extensions $\vdash$ of $\vdash_L$, for $L \in \mathrm{NExt}\mathbf{S4.3}$, can be obtained by adding passive rules only.



Splitting of NExt**S4.3**



Splitting of $\mathrm{NExt}\mathbf{S4.3}$

For $L \in \mathrm{NExt}\mathbf{S4.3M}$, $L \mapsto \vdash_L$ is a bijection (a lattice iso).

FACT. If α is not unifiable and $Var(\alpha) \subseteq \{p_1, \ldots, p_n\}$, then $\alpha \vdash_{S4} (\Diamond p_1 \wedge \Diamond \sim p_1) \vee \cdots \vee (\Diamond p_n \wedge \Diamond \sim p_n)$.

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where $\sigma \colon \{1, \ldots, n\} \to \{0, 1\}$, and $p^0 = p$, and $p^1 = \sim p$. There are 2^n boolean atoms in Fm_n, denoted by: $\theta_1, \ldots, \theta_{2^n}$. Let $\vdash_n$ be the extension of $\vdash_{S4.3}$ with the rule

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The above rule is valid in any $2^n - 1$ (or less) element cluster, and it is not valid in the 2^n element cluster. Hence, for $n \in \omega$,

$\vdash_{S4.3} < \quad \cdots \quad < \quad \vdash_n \quad < \quad \cdots \quad < \quad \vdash_1 = \vdash_{S4.3} + P_2$ and $\vdash_{S4.3} + P_2 \in SC$.

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Each passive rule is equivalent over **S4.3** to a subrule of P_2, to

$$\frac{\Diamond\gamma \wedge \Diamond\sim\gamma}{\delta} \qquad \text{for some } \gamma, \delta.$$

(Rybakov) P_2 forms a basis for admissible (passive) rules over **S4.3.** All passive rules are consequences of P_2 and hence, (see Rybakov):
The modal consequence relation resulting by extending a modal logic $L \supseteq$ **S4.3** with the rule P_2 is structurally complete.

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The modal consequence relation resulting by extending a modal logic $L \supseteq$ **S4.3** with the rule P_2 is structurally complete.

Theorem

Each consequence relation over **S4.3** *can be given by extending a normal modal logic with a collection of passive rules of the form:*

$$\frac{\Diamond\theta_1 \wedge \cdots \wedge \Diamond\theta_s}{\delta}$$

$2 \leq s \leq 2^n$ *and where* $\{p_1, \ldots, p_n\} \cap \mathit{Var}(\delta) = \emptyset$

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TASK: Find a class $\mathbb{L}$ of algebras which is strongly adequate for $\vdash$, i.e. such that for each finite X and each α

$$X \vdash \alpha \qquad \text{iff} \qquad X \models_{\mathbb{L}} \alpha \qquad \big(\text{ iff } \qquad X \models_{\mathcal{B}} \alpha, \ \text{ for each } \mathcal{B} \in \mathbb{L}\big)$$

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Let $\mathbb{K}^{\vdash} = \{\mathcal{B} \in \mathbb{K} : \ \vdash \ \leq \ \models_{\mathcal{B}}\}$ be the class of algebras from $\mathbb{K}$ which are models for $\vdash$. $\mathbb{K}^{\vdash}$ is not sufficient to characterize $\vdash$.

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$$\alpha \models_{\mathbb{K}} \beta \qquad \textit{iff} \qquad \Box\alpha \to \beta \in \mathrm{Log}(\mathbb{K}), \qquad \textit{for each } \alpha, \beta$$

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No class of s.i. $S4.3$-algebras can be strongly adequate for any proper extension of $\vdash_L$ with passive rules. To get models for $\vdash$ products of s.i. algebras with Henle algebras are necessary.

Theorem

Let $\vdash$ be an extension of $\vdash_L$, for $L \in \mathrm{NExt}\mathbf{S4.3}$, with some passive rules and let $\mathbb{K}$ be a class of sub. irr. alg. strongly adequate for $\vdash_L$. Then

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Theorem

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Moreover there are classes $\mathbb{K}_1, \mathbb{K}_2, \ldots, \mathbb{K}_m$ *such that* $S(\mathbb{K}) \supseteq \mathbb{K}_1 \supseteq \mathbb{K}_2 \supseteq \cdots \supseteq \mathbb{K}_m$ *and*

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Corollary

Every finitary modal consequence relation extending **S4.3** *has the strongly finite model property.*

Decidability, SC extensions, $\mathrm{EXT}(\mathbf{S4.3})$ Distributive

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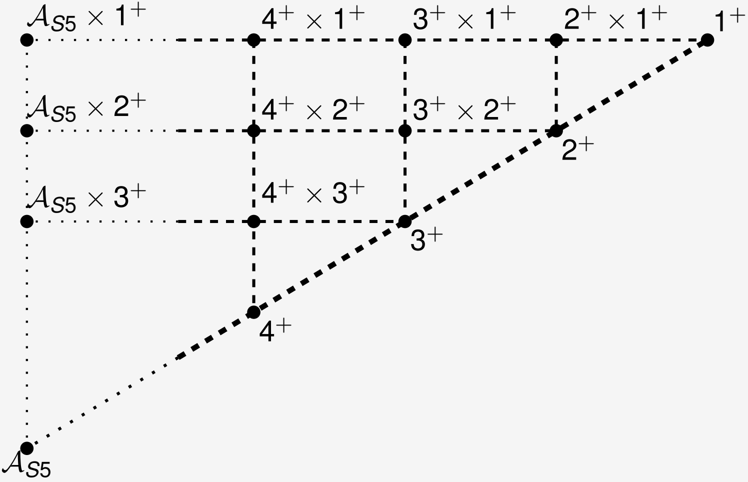
$\mathcal{A}_{S5} \times 1^+$
$4^+ \times 1^+$
$3^+ \times 1^+$
$2^+ \times 1^+$
1^+
$\mathcal{A}_{S5} \times 2^+$
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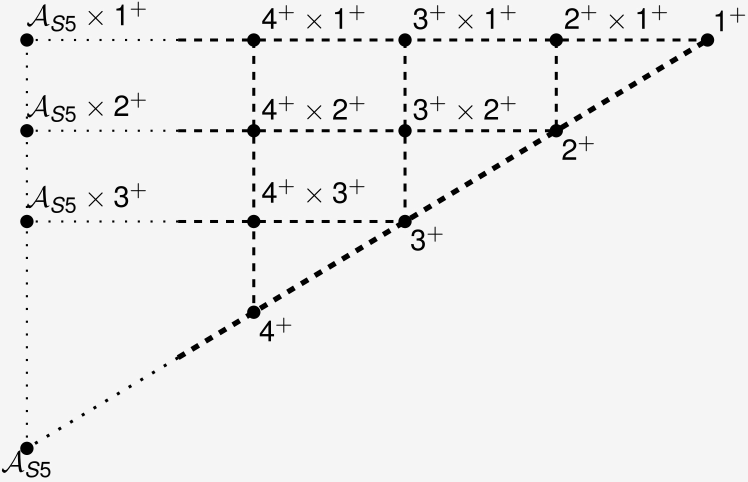

$\mathcal{A}_{S5}$ a countable BA with the Henle operator $\Box$
strongly adequate for S5 (R.Suszko, 70's)