SUBFRAMIZATION AND STABILIZATION FOR SUPERINTUITIONISTIC LOGICS

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ABSTRACT. With each superintuitionistic logic (si-logic for sort), we associate its downward and upward subframizations, and characterize them by means of Zakharyaschev's canonical formulas, as well as by embedding si-logics into the extensions of the propositional lax logic PLL. In an analogous fashion, with each si-logic, we associate its downward and upward stabilizations, and characterize them by means of stable canonical formulas, as well as by embedding si-logics into extensions of the intuitionistic S4.

1. INTRODUCTION

Subframe logics form a well-behaved class of normal uni-modal logics (see, e.g., [19, 30, 31, 14]). They are characterized by a class of descriptive frames closed under subframes, which algebraically correspond to relativizations; all transitive subframe logics have the finite model property (fmp for short), and a transitive logic is a subframe logic iff it is axiomatized by subframe formulas (see, e.g., [31], [14, Sec. 11.3]).

Subframe logics form a complete sublattice of the lattice of all normal modal logics. Therefore, for each modal logic L, there is a greatest subframe logic contained in L, and a least subframe logic containing L, called the *downward* and *upward subframizations* of L. They were studied by Wolter [30, 31] who characterized the downward and upward subframizations in terms of relativizations.

Superintuitionistic logics (si-logics for short) are extensions of intuitionistic propositional calculus IPC. Subframe si-logics have similar properties to transitive subframe logics. They were studied by Zakharyaschev [33] who proved that they are exactly the si-logics axiomatized by $(\wedge, \to)$-formulas.

Another well-behaved class of logics is that of stable logics. Stable si-logics were introduced in [6] and stable modal logics in [8]. Stable logics parallel subframe logics in many ways. While subframe logics are characterized by a class of descriptive frames closed under subframes, stable logics are characterized by a class of rooted descriptive frames closed under relation-preserving images. Transitive subframe logics are those modal logics for which the fmp can be proved via the selective filtration method, while stable logics are those modal logics for which the fmp can be proved via the standard filtration method. The same parallel holds in the intuitionistic setting.

From the point of view of epistemic logic, subframe logics are exactly the logics admitting epistemic updates (cf. [3, Ch. 2], [15, Sec. 7.4–7.5]), while stable logics are the ones admitting epistemic abstraction [2].

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From the algebraic perspective, subframe si-logics are characterized by a class of Heyting algebras closed under $(\wedge, \to)$-subalgebras, while stable si-logics are characterized by a class of Heyting algebras closed under $(\wedge, \vee, 0, 1)$-subalgebras. Subframe si-logics are axiomatized by subframe formulas, which are obtained from Zakharyaschev's canonical formulas by deleting the parameter of closed domains. Similarly, stable si-logics are axiomatized by stable formulas, which are obtained from stable canonical formulas of [6] by deleting the parameter of closed domains. Both classes of subframe and stable si-logics have the cardinality that of the continuum [14, 6].

In this paper we observe further analogies between the classes of subframe and stable si-logics, which come from moving to the realm of intuitionistic modal logic. We show that subframe si-logics are related to the propositional lax logic PLL [17] and stable si-logics to IS4, the intuitionistic S4 [27].

As was shown by Goldblatt [21], PLL provides a link between Grothendieck topology and geometric modality. This in turn gives rise to nuclei on Heyting algebras, which yield algebraic semantics for PLL. By [10], nuclei on a Heyting algebra A correspond to subframes of the frame of prime filters of A.

We can translate IPC into PLL via a version of the Gödel-Gentzen translation. This yields two natural embeddings of si-logics into extensions of PLL. Via these embeddings we obtain a new characterization of subframe si-logics.

Similarly a version of the Gödel-McKinsey-Tarski translation yields embeddings of si-logics into extensions of IS4. Since rootedness of a frame is captured by the multiple-conclusion rule $p \vee q/p, q$ (cf. [8, Thm. 8.6]), we embed stable si-logics into multiple-conclusion consequence relations extending $\mathsf{IS4} + p \vee q/p, q$. We show that there are two natural embeddings of si-logics into extensions of $\mathsf{IS4} + p \vee q/p, q$, which yield a new characterization of stable si-logics.

We also investigate the intuitionistic analogues of the operations of downward and upward subframizations of Wolter [30, 31]. Since in the intuitionistic setting subframes do not correspond to relativizations, but rather to nuclei, a characterization of subframization requires a different technique. We give such a characterization for si-logics in terms of Zakharyaschev's canonical formulas. We also characterize the downward subframization via the embeddings into PLL. Analogously, we give a characterization of stabilization for si-logics in terms of stable canonical formulas, and characterize the downward stabilization via the embeddings into $\mathsf{IS4} + p \vee q/p, q$.

In Table 1 we summarize the parallels between subframe and stable si-logics. The table aims to serve as an orientation to the reader. Whereas the first three rows contain known results, the last two rows contain the results obtained in this paper.

The paper is organized as follows. In Section 2 we recall the algebraic and relational semantics of si-logics, and in Section 3 we recall known facts about subframe and stable si-logics, as well as canonical formulas and stable canonical formulas for si-logics. Section 4 introduces the notion of subframization for si-logics and discusses the connection to Wolter's describable operations. We also compute subframizations of many well-known si-logics. In Section 5 we recall the propositional lax logic PLL and review its algebraic and relational semantics. In Section 6 we define two embeddings of si-logics into the extensions of PLL and discuss their properties. Using these embeddings we obtain new characterizations of subframe si-logics. Section 7 parallels Section 4 for stable si-logics. In Section 8 we define two embeddings of si-logics into the extensions of intuitionistic S4 and characterize stable

	Subframe logics	**Stable logics**
A generating class is persistent under	subframes (Theorem 3.2)	stable images (Theorem 3.7)
Corresponding Heyting operator	nucleus (Section 5)	interior operator (Definition 8.2 and discussion thereafter)
Axiomatization in terms of	subframe formulas (Theorem 3.3)	stable formulas (Theorem 3.9)
Characterization via Wolter's describable operations	Zakharyaschev's canonical formulas (Theorem 4.4, Remark 4.6)	stable canonical formulas (Theorem 7.4, Remark 7.6)
Characterized by translations into	PLL via nucleic Gödel-Gentzen (Theorem 6.18)	$\mathsf{IS4}$ via Gödel-McKinsey-Tarski (Theorem 8.11)

TABLE 1. Parallels between subframe and stable si-logics

si-logics via these embeddings. Finally, in Section 9 we summarize the main results of the paper.

2. Superintuitionistic logics

In this preliminary section we recall algebraic and relational semantics of IPC. We use [28] and [14] as our basic references.

2.1. **Algebraic semantics.** Algebraic semantics of IPC is given by Heyting algebras. A *Heyting algebra* is a bounded distributive lattice A with an additional binary operation $\to$, called *Heyting implication*, that is residual to $\wedge$; that is, for all $a, b, x \in A$, we have:

$$x \leq a \to b \text{ iff } x \wedge a \leq b.$$

A *valuation* v on a Heyting algebra A assigns to propositional letters elements of A, and evaluates the logical connectives $\wedge, \vee, \to, \neg$ as the corresponding operations of A. Then for a formula φ, we have $v(\varphi) \in A$, and φ is *true* in A under v if $v(\varphi) = 1$. We say that φ is *valid* in A, and write $A \models \varphi$, provided φ is true in A under every valuation. A set of formulas Γ is *valid* in A, written $A \models \Gamma$, provided each $\varphi \in \Gamma$ is valid in A.

For a class K of Heyting algebras, φ is *valid* in K, written $K \models \varphi$, provided φ is valid in each $A \in K$; and a set of formulas Γ is *valid* in K, written $K \models \Gamma$, provided $K \models \varphi$ for each $\varphi \in \Gamma$.

For each si-logic L, let $Alg(\mathsf{L}) = \{A \mid A \models \mathsf{L}\}$ be the class of algebraic models of L. Then $Alg(\mathsf{L})$ is a *variety*; that is, it is closed under homomorphic images, subalgebras, and direct products. By the standard Lindenbaum-Tarski construction, each si-logic L is complete with respect to $Alg(\mathsf{L})$. Conversely, each variety K of Heyting algebras gives rise to the si-logic $\mathsf{Log}(K) := \{\varphi \mid K \models \varphi\}$, and we have $\mathsf{Log}(Alg(\mathsf{L})) = \mathsf{L}$ and $Alg(\mathsf{Log}(K)) = K$ (see, e.g., [14, Sec. 7.6]).

2.2. **Relational semantics.** Relational semantics of IPC is given by partial orders $(X, \leq)$, called *(intuitionistic) Kripke frames.* Let $\mathfrak{F} = (X, \leq)$ be an intuitionistic Kripke frame. For

$x \in X$ and $Y \subseteq X$, let

$$\uparrow x = \{y \mid x \le y\} \text{ and } \uparrow Y = \bigcup\{\uparrow y \mid y \in Y\}.$$

We define $\downarrow x$ and $\downarrow Y$ dually. A subset $Y \subseteq X$ is called an *upward closed set* or an *upset* if $y \in Y$ implies $\uparrow y \subseteq Y$. *Downward closed sets* or *downsets* are defined dually.

Let $\mathsf{Up}(\mathfrak{F})$ be the set of upsets of $\mathfrak{F}$. Then $\mathsf{Up}(\mathfrak{F})$ is a Heyting algebra with respect to the operations $\cap, \cup, \to, \neg$ where

$$U \to V := \{x \mid \uparrow x \cap U \subseteq V\} \text{ and } \neg U := U \to \varnothing.$$

A *valuation* v on $\mathfrak{F}$ is a valuation on the Heyting algebra $\mathsf{Up}(\mathfrak{F})$. For $x \in X$ and a formula φ, we write $\mathfrak{F}, x \models_v \varphi$ provided $x \in v(\varphi)$. We then have:

$$\begin{array}{lll} \mathfrak{F}, x \models_v \varphi \wedge \psi & \text{iff} & \mathfrak{F}, x \models_v \varphi \text{ and } \mathfrak{F}, x \models_v \psi \\ \mathfrak{F}, x \models_v \varphi \vee \psi & \text{iff} & \mathfrak{F}, x \models_v \varphi \text{ or } \mathfrak{F}, x \models_v \psi \\ \mathfrak{F}, x \models_v \varphi \to \psi & \text{iff} & \mathfrak{F}, y \models_v \varphi \text{ implies } \mathfrak{F}, y \models_v \psi \text{ for every } y \ge x. \end{array}$$

FIGURE 1. Evaluating formulas

With every Heyting algebra A we can associate the intuitionistic Kripke frame $\mathfrak{F}_A$ of prime filters of A ordered by set-theoretic inclusion. Then A embeds in $\mathsf{Up}(\mathfrak{F}_A)$ via the map $\alpha : A \to \mathsf{Up}(\mathfrak{F}_A)$ given by

$$\alpha(a) = \{x \mid a \in x\}.$$

Therefore, $\alpha[A]$ is a subalgebra of $\mathsf{Up}(\mathfrak{F}_A)$. If we define a topology on $\mathfrak{F}_A$ by letting

$$\mathcal{B} := \{\alpha(a) \setminus \alpha(b) \mid a, b \in A\}$$

be a basis for the topology, then $\mathfrak{F}_A$ becomes a Stone space (zero-dimensional compact Hausdorff space) such that $\uparrow x$ is closed for each x and $\downarrow U$ is clopen for each clopen U. Such spaces are known as *Esakia spaces* or *Esakia frames*.

The image $\alpha[A]$ is then exactly the Heyting subalgebra of $\mathsf{Up}(\mathfrak{F}_A)$ consisting of clopen upsets of $\mathfrak{F}_A$. This yields the Esakia representation for Heyting algebras:

Theorem 2.1. [16] *Each Heyting algebra is isomorphic to the Heyting algebra of clopen upsets of an Esakia frame.*

In fact, Esakia duality establishes that the category of Heyting algebras and Heyting algebra homomorphisms is dually equivalent to the category of Esakia frames and *Esakia morphisms* (that is, continuous order preserving maps $f : X \to Y$ such that $\uparrow f(x) = f(\uparrow x)$ for each $x \in X$).

Remark 2.2. The category of Esakia frames is isomorphic to the category of intuitionistic descriptive frames. We recall (see, e.g., [14, Ch. 8]) that a *general* (*intuitionistic*) *frame* is a tuple $\mathfrak{F} = (X, \le, \mathcal{P})$, where $(X, \le)$ is an intuitionistic Kripke frame and $\mathcal{P}$ is a Heyting subalgebra of the upsets of $(X, \le)$; a general frame $\mathfrak{F}$ is *descriptive* if it is *tight* ($x \not\le y$ implies there is $U \in \mathcal{P}$ with $x \in U$ and $y \notin U$) and *compact* (if $\bigcap_{i \in I}(U_i \setminus V_i) = \varnothing$ for $U_i, V_i \in \mathcal{P}$, then this holds already for some finite subset J of I). The isomorphism is obtained as follows. For a descriptive frame $\mathfrak{F} = (X, \le, \mathcal{P})$, let τ be the topology generated by $\{U \setminus V \mid U, V \in \mathcal{P}\}$. Then $(X, \le, \tau)$ is an Esakia frame. Conversely, for an Esakia frame

$\mathfrak{F} = (X, \leq, \tau)$, let $\mathcal{P}$ be the Heyting subalgebra of $\mathsf{Up}(\mathfrak{F})$ consisting of clopen upsets of $\mathfrak{F}$. Then $(X, \leq, \mathcal{P})$ is a descriptive frame, and these correspondences are one-to-one. Moreover, $f : X \to Y$ is an Esakia morphism iff it is a descriptive frame morphism, thus yielding the desired isomorphism between the categories of descriptive frames and Esakia frames.

Let A be a Heyting algebra. Then A is *subdirectly irreducible* (s.i. for short) if A has a second largest element; that is, there is $c \neq 1 \in A$ such that $b \leq c$ for each $b \neq 1$. Moreover, A is *well-connected* if $a \vee b = 1$ implies $a = 1$ or $b = 1$ for all $a, b \in A$. It is easy to see that every s.i. Heyting algebra is well-connected, and that the two notions coincide for finite Heyting algebras. An Esakia frame $\mathfrak{F} = (X, \leq)$ is *rooted* if there is $x \in X$ such that $X = {\uparrow}x$, and it is *strongly rooted* if it is rooted and the root x is an isolated point. By Esakia duality, A is well-connected iff $\mathfrak{F}_A$ is rooted, and A is s.i. iff $\mathfrak{F}_A$ is strongly rooted. Consequently, if A is finite, then A is s.i. iff $\mathfrak{F}_A$ is rooted.

By Esakia duality, homomorphic images of A correspond to generated subframes of the dual Esakia frame $\mathfrak{F}_A$, where a *generated subframe* is a closed upset of an Esakia frame. By a *point-generated subframe* of an Esakia frame $\mathfrak{F}$ we mean a rooted generated subframe. By the above, well-connected homomorphic images of A correspond to point-generated subframes of $\mathfrak{F}_A$.

Since every si-logic L is complete with respect to the s.i. Heyting algebras that validate L, from the above we deduce that L is complete with respect to the rooted Esakia frames that validate L.

3. Subframe logics and stable logics

In this section we summarize known facts about subframe and stable si-logics. We also recall the definition of Zakharyaschev's canonical formulas for IPC and stable canonical formulas of [6], and discuss their connection.

3.1. **Subframes, subframe logics, and canonical formulas.** Let $\mathfrak{F} = (X, \leq)$ and $\mathfrak{G} = (X', \leq')$ be Esakia frames. We recall [33, 10] that $\mathfrak{G}$ is a *subframe* of $\mathfrak{F}$ if X' is a closed subspace of X, $\leq'$ is the restriction of $\leq$, and for each clopen U of X', the set ${\downarrow}U$ is clopen in X .

Definition 3.1. A si-logic L is called a *subframe logic* if its class of Esakia frames is closed under subframes; that is, if $\mathfrak{F}$ is an L-frame, then so is every subframe of $\mathfrak{F}$.

In the following theorem, we summarize the properties of subframe logics that we will often use throughout the paper. For a proof of the theorem, see for example [14, Section 11.3]. We refer to formulas using only the connectives $\wedge$ and $\to$ as *$(\wedge, \to)$-formulas*.

Theorem 3.2 (Zakharyaschev)**.** *For a si-logic L, the following are equivalent:*

(i) L *is a subframe logic.*
(ii) L *is characterized by a class of Esakia frames closed under subframes.*
(iii) L *is axiomatizable by $(\wedge, \to)$-formulas.*

Moreover, all subframe logics have the fmp, and they form a complete sublattice of the lattice of all si-logics.

Next we recall the definition of canonical formulas and subframe formulas for IPC. The key property of canonical formulas is that they axiomatize all si-logics in a uniform way. This has the advantage that many properties of logics such as the fmp or decidability can be investigated by considering canonical formulas only. We refer the reader to [14] for motivation and many applications of canonical formulas.

Let $\mathfrak{H} = (Y, \leq)$ be a finite rooted frame and $\mathfrak{D}$ a family of upsets of $\mathfrak{H}$, called *closed domains*. For each $x \in Y$ we introduce a new propositional variable p_x and define the *canonical formula* associated with $(\mathfrak{H}, \mathfrak{D})$ as

$$\beta(\mathfrak{H}, \mathfrak{D}) = \bigwedge_{x \leq y}[(\bigwedge_{y \not\leq z} p_z \to p_y) \to p_x] \wedge \bigwedge_{d \in \mathfrak{D}}[\bigwedge_{x \notin d}(\bigwedge_{x \not\leq z} p_z \to p_x) \to \bigvee_{y \in d} p_y] \to p_r,$$

where $x, y, z \in Y$ and r is the root of $\mathfrak{H}$.

Let $\mathfrak{F} = (X, \leq)$ be an Esakia frame. We say that a p-morphism f from a subframe $\mathfrak{G} = (S, \leq)$ of $\mathfrak{F}$ onto $\mathfrak{H}$ satisfies the *closed domain condition* (CDC) for $\mathfrak{D}$ provided

$$x \in {\uparrow}S \text{ and } f({\uparrow}x) \in \mathfrak{D} \text{ imply } x \in S.$$

The following is an important property of canonical formulas:

$$(1) \qquad \mathfrak{F} \not\models \beta(\mathfrak{H}, \mathfrak{D}) \text{ iff } \begin{array}{l}\text{there is a p-morphism from a subframe of } \mathfrak{F} \text{ onto } \mathfrak{H} \\ \text{satisfying CDC for } \mathfrak{D}.\end{array}$$

If $\mathfrak{D} = \varnothing$, then $\beta(\mathfrak{H}, \mathfrak{D})$ is denoted by $\beta(\mathfrak{H})$ and is called the *subframe formula of* $\mathfrak{H}$. From the above we obtain:

$$(2) \qquad \mathfrak{F} \not\models \beta(\mathfrak{H}) \quad \text{iff there is a p-morphism from a subframe of } \mathfrak{F} \text{ onto } \mathfrak{H}.$$

Theorem 3.3 (Zakharyaschev)**.**
(i) *Every si-logic is axiomatizable by canonical formulas.*
(ii) *A si-logic is a subframe logic iff it is axiomatizable by subframe formulas.*

Remark 3.4. The above presentation of canonical formulas follows Jeřábek's account [23, Sec. 3], which is slightly different from Zakharyaschev's approach. Namely, our closed domains are upsets rather than antichains. Also, closed domains may be empty, which allows us to work with subframes rather than cofinal subframes (see [23, Rem. 3.7]).

An algebraic formulation of Zakharyaschev's canonical formulas was provided in [5] (some of the ideas can be traced back to [29]). We recall some details since it will enable us to see most easily the analogy with stable canonical formulas. In order to match Jeřábek's account, we slightly alter the approach of [5], so that our presentation of the canonical formula $\beta(B, D)$ combines the formulas $\beta(B, D, \bot)$ and $\beta(B, D)$ of [5].

Let B be a finite s.i. Heyting algebra with the second largest element s, let $* \notin B^2$, and let D be a subset of $B^2 \cup \{*\}$. For each $a \in B$, we introduce a new variable p_a, set

$$\begin{aligned}\Gamma = \{&p_{a \wedge b} \leftrightarrow p_a \wedge p_b \mid a, b \in B\} \cup \\ \{&p_{a \to b} \leftrightarrow p_a \to p_b \mid a, b \in B\} \cup \\ \{&p_{a \vee b} \leftrightarrow p_a \vee p_b \mid (a, b) \in D\} \cup \\ \{&p_0 \leftrightarrow 0 \mid * \in D\},\end{aligned}$$

and define the canonical formula of (B, D) as

$$\beta(B, D) = \bigwedge \Gamma \to p_s.$$

It follows from [5] that for any Heyting algebra A, we have: $A \not\models \beta(B, D)$ iff there is a homomorphic image C of A and a $(\wedge, \to)$-embedding $h : B \to C$ such that $h(a \vee b) = h(a) \vee h(b)$ for all $(a, b) \in D$ and $h(0) = 0$ if $* \in D$.

Roughly speaking, $\beta(B, D)$ describes fully the $(\wedge, \to)$-structure of B, and the $(\vee, 0)$-structure only partially on the declared subset D.

Although the formulas $\beta(B, D)$ and $\beta(\mathfrak{H}, \mathfrak{D})$ look syntactically quite different, they are equivalent as discussed in [5, Remark 5.6]. In particular, if B is a finite s.i. Heyting algebra whose dual is $\mathfrak{H}$ and $D \subseteq B^2 \cup \{*\}$, then there is a collection $\mathfrak{D}$ of upsets of $\mathfrak{H}$ such that for any Esakia frame $\mathfrak{F}$, we have:

$$\mathfrak{F} \models \beta(B, D) \quad \text{iff} \quad \mathfrak{F} \models \beta(\mathfrak{H}, \mathfrak{D}).$$

Remark 3.5. From the algebraic and frame-theoretic characterization of canonical formulas discussed above one may get the impression that $(\wedge, \to)$-subalgebras are dual to subframes. The correspondence is a bit more subtle, as we will explain in Section 5.

3.2. **Stable logics and stable canonical formulas.** We recall that a map between Esakia frames is *stable* if it is continuous and order preserving, and that an Esakia frame $\mathfrak{G}$ is a *stable image* of an Esakia frame $\mathfrak{F}$ if there is a stable map from $\mathfrak{F}$ onto $\mathfrak{G}$. It is easy to see that stable images of rooted frames are rooted.

Definition 3.6. A si-logic L is *stable* if the class of rooted Esakia frames validating L is closed under stable images; that is, if $\mathfrak{F}$ is a rooted L-frame, then so is every stable image of $\mathfrak{F}$.

In analogy with subframe logics, stable logics have the following properties (cf. Theorem 3.2):

Theorem 3.7. *For a si-logic L, the following are equivalent:*

(i) L *is stable.*
(ii) L *is characterized by a class of rooted Esakia frames closed under stable images.*
(iii) *The rooted L-frames are closed under finite stable images.*

Moreover, all stable logics have the fmp, and they form a complete sublattice of the lattice of all si-logics.

Proof. The equivalence of (i) and (ii) is proved in [9, Thm. 5.3]. It is clear that (i) implies (iii). To see that (iii) implies (i), let $\mathfrak{F}$ be a rooted L-frame and let $\mathfrak{G}$ be a stable image of $\mathfrak{F}$. If $\mathfrak{G}$ is not an L-frame, then $\mathfrak{G} \not\models \varphi$ for some $\varphi \in \mathsf{L}$. By [6, Lem. 3.6], there is a finite stable image $\mathfrak{H}$ of $\mathfrak{G}$ such that $\mathfrak{H} \not\models \varphi$. Therefore, $\mathfrak{H}$ is a finite stable image of $\mathfrak{F}$. By (iii), $\mathfrak{H}$ is an L-frame, contradicting $\mathfrak{H} \not\models \varphi$. Thus, $\mathfrak{G}$ is an L-frame, and hence L is stable.

That all stable logics have the fmp is shown in [6, Thm. 6.8]. We show that they form a complete sublattice of the lattice of all si-logics. Let $\{\mathsf{L}_i \mid i \in I\}$ be a family of stable logics. Then the classes of rooted L_i-frames are stable. Therefore, so are the classes $\bigcap_{i\in I}\{\mathfrak{F} \mid \mathfrak{F} \text{ is a rooted } \mathsf{L}_i\text{-frame}\}$ and $\bigcup_{i\in I}\{\mathfrak{F} \mid \mathfrak{F} \text{ is a rooted } \mathsf{L}_i\text{-frame}\}$. The intersection $\bigcap_{i\in I}\{\mathfrak{F} \mid \mathfrak{F} \text{ is a rooted } \mathsf{L}_i\text{-frame}\}$ is exactly the class of all rooted $(\bigvee_{i\in I} \mathsf{L}_i)$-frames. Since every si-logic is characterized by its rooted frames, $\bigvee_{i\in I} \mathsf{L}_i$ is characterized by $\bigcap_{i\in I}\{\mathfrak{F} \mid \mathfrak{F} \text{ is a rooted } \mathsf{L}_i\text{-frame}\}$. Thus, by (ii), $\bigvee_{i\in I} \mathsf{L}_i$ is stable. The logic $\bigwedge_{i\in I} \mathsf{L}_i$ is characterized by $\bigcup_{i\in I}\{\mathfrak{F} \mid \mathfrak{F} \text{ is a rooted } \mathsf{L}_i\text{-frame}\}$ (see, e.g., [14, Sec. 4]). Therefore, $\bigwedge_{i\in I} \mathsf{L}_i$ is also stable. □

Remark 3.8. Subframe logics have the property that *all* their frames are closed under subframes. An analogue of this statement is not true for stable logics as only the rooted frames are required to be closed under stable images (compare Theorems 3.2 and 3.7). In fact, IPC is the only stable logic such that all its frames are closed under stable images (see [9, Remark 5.6]). This discrepancy between subframe and stable logics will often occur in this paper.

Stable canonical formulas were introduced in [6] as an alternative to Zakharyaschev's canonical formulas. They can most easily be described in algebraic terms. As we pointed out in Section 3.1, from the algebraic perspective, Zakharyaschev's canonical formulas encode the $(\wedge, \to)$-structure of a finite s.i. Heyting algebra B and encode the structure of the "missing" operations $\vee$ and 0 only partially on the designated set $D \subseteq B^2 \cup \{*\}$. In contrast, stable canonical formulas encode the $(\wedge, \vee, 0, 1)$-structure of a finite s.i. Heyting algebra B, and the structure of the missing operation $\to$ partially on a designated subset $D \subseteq B^2$.

Given a finite s.i. Heyting algebra B and $D \subseteq B^2$, we introduce a new variable p_a for each $a \in B$, set

$$\begin{aligned}\Gamma = &\{p_0 \leftrightarrow \bot\} \cup \{p_1 \leftrightarrow \top\} \cup \\ &\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b \mid a, b \in B\} \cup \\ &\{p_{a\vee b} \leftrightarrow p_a \vee p_b \mid a, b \in B\} \cup \\ &\{p_{a\to b} \leftrightarrow p_a \to p_b \mid (a, b) \in D\}, \\ \Delta = &\{p_a \to p_b \mid a, b \in B \text{ with } a \not\leq b\},\end{aligned}$$

and define the *stable canonical formula* of (B, D) as

$$\gamma(B, D) = \bigwedge \Gamma \to \bigvee \Delta.$$

By [6, Thm. 3.4], for any Heyting algebra A, we have: $A \not\models \gamma(B, D)$ iff there is a well-connected homomorphic image C of A and a bounded lattice embedding $h : B \to C$ such that $h(a \to b) = h(a) \to h(b)$ for all $(a, b) \in D$.

This can easily be translated into frame-theoretic terms. Let $\mathfrak{G}$ and $\mathfrak{H}$ be Esakia frames, with $\mathfrak{H}$ finite, and let $\mathfrak{D}$ be a set of subsets of $\mathfrak{H}$. We say that a stable onto map $f : \mathfrak{G} \to \mathfrak{H}$ satisfies the *stable closed domain condition* (SCDC) for $\mathfrak{D}$ provided

$$\uparrow f(x) \cap d \neq \varnothing \ \Rightarrow \ f[\uparrow x] \cap d \neq \varnothing \ \text{ for all } d \in \mathfrak{D}.$$

We write $\gamma(\mathfrak{H}, \mathfrak{D})$ for the canonical formula $\gamma(B, D)$, where B is the dual Heyting algebra of $\mathfrak{H}$ and $D = \{(U, V) \mid U \setminus V \in \mathfrak{D}\}$ for upsets U, V of $\mathfrak{H}$. Then for every Esakia frame $\mathfrak{F}$ we have:

(3) $\mathfrak{F} \not\models \gamma(\mathfrak{H}, \mathfrak{D})$ iff there are a point-generated subframe $\mathfrak{G}$ of $\mathfrak{F}$ and a stable onto map $f : \mathfrak{G} \to \mathfrak{H}$ satisfying SCDC for $\mathfrak{D}$.

Stable canonical formulas of the form $\gamma(\mathfrak{H}, \varnothing)$ are called *stable formulas* and are denoted by $\gamma(\mathfrak{H})$. As follows from the above,

(4) $\mathfrak{F} \not\models \gamma(\mathfrak{H})$ iff there are a point-generated subframe $\mathfrak{G}$ of $\mathfrak{F}$ and a stable onto map $f : \mathfrak{G} \to \mathfrak{H}$.

An analogy between stable canonical formulas and Zakharyaschev's canonical formulas, as well as between stable formulas and subframe formulas can be seen from the following theorem (cf. Theorem 3.3).

Theorem 3.9 ([6]).
(i) *Every si-logic is axiomatizable by stable canonical formulas.*

(ii) *A si-logic is stable iff it is axiomatizable by stable formulas.*

Remark 3.10. The definition of stable logics (Definition 3.6) is slightly different from [6, Def. 6.6], but it follows from [9, Thm. 5.3] that the two are equivalent. Similarly, in the refutation criterion [6, Thm. 3.4] for stable canonical formulas we replaced subdirectly irreducibles algebras by well-connected ones. That this is legitimate can be easily verified. The reason for this deviation is that we find it more natural to work with rooted frames as opposed to strongly rooted ones.

4. Subframization

Let Λ_{Subf} be the class of subframe logics. Since Λ_{Subf} is a complete sublattice of the lattice of all si-logics, for each si-logic L, there is a greatest subframe logic contained in L and a least subframe logic containing L (cf. [30, 31]).

Definition 4.1. For a si-logic L, define the *downward subframization of* L as

$$\mathsf{Subf}_{\downarrow}(\mathsf{L}) := \bigvee\{\mathsf{L}' \in \Lambda_{\mathsf{Subf}} \mid \mathsf{L}' \subseteq \mathsf{L}\}$$

and the *upward subframization of* L as

$$\mathsf{Subf}_{\uparrow}(\mathsf{L}) := \bigwedge\{\mathsf{L}' \in \Lambda_{\mathsf{Subf}} \mid \mathsf{L} \subseteq \mathsf{L}'\}.$$

We summarize some rather obvious facts about the downward and upward subframizations that we will use throughout the paper.

Lemma 4.2.

(i) $\mathsf{Subf}_{\downarrow}$ *is an interior operator and* $\mathsf{Subf}_{\uparrow}$ *is a closure operator on the lattice of si-logics.*
(ii) $\mathsf{Sub}_{\downarrow}(\mathsf{L}) = \mathsf{IPC} + \{\varphi \mid \varphi$ *is a* $(\wedge, \to)$*-formula and* $\mathsf{L} \vdash \varphi\}$.
(iii) $\mathsf{Subf}_{\downarrow}(\mathsf{L}) = \mathsf{IPC}$ *iff for every* $(\wedge, \to)$*-formula* φ*,* $\mathsf{L} \vdash \varphi$ *iff* $\mathsf{IPC} \vdash \varphi$*.*

Proof. (i). Straightforward from the definition.

(ii). By Theorem 3.2, every subframe logic is axiomatizable by $(\wedge, \to)$-formulas. Therefore, every subframe logic contained in L is axiomatizable by a set of $(\wedge, \to)$-formulas that are provable in L. Thus, the set $\{\varphi \mid \varphi$ is a $(\wedge, \to)$-formula and $\mathsf{L} \vdash \varphi\}$ axiomatizes the largest subframe logic contained in L.

(iii). Apply (ii). □

We next give a semantic characterization of the downward and upward subframizations of a si-logic L. Recall that if K is a class of frames, then $\mathsf{Log}(K) = \{\varphi \mid K \models \varphi\}$ is the si-logic of K.

Proposition 4.3. *Let* L *be a si-logic such that* $\mathsf{L} = \mathsf{Log}(K)$ *for some class* K *of Esakia frames. Then*

(i) $\mathsf{Subf}_{\downarrow}(\mathsf{L}) = \mathsf{Log}\left(\{\mathfrak{G} \mid \mathfrak{G} \text{ is a subframe of some } \mathfrak{F} \in K\}\right)$.
(ii) $\mathsf{Subf}_{\uparrow}(\mathsf{L}) = \mathsf{Log}\left(\{\mathfrak{F} \mid \mathfrak{G} \models \mathsf{L} \text{ for all subframes } \mathfrak{G} \text{ of } \mathfrak{F}\}\right)$
$= \mathsf{Log}\left(\{\mathfrak{F} \mid \mathfrak{F} \text{ is finite and } \mathfrak{G} \models \mathsf{L} \text{ for all subframes } \mathfrak{G} \text{ of } \mathfrak{F}\}\right)$.

Proof. (i). Let $K' = \{\mathfrak{G} \mid \mathfrak{G}$ is a subframe of some $\mathfrak{F} \in K\}$. Then $K \subseteq K'$, so $\mathsf{Log}(K') \subseteq \mathsf{Log}(K) = \mathsf{L}$. Since K' is closed under subframes, $\mathsf{Log}(K')$ is a subframe logic by Theorem 3.2. If L' is a subframe logic contained in L, then $K \models \mathsf{L}'$, so $K' \models \mathsf{L}'$ as L' is a subframe logic. Therefore, $\mathsf{L}' \subseteq \mathsf{Log}(K')$. Thus, $\mathsf{Log}(K')$ is the largest subframe logic contained in L, and hence $\mathsf{Subf}_{\downarrow}(\mathsf{L}) = \mathsf{Log}(K')$.

(ii). Let $K' = \{\mathfrak{F} \mid \mathfrak{G} \models \mathsf{L}$ for all subframes $\mathfrak{G}$ of $\mathfrak{F}\}$ and $K'' = \{\mathfrak{F} \mid \mathfrak{F}$ is finite and $\mathfrak{G} \models \mathsf{L}$ for all subframes $\mathfrak{G}$ of $\mathfrak{F}\}$. We show that $\mathsf{Subf}_{\uparrow}(\mathsf{L}) = \mathsf{Log}(K') = \mathsf{Log}(K'')$. It is clear that $K'' \subseteq K'$, so $\mathsf{Log}(K') \subseteq \mathsf{Log}(K'')$. It is obvious that both K' and K'' are closed under subframes, so both $\mathsf{Log}(K'), \mathsf{Log}(K'')$ are subframe logics by Theorem 3.2. Moreover, $K', K'' \models \mathsf{L}$ imply $\mathsf{L} \subseteq \mathsf{Log}(K'), \mathsf{Log}(K'')$. Therefore, $\mathsf{Subf}_{\uparrow}(\mathsf{L}) \subseteq \mathsf{Log}(K'), \mathsf{Log}(K'')$. It is left to prove that $\mathsf{Log}(K'') \subseteq \mathsf{Subf}_{\uparrow}(\mathsf{L})$.

Let L' be a subframe logic containing L. If $\mathfrak{F}$ is a finite frame such that $\mathfrak{F} \models \mathsf{L}'$, then since L' is a subframe logic, $\mathfrak{G} \models \mathsf{L}'$ for every subframe $\mathfrak{G}$ of $\mathfrak{F}$. But then $\mathfrak{G} \models \mathsf{L}$ as $\mathsf{L} \subseteq \mathsf{L}'$, so $\mathfrak{F} \in K''$. Therefore, every finite L'-frame is contained in K''. Since L' is a subframe logic, it is the logic of its finite frames, so we have $\mathsf{Log}(K'') \subseteq \mathsf{L}'$. Thus, $\mathsf{Log}(K'')$ is the smallest subframe logic containing L, and hence $\mathsf{Subf}_{\uparrow}(\mathsf{L}) = \mathsf{Log}(K'')$. ☐

We use Proposition 4.3 and Zakharyaschev's canonical formulas (see Section 3.1) to give a syntactic characterization of the downward and upward subframizations of a si-logic L.

Theorem 4.4. *Let* $\mathsf{L} = \mathsf{IPC} + \{\beta(\mathfrak{H}_i, \mathfrak{D}_i) \mid i \in I\}$ *be a si-logic.*

(i) $\mathsf{Subf}_{\downarrow}(\mathsf{L}) = \mathsf{IPC} + \{\beta(\mathfrak{H}) \mid \mathsf{L} \vdash \beta(\mathfrak{H})\}$.

(ii) $\mathsf{Subf}_{\uparrow}(\mathsf{L}) = \mathsf{IPC} + \{\beta(\mathfrak{H}_i) \mid i \in I)\}$.

Proof. (i). By Theorem 3.3, every subframe logic is axiomatizable by subframe formulas. Therefore, every subframe logic contained in L is axiomatizable by a set of subframe formulas that are provable in L. Thus, $\mathsf{IPC} + \{\beta(\mathfrak{H}) \mid \mathsf{L} \vdash \beta(\mathfrak{H})\}$ is the largest subframe logic contained in L.

(ii). Let $\mathsf{M} = \mathsf{IPC} + \{\beta(\mathfrak{H}_i) \mid i \in I\}$. If $\mathfrak{F}$ is an M-frame, then $\mathfrak{F} \models \beta(\mathfrak{H}_i)$ for all $i \in I$. Therefore, by (1) and (2), $\mathfrak{F} \models \beta(\mathfrak{H}_i, \mathfrak{D}_i)$ for all $i \in I$. Thus, $\mathfrak{F}$ is an L-frame, and so $\mathsf{L} \subseteq \mathsf{M}$. Since M is axiomatized by subframe formulas, M is a subframe logic by Theorem 3.3. It remains to show that M is the least subframe logic containing L. If not, then there is a subframe logic $\mathsf{L}' \supseteq \mathsf{L}$ and an L'-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \mathsf{M}$. Therefore, $\mathfrak{F} \not\models \beta(\mathfrak{H}_i)$ for some $i \in I$. By (2), $\mathfrak{H}_i$ is a p-morphic image of a subframe $\mathfrak{G}$ of $\mathfrak{F}$. Since L' is a subframe logic, $\mathfrak{G}$ is an L'-frame. Thus, $\mathfrak{H}_i$ is also an L'-frame. But $\mathfrak{H}_i \not\models \beta(\mathfrak{H}_i, \mathfrak{D}_i)$ by (1) because the identity map is a p-morphism from $\mathfrak{H}_i$ onto itself that satisfies CDC for any set of closed domains. Consequently, $\mathfrak{H}_i$ is not an L-frame, which is a contradiction since $\mathsf{L}' \supseteq \mathsf{L}$. ☐

Remark 4.5.

(i) It follows from Theorem 4.4(ii) that if L is a si-logic axiomatized by a set of formulas Γ, then the upward subframization $\mathsf{Subf}_{\uparrow}(\mathsf{L})$ of L can be calculated effectively from Γ as follows: First use Zakharyaschev's theorem to transform Γ into an equivalent set of canonical formulas; then delete the additional parameters $\mathfrak{D}_i$ in the resulting canonical formulas; and finally apply Theorem 4.4(ii).

(ii) On the other hand, Theorem 4.4(i) does not provide an effective axiomatization of the downward subframization $\mathsf{Subf}_{\downarrow}(\mathsf{L})$ of L. We will come back to this issue at the end of Section 6.

Remark 4.6. In [30] Wolter studied *describable operations* on varieties of modal algebras. This translates to Esakia frames as follows. A map $\mathbf{C}$ that associates with each Esakia frame $\mathfrak{F}$ a class $\mathbf{C}(\mathfrak{F})$ of Esakia frames is *describable* if there is a map $(-)^c$ on the set of formulas of IPC such that for each Esakia frame $\mathfrak{F}$ and each formula φ,

$$\mathfrak{F} \models \varphi^c \text{ iff } \mathbf{C}(\mathfrak{F}) \models \varphi.$$

As follows from [30, p. 23], if L is the logic of a class K of Esakia frames, then the logic of $\mathbf{C}(K)$ is axiomatized by $\{\varphi^c \mid \mathsf{L} \vdash \varphi^c\}$, and the logic of $\{\mathfrak{F} \in K \mid \mathbf{C}(\mathfrak{F}) \subseteq K\}$ is axiomatized by $\{\varphi^c \mid \mathsf{L} \vdash \varphi\}$.

Now let $\mathbf{C}(\mathfrak{F}) = \{\mathfrak{G} \mid \mathfrak{G} \text{ is a subframe of } \mathfrak{F}\}$. Since canonical formulas axiomatize every si-logic, we restrict our attention to the set of canonical formulas. We show that

$$\mathfrak{F} \models \beta(\mathfrak{H}) \text{ iff } \mathbf{C}(\mathfrak{F}) \models \beta(\mathfrak{H}, \mathfrak{D}). \tag{5}$$

The left to right direction is obvious. For the right to left direction, suppose $\mathfrak{F} \not\models \beta(\mathfrak{H})$. Then there is a subframe $\mathfrak{G}$ of $\mathfrak{F}$ which is p-morphically mapped onto $\mathfrak{H}$. Since $\mathfrak{H} \not\models \beta(\mathfrak{H}, \mathfrak{D})$, we have $\mathfrak{G} \not\models \beta(\mathfrak{H}, \mathfrak{D})$. Therefore, we found $\mathfrak{G} \in \mathbf{C}(\mathfrak{F})$ such that $\mathfrak{G} \not\models \beta(\mathfrak{H}, \mathfrak{D})$.

From (5) we obtain that the map defined by $(\beta(\mathfrak{H}, \mathfrak{D}))^c = \beta(\mathfrak{H})$ describes the operation $\mathbf{C}$. Thus, applying Wolter's result to Proposition 4.3 yields an alternative proof of Theorem 4.4.

We conclude this section by providing the downward and and upward subframizations of many well-known si-logics. We will utilize known axiomatizations of these logics via canonical formulas. As in [14, Sec. 9.4], for a finite rooted frame $\mathfrak{H}$, we write $\beta^\sharp(\mathfrak{H})$ for the canonical formula $\beta(\mathfrak{H}, \mathfrak{D})$, where $\mathfrak{D}$ is the set of all nonempty upsets of $\mathfrak{H}$, and we write $\chi(\mathfrak{H})$ for the canonical formula $\beta(\mathfrak{H}, \mathfrak{D})$, where $\mathfrak{D}$ is the set of all (including empty) upsets of $\mathfrak{H}$. The formula $\chi(\mathfrak{H})$ is called the *frame formula* or *Jankov-de Jongh formula* of $\mathfrak{H}$, and $\beta^\sharp(\mathfrak{H})$ is called the *negation free frame formula* or *negation free Jankov-de Jongh formula* of $\mathfrak{H}$.

Following [7], we denote by $\mathfrak{L}$ the Rieger-Nishimura ladder (the dual Esakia frame of the free cyclic Heyting algebra, see Figure 2).

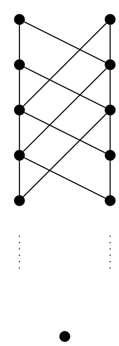


FIGURE 2. The Rieger-Nishimura ladder $\mathfrak{L}$

For Esakia frames $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$, we denote their *ordered sum* by $\bigoplus_{i=1}^n \mathfrak{F}_i$ [7, Sec. 2.2]. We consider the following logics:

- The Rieger-Nishimura logic RN, which is the logic of the Rieger-Nishimura ladder $\mathfrak{L}$.
- The Kuznetsov-Gerciu logic $\mathsf{KG} = (p \to q) \vee (q \to r) \vee ((q \to r) \to r) \vee (r \to (p \vee q))$, which is the logic of $\bigoplus_{i=1}^n \mathfrak{F}_i$, where each $\mathfrak{F}_i$ is a generated subframe of $\mathfrak{L}$.
- The Kreisel-Putnam logic $\mathsf{KP} = \mathsf{IPC} + (\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$.
- The Gabbay-de Jongh logics T_n, where T_n is the logic of finite trees of branching $\leq n$.

- The logics BW_n of finite frames of width $\leq n$. In particular, BW_1 is the Gödel-Dummett logic $\mathsf{LC} = \mathsf{IPC} + (p \to q) \vee (q \to p)$ of finite linear frames.
- The logics BTW_n of finite frames of top width $\leq n$. In particular, BTW_1 is the logic KC of weak excluded middle, which is the logic of finite directed frames.
- Maksimova's logics $\mathsf{ND}_n = \mathsf{IPC} + (\neg p \to \bigvee_{1 \leq i \leq n} \neg q_i) \to \bigvee_{1 \leq i \leq n} (\neg p \to \neg q_i)$.

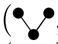
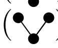
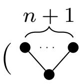
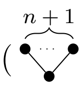
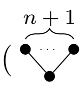
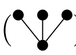
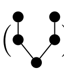
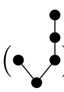
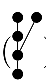
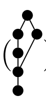
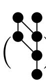
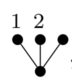
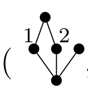
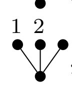
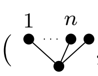
KC	$=$	$\mathsf{IPC} + \beta($$, \{\varnothing\})$
LC	$=$	$\mathsf{IPC} + \beta($$)$
BTW_n	$=$	$\mathsf{IPC} + \beta($$, \{\varnothing\})$
BW_n	$=$	$\mathsf{IPC} + \beta($$)$
T_n	$=$	$\mathsf{IPC} + \beta^\sharp($$)$
KG	$=$	$\mathsf{IPC} + \beta($$) + \beta($$) + \beta($$)$
RN	$=$	$\mathsf{KG} + \chi($$) + \chi($$) + \chi($$)$
KP	$=$	$\mathsf{IPC} + \beta($$, \{\varnothing, \{1,2\}\}) + \beta($$, \{\varnothing, \{1,2\}\})$
ND_n	$=$	$\mathsf{IPC} + \beta($$, \{\varnothing, \{1,2\}\}) + \cdots + \beta($$, \{\varnothing, \{1, \ldots, n\}\})$

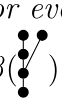
Table 2. Axiomatizations in terms of canonical formulas (see [7, Thm. 3.13] and [7, Thm. 4.33] for the axiomatizations of KG and RN, respectively, and see [14, Table 9.7] for the other cases).

Proposition 4.7.

(i) $\mathsf{Subf}_\downarrow(\mathsf{KC}) = \mathsf{IPC}$ *and* $\mathsf{Subf}_\uparrow(\mathsf{KC}) = \mathsf{LC}$.

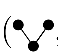
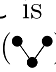
(ii) $\mathsf{Subf}_\downarrow(\mathsf{BTW}_n) = \mathsf{IPC}$ *and* $\mathsf{Subf}_\uparrow(\mathsf{BTW}_n) = \mathsf{BW}_n$ *for every* $n \geq 2$.

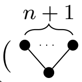
(iii) $\mathsf{Subf}_\downarrow(\mathsf{T}_n) = \mathsf{IPC}$ *and* $\mathsf{Subf}_\uparrow(\mathsf{T}_n) = \mathsf{BW}_n$ *for every* $n \geq 2$.

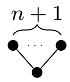
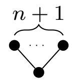
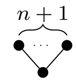
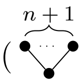
(iv) $\mathsf{Subf}_\downarrow(\mathsf{RN}) = \mathsf{KG}$ *and* $\mathsf{Subf}_\uparrow(\mathsf{RN}) = \mathsf{KG} + \beta($$)$.

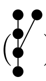
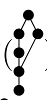
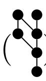
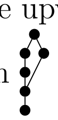
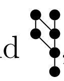
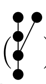
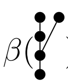
(v) $\mathsf{Subf}_\downarrow(\mathsf{KP}) = \mathsf{IPC}$ *and* $\mathsf{Subf}_\uparrow(\mathsf{KP}) = \mathsf{BW}_2$.

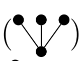
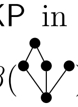
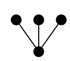
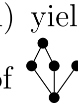
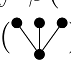
(vi) $\mathsf{Subf}_\downarrow(\mathsf{ND}_n) = \mathsf{IPC}$ *and* $\mathsf{Subf}_\uparrow(\mathsf{ND}_n) = \mathsf{BW}_2$ *for every* $n \geq 2$.

Proof. (i). Since KC is axiomatized by $\beta($$, \{\varnothing\})$, it follows from Theorem 4.4(ii) that $\mathsf{Subf}_\uparrow(\mathsf{KC}) = \mathsf{IPC} + \beta($$) = \mathsf{LC}$. To calculate the downward subframization of KC, we utilize Proposition 4.3(i). It is well known that IPC is the logic of all finite frames and that KC is the logic of all finite directed frames. Moreover, adding a new top to a finite frame $\mathfrak{F}$ results in a finite directed frame $\mathfrak{G}$ containing $\mathfrak{F}$ as a subframe. Therefore, by Proposition 4.3(i), $\mathsf{Subf}_\downarrow(\mathsf{KC}) = \mathsf{IPC}$.

(ii). From the axiomatization of BTW_n in Table 1 and Theorem 4.4(ii) it follows that $\mathsf{Subf}_{\uparrow}(\mathsf{BTW}_n) = \mathsf{IPC} + \beta($$) = \mathsf{BW}_n$. To see that $\mathsf{Subf}_{\downarrow}(\mathsf{BTW}_n) = \mathsf{IPC}$ observe that $\mathsf{BTW}_n \subseteq \mathsf{KC}$ and apply (i) and Lemma 4.2(i).

(iii). It follows from Table 1 that T_n is axiomatized by the negation-free frame formula $\beta^{\sharp}($$) = \beta($$, \mathfrak{D})$, where $\mathfrak{D}$ is the set of all nonempty upsets of . Therefore, $\mathsf{Subf}_{\uparrow}(\mathsf{T}_n) = \mathsf{IPC} + \beta($$) = \mathsf{BW}_n$. To determine the downward subframization, since T_n has the disjunction property [20] and every si-logic with the disjunction property proves the same disjunction-free formulas as IPC [26, 32], we conclude that T_n proves the same $(\wedge, \to)$-formulas as IPC. Thus, by Lemma 4.2(iii), $\mathsf{Subf}_{\downarrow}(\mathsf{T}_n) = \mathsf{IPC}$.

(iv). Since KG is a subframe logic contained in RN (see, e.g., [7, Sec. 3]), it follows from the axiomatization of RN in Table 1 and Theorem 4.4(ii) that the upward subframization of RN is $\mathsf{KG} + \beta($$) + \beta($$) + \beta($$)$. Since  is a subframe of both  and , the latter logic is equal to $\mathsf{KG} + \beta($$)$. Therefore, $\mathsf{Subf}_{\uparrow}(\mathsf{RN}) = \mathsf{KG} + \beta($$)$. To determine the downward subframization, $\mathsf{KG} \subseteq \mathsf{Subf}_{\downarrow}(\mathsf{RN})$ since KG is a subframe logic contained in RN. For the reverse inclusion, since KG is the logic of its finite rooted frames, by Proposition 4.3(i), it is sufficient to show that every finite rooted KG-frame is a subframe of the Rieger-Nishimura ladder $\mathfrak{L}$. First note that the subframe of $\mathfrak{L}$ obtained by deleting the first k layers of $\mathfrak{L}$ is isomorphic to $\mathfrak{L}$. Using this it is easy to see that every finite generated subframe of $\mathfrak{L}$ can be realized as a subframe of $\mathfrak{L}$ at an arbitrary depth, i.e., as a subframe of $\mathfrak{L}$ that does not contain the first k-layers of $\mathfrak{L}$ for any $k \in \mathbb{N}$. Therefore, a finite rooted KG-frame $\bigoplus_{i=1}^{n} \mathfrak{F}_i$ can be realized as a subframe of $\mathfrak{L}$ by embedding $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ below each other so that the two subsequent points in $\mathfrak{L}$ between the embeddings of $\mathfrak{F}_i$ and $\mathfrak{F}_{i+1}$ are skipped.

(v). The axiomatization of KP in Table 1 and Theorem 4.4(ii) yield that $\mathsf{Subf}_{\uparrow}(\mathsf{KP})$ is axiomatized by $\beta($$)$ and $\beta($$)$. But  is a subframe of , so $\mathsf{Subf}_{\uparrow}(\mathsf{KP})$ is axiomatized by $\beta($$)$, and hence $\mathsf{Subf}_{\uparrow}(\mathsf{KP}) = \mathsf{BW}_2$. Since KP has the disjunction property, $\mathsf{Subf}_{\downarrow}(\mathsf{KP}) = \mathsf{IPC}$ by the same argument as in (iii).

(vi). Since the 3-fork is a subframe of the n-fork for $n \geq 3$, it follows from the axiomatization of ND_n in Table 1 and Theorem 4.4(ii) that $\mathsf{Subf}_{\uparrow}(\mathsf{ND}_n) = \mathsf{BW}_2$ for $n \geq 2$. Since ND_n has the disjunction property, $\mathsf{Subf}_{\downarrow}(\mathsf{ND}_n) = \mathsf{IPC}$ by the same argument as in (iii). $\square$

5. Subframe logics and lax logics

In this section we recall the correspondence between subframes and nuclei on Heyting algebras from [10] that leads to a correspondence between nuclear Heyting algebras and S-frames. We show how S-frames provide a new semantics for the intuitionistic modal logic PLL and compare our semantics to those of [21] and [17].

We start by recalling that a *nucleus* on a Heyting algebra A is a unary function $j : A \to A$ satisfying $a \leq ja$, $jja \leq ja$, and $j(a \wedge b) = ja \wedge jb$. It is well known (see, e.g., [10, pg. 88] and the references therein) that nuclei on A correspond to localizations of A, where a *localization* of A is a triple (L, i, l) such that the Heyting algebra L is a $(\wedge, 1)$-subalgebra of A and the inclusion $i : L \to A$ has a left exact left adjoint $l : A \to L$; that is, l is a $(\wedge, 1)$-homomorphism that is left adjoint of i (meaning that $l(a) \le b$ iff $a \le i(b)$ for all $a \in A$ and $b \in L$). This in particular implies that l also preserves Heyting implication.

The one-to-one correspondence between nuclei on A and localizations of A is obtained as follows. Given a nucleus $j : A \to A$, we have that the fixpoints $A_j := \{a \in A \mid a = ja\}$ form a $(\wedge, 1)$-subalgebra of A and the inclusion $A_j \hookrightarrow A$ has j as a left exact left adjoint. Conversely, given a localization (L, i, l) of A, we have that $i \circ l$ is a nucleus on A. Moreover, the two correspondences are inverse to each other. This is parallel to the one-to-one correspondence between local operators and subtopoi in an elementary topos [25, pg. 201, A.4.4.8].

The fixpoints A_j form a Heyting algebra with respect to the operations $a \wedge_j b = a \wedge b$, $a \to_j b = a \to b$, $a \vee_j b = j(a \vee b)$, and $0_j = j0$. The Heyting algebra A_j is a $(\wedge, \to)$-subalgebra of A, but in general A_j is not a Heyting subalgebra of A.

As was observed in [10, Sec. 5], subframes of an Esakia frame $\mathfrak{F}$ correspond to nuclei on the dual Heyting algebra A of clopen upsets of $\mathfrak{F}$. If $\mathfrak{G} = (S, \le)$ is a subframe of $\mathfrak{F} = (X, \le)$, then j given by

$$jU = X \setminus {\downarrow}(S \setminus U) \tag{6}$$

is a nucleus on A, and every nucleus on A is obtained this way. Moreover, the dual Esakia frame of A_j is isomorphic to $\mathfrak{G}$. This motivates the following definition.

Definition 5.1.
- (i) A *nuclear algebra* is a pair (A, j) consisting of a Heyting algebra A and a nucleus j on A.
- (ii) An *S-frame* is a pair $(\mathfrak{F}, \mathfrak{G})$ consisting of an Esakia frame $\mathfrak{F}$ and a subframe $\mathfrak{G}$ of $\mathfrak{F}$.

Remark 5.2.
- (i) Nuclear algebras are also called local algebras (see, e.g., [21]).
- (ii) In the definition of an S-frame, "S" stands for subframe.

Remark 5.3. The one-to-one correspondence between nuclear Heyting algebras and S-frames can also be seen via classical arguments from residuation theory. This is spelled out in [24, Thm. 6.5.5].

Throughout the paper we will use the following notational convention.

Notation 5.4. For an S-frame $(\mathfrak{F}, \mathfrak{G})$, we always assume that $\mathfrak{F} = (X, \le)$ and $\mathfrak{G} = (S, \le)$.

We next recall the definition of the intuitionistic modal logic PLL and explain how nuclear Heyting algebras and S-frames serve as adequate semantics for it. Let $\mathcal{L}_{\mathsf{IPC}}$ be the propositional language of IPC and let $\mathcal{L}_{\mathsf{PLL}}$ be obtained by enriching $\mathcal{L}_{\mathsf{IPC}}$ with an extra modal operator $\bigcirc$.

Definition 5.5. The *propositional lax logic* (PLL) is the least set of formulas of $\mathcal{L}_{\mathsf{PLL}}$ containing (the axioms of) IPC, the axioms

$$\bigcirc(p \to q) \to (\bigcirc p \to \bigcirc q),\ p \to \bigcirc p,\ \bigcirc\bigcirc p \to \bigcirc p,$$

and being closed under the rules of substitution, modus ponens, and

$$\varphi \to \psi / \bigcirc\varphi \to \bigcirc\psi.$$

We refer to the modality $\bigcirc$ as the *lax modality.*

By interpreting $\bigcirc$ as the nucleus j, nuclear Heyting algebras provide semantics for PLL. This semantics is sound and complete since the defining axioms of $\bigcirc$ match the defining axioms of nuclei, as was already pointed out by Goldblatt [21].

Proposition 5.6 (Goldblatt)**.** PLL *is sound and complete with respect to nuclear Heyting algebras.*

Esakia duality coupled with the one-to-one correspondence between nuclei on Heyting algebras and subframes of Esakia frames yields a one-to-one correspondence between nuclear algebras and S-frames. Thus, S-frames provide sound and complete semantics for PLL. In more detail, if $(\mathfrak{F}, \mathfrak{S})$ is an S-frame, then as in the case of IPC, a *valuation* v *on* $(\mathfrak{F}, \mathfrak{S})$ interprets propositional letters as clopen upsets of $\mathfrak{F}$ and intuitionistic connectives as the corresponding operations of the Heyting algebra of clopen upsets of $\mathfrak{F}$. In addition, the lax modality $\bigcirc$ is interpreted as the nucleus j given by (6).

It is easy to verify that (6) translates into the following semantics on points extending the clauses of Figure 1. If v is a valuation on $(\mathfrak{F}, \mathfrak{S})$ and $x \in X$, then

$$x \models_v \bigcirc\varphi \text{ iff } y \models_v \varphi \text{ for all } y \in {\uparrow}x \cap S. \tag{7}$$

We will use the same notations as in the intuitionistic case; e.g., we write $(A, j) \models \varphi$ provided $v(\varphi) = 1$ for each valuation v on (A, j), and we write $(\mathfrak{F}, \mathfrak{S}) \models \varphi$ provided $x \models_v \varphi$ for each $x \in X$ and each valuation v on $(\mathfrak{F}, \mathfrak{S})$. The multiple usage of $\models$ should not lead to ambiguity since in each case it should be clear from the context what we are referring to.

As an immediate corollary of the above, we obtain:

Corollary 5.7. PLL *is sound and complete with respect to S-frames.*

The semantics via S-frames and nuclear Heyting algebras is closely related to the frame-based semantics of PLL developed by Goldblatt [21] and Fairtlough and Mendler [17] (see also [11]). We explain the precise connections.

We recall that a *Goldblatt frame* is a tuple $\mathfrak{F} = (X, \leq, R)$, where $(X, \leq)$ is a partially ordered set and R is a binary relation on X such that (i) $x \leq yRz$ implies xRz, (ii) xRy implies $x \leq y$, and (iii) xRy implies $xRzRy$ for some $z \in X$. The language of PLL is interpreted in a Goldblatt frame $\mathfrak{F}$ by interpreting propositional letters as upsets of $\mathfrak{F}$, intuitionistic connectives as the corresponding operations of the Heyting algebra of upsets of $\mathfrak{F}$, and $\bigcirc$ as the nucleus j_R given by

$$j_R U = X \setminus R^{-1}(X \setminus U). \tag{8}$$

Thus, every Goldblatt frame $\mathfrak{F}$ can be turned into a nuclear Heyting algebra that validates the same formulas as $\mathfrak{F}$. However, not every nuclear Heyting algebra can be obtained in this way. This constitutes a similar discrepancy as the one between intuitionistic Kripke frames and Heyting algebras. Due to the one-to-one correspondence between nuclear Heyting algebras and S-frames, S-frames can be thought of as a "descriptive version" of Goldblatt frames. As discussed in [10], an S-frame $(\mathfrak{F}, \mathfrak{S})$ can be turned into a Goldblatt frame by forgetting the topology on X and defining a relation R on X by xRy iff $x \leq s \leq y$ for some $s \in S$. Using that $R[x] = {\uparrow}({\uparrow}x \cap S)$, we see that $j_R U = jU$ for each clopen upset U of $\mathfrak{F}$. The Goldblatt frames obtained in this way satisfy the additional condition that xRy iff $(\exists z \in X)(zRz$ and $x \leq z \leq y)$ which does not hold in all Goldblatt frames (see [10, Rem. 24]). Thus, not every Goldblatt frame underlies an S-frame.

We also recall that an *FM-frame* (Fairtlough-Mendler frame) is a tuple $\mathfrak{F} = (X, \leq, \preceq, F)$ such that $\leq, \preceq$ are partial orders on X, $x \preceq y$ implies $x \leq y$, and F is an $\leq$-upset of X. The language of PLL is interpreted in an FM-frame slightly differently than in a Goldblatt frame. Instead of working with the Heyting algebra of all upsets of $\mathfrak{F}$, we work with the Heyting algebra of the upsets of $\mathfrak{F}$ containing F. Therefore, propositional letters are interpreted as upsets of $\mathfrak{F}$ containing F, intuitionistic connectives as the corresponding operations in this relativized Heyting algebra, and $\bigcirc$ is interpreted as the nucleus $j_{\leq\preceq}$ given by

$$j_{\leq\preceq}U = \{x \in X \mid \forall y(x \leq y \Rightarrow \exists z(y \preceq z \text{ and } z \in U))\}. \tag{9}$$

If $(\mathfrak{F}, \mathfrak{G})$ is an S-frame, then define $\mathfrak{F}^*_{\mathfrak{G}} = (X^*, \leq^*, \preceq^*, F^*)$ as follows. Set $X^* = X \cup \{m\}$, where $m \notin X$. Let $\leq^*$ extend $\leq$ so that m is the maximum of X^*. Set $F^* = \{m\}$ and define $x \preceq^* y$ iff $x = y$ or $x \in X \setminus S$ and $y = m$. It is straightforward to verify that $\mathfrak{F}^*_{\mathfrak{G}}$ is an FM-frame. Moreover, if for a clopen upset U of $\mathfrak{F}$, we let $U^* = U \cup \{m\}$, then U^* is an upset of $\mathfrak{F}^*_{\mathfrak{G}}$ and $j_{\leq\preceq}(U^*) = (jU)^*$.

Clearly not every FM-frame is of the form $\mathfrak{F}^*_{\mathfrak{G}}$ for some S-frame $(\mathfrak{F}, \mathfrak{G})$. For example, if in the FM-frame $(X, \leq, \preceq, F)$ the upset F consists of more than one point, then $(X, \leq, \preceq, F)$ is not of the form $\mathfrak{F}^*_{\mathfrak{G}}$. However, the FM-frames $\mathfrak{F}^*_{\mathfrak{G}}$ are sufficient for representing nuclear Heyting algebras.

6. Superintuitionistic logics and lax logics

In this section we define a translation τ from $\mathcal{L}_{\mathsf{IPC}}$ into $\mathcal{L}_{\mathsf{PLL}}$ which gives rise to two embeddings from the lattice of si-logics into the lattice of extensions of PLL. We will study some elementary properties of these embeddings and show that they provide us with new characterizations of subframe logics. As we explained in Remark 4.5, the upward subframization of a si-logic $\mathsf{L} = \mathsf{IPC} + \Gamma$ can be calculated effectively from Γ. In this section we show how to calculate the downward subframization of L by utilizing the translation τ.

Definition 6.1. Define a translation $\tau : \mathcal{L}_{\mathsf{IPC}} \to \mathcal{L}_{\mathsf{PLL}}$ by

- $\tau(p) = \bigcirc p$ for a propositional letter p,
- $\tau(\bot) = \bigcirc\bot$,
- $\tau(\varphi \wedge \psi) = \tau(\varphi) \wedge \tau(\psi)$,
- $\tau(\varphi \to \psi) = \tau(\varphi) \to \tau(\psi)$,
- $\tau(\varphi \vee \psi) = \bigcirc(\tau(\varphi) \vee \tau(\psi))$.

Remark 6.2. The translation τ is a version of the Gödel-Gentzen translation (see, e.g., [18]). It has been pointed out by Aczel [1] that every lax modality which is definable within IPC (for example, the double negation) provides a translation from IPC to itself. The translation τ can be seen as a generalization of this, where the lax modality is not necessarily definable within IPC. It is used in [12, Sec. 3.2.2] to explain verificationist interpretation of intuitionistic logic in terms of nuclei.

Recall that by Notation 5.4, given an S-frame $(\mathfrak{F}, \mathfrak{G})$, we always assume that $\mathfrak{F} = (X, \leq)$ and $\mathfrak{G} = (S, \leq)$.

Lemma 6.3. *Let v be a valuation on an S-frame $(\mathfrak{F}, \mathfrak{G})$. Define a valuation $v_{\mathfrak{G}}$ on $\mathfrak{G}$ by $v_{\mathfrak{G}}(p) = v(p) \cap S$. For every $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ and $x \in X$,*

$$x \models_v \tau(\varphi) \text{ iff } y \models_{v_{\mathfrak{G}}} \varphi \text{ for all } y \in {\uparrow}x \cap S.$$

Proof. The proof is by induction on the complexity of $\varphi \in \mathcal{L}_{\mathbf{IPC}}$.

If $\varphi = p$, then $\tau(\varphi) = \bigcirc p$. Therefore, by (7) and the definition of $v_{\mathfrak{G}}$,

$$\begin{aligned} x \models_v \bigcirc p &\text{ iff } y \models_v p \text{ for all } y \in {\uparrow}x \cap S \\ &\text{ iff } y \models_{v_{\mathfrak{G}}} p \text{ for all } y \in {\uparrow}x \cap S. \end{aligned}$$

If $\varphi = \bot$, then $\tau(\varphi) = \bigcirc\bot$. Therefore, $x \models_v \bigcirc\bot$ iff ${\uparrow}x \cap S = \varnothing$. Thus, $x \models_v \bigcirc\bot$ iff $y \models_{v_{\mathfrak{G}}} \bot$ for all $y \in {\uparrow}x \cap S$.

If $\varphi = \psi \wedge \chi$, then $\tau(\psi \wedge \chi) = \tau(\psi) \wedge \tau(\chi)$. Therefore,

$$\begin{aligned} x \models_v \tau(\psi \wedge \chi) &\text{ iff } x \models_v \tau(\psi) \text{ and } x \models_v \tau(\chi) \\ &\text{ iff } y \models_{v_{\mathfrak{G}}} \psi \text{ and } y \models_{v_{\mathfrak{G}}} \chi \text{ for all } y \in {\uparrow}x \cap S \\ &\text{ iff } y \models_{v_{\mathfrak{G}}} \psi \wedge \chi \text{ for all } y \in {\uparrow}x \cap S. \end{aligned}$$

If $\varphi = \psi \to \chi$, then $\tau(\psi \to \chi) = \tau(\psi) \to \tau(\chi)$. Therefore,

$$\begin{aligned} x \models_v \tau(\psi) \to \tau(\chi) &\text{ iff } z \models_v \tau(\psi) \text{ implies } z \models_v \tau(\chi) \text{ for all } z \geq x \\ &\text{ iff } (w \models_{v_{\mathfrak{G}}} \psi \text{ implies } w \models_{v_{\mathfrak{G}}} \chi \text{ for all } w \in {\uparrow}z \cap S) \text{ for all } z \geq x \\ &\text{ iff } (w \models_{v_{\mathfrak{G}}} \psi \text{ implies } w \models_{v_{\mathfrak{G}}} \chi) \text{ for all } w \in {\uparrow}x \cap S. \end{aligned}$$

If $\varphi = \psi \vee \chi$, then $\tau(\psi \vee \chi) = \bigcirc(\tau(\psi) \vee \tau(\chi))$. Therefore,

$$\begin{aligned} x \models_v \bigcirc(\tau(\psi) \vee \tau(\chi)) &\text{ iff } y \models_v \tau(\psi) \vee \tau(\chi) \text{ for all } y \in {\uparrow}x \cap S \\ &\text{ iff } y \models_v \tau(\psi) \text{ or } y \models_v \tau(\chi) \text{ for all } y \in {\uparrow}x \cap S \\ &\text{ iff } (z \models_{v_{\mathfrak{G}}} \psi \text{ or } z \models_{v_{\mathfrak{G}}} \chi \text{ for all } z \in {\uparrow}y \cap S) \text{ for all } y \in {\uparrow}x \cap S \\ &\text{ iff } z \models_{v_{\mathfrak{G}}} \psi \vee \chi \text{ for all } z \in {\uparrow}x \cap S. \end{aligned}$$

□

Lemma 6.4. *Let $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ and $(\mathfrak{F}, \mathfrak{G})$ be an S-frame.*

(i) $(\mathfrak{F}, \mathfrak{G}) \models \varphi$ *iff* $\mathfrak{F} \models \varphi$.
(ii) $(\mathfrak{F}, \mathfrak{G}) \models \tau(\varphi)$ *iff* $\mathfrak{G} \models \varphi$.

Proof. (i). This is obvious since φ contains no occurrences of $\bigcirc$.

(ii). For the right to left direction, suppose v is a valuation on $(\mathfrak{F}, \mathfrak{G})$ that refutes $\tau(\varphi)$. Define a valuation v' on $\mathfrak{G}$ by $v'(p) = v(p) \cap S$. By Lemma 6.3, v' refutes φ on $\mathfrak{G}$. For the left to right direction, suppose v' is a valuation on $\mathfrak{G}$ that refutes φ. Define a valuation v on $\mathfrak{F}$ by $v(p) = X \setminus {\downarrow}(S \setminus v'(p))$. Then $v'(p) = v(p) \cap S$ for every propositional letter p. Applying Lemma 6.3 again yields that v refutes $\tau(\varphi)$ on $(\mathfrak{F}, \mathfrak{G})$. □

Remark 6.5. An algebraic reformulation of Lemma 6.4 is as follows. If $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ and (A, j) is a nuclear Heyting algebra, then

(i) $(A, j) \models \varphi$ iff $A \models \varphi$.
(ii) $(A, j) \models \tau(\varphi)$ iff $A_j \models \varphi$.

We call a set M of formulas of $\mathcal{L}_{\mathsf{PLL}}$ a *lax logic* if $\mathsf{PLL} \subseteq \mathsf{M}$ and M is closed under the rules of substitution, modus ponens, and $\varphi \to \psi / \bigcirc\varphi \to \bigcirc\psi$. As in the case of si-logics, lax logics form a complete lattice. Each lax logic M is sound and complete with respect to the variety $Alg(\mathsf{M}) := \{(A, j) \mid (A, j) \models \mathsf{M}\}$ of nuclear algebras, as well as with respect to the corresponding class of S-frames.

Definition 6.6. Let L be a si-logic and let M be a lax logic.

(i) We say that L is the *intuitionistic fragment* of M if for all $\varphi \in \mathcal{L}_{\mathsf{IPC}}$,
$$\varphi \in \mathsf{L} \text{ iff } \varphi \in \mathsf{M}.$$

(ii) We say that L is the *lax fragment* of M if for all $\varphi \in \mathcal{L}_{\mathsf{IPC}}$,
$$\varphi \in \mathsf{L} \text{ iff } \tau(\varphi) \in \mathsf{M}.$$

Definition 6.7. For a lax logic M, we define
$$\rho_1(\mathsf{M}) = \{\varphi \in \mathcal{L}_{\mathsf{IPC}} \mid \varphi \in \mathsf{M}\},$$
$$\rho_2(\mathsf{M}) = \{\varphi \in \mathcal{L}_{\mathsf{IPC}} \mid \tau(\varphi) \in \mathsf{M}\}.$$

Lemma 6.8. *Let M be a lax logic.*

(i) *$\rho_1(\mathsf{M})$ is the intuitionistic fragment of M and*
$$\rho_1(\mathsf{M}) = \mathsf{Log}\left(\{\mathfrak{F} \mid (\mathfrak{F}, \mathfrak{G}) \models \mathsf{M} \textit{ for some subframe } \mathfrak{G} \textit{ of } \mathfrak{F}\}\right).$$

(ii) *$\rho_2(\mathsf{M})$ is the lax fragment of M and*
$$\rho_2(\mathsf{M}) = \mathsf{Log}\left(\{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \mathsf{M}\}\right).$$

Proof. We only show (ii) as (i) is proved similarly but uses Lemma 6.4(i) instead of Lemma 6.4(ii). For $\varphi \in \mathcal{L}_{\mathsf{IPC}}$, using Lemma 6.4(ii), we have
$$\begin{aligned} \tau(\varphi) \in \mathsf{M} &\Leftrightarrow (\mathfrak{F}, \mathfrak{G}) \models \tau(\varphi) \text{ for all } (\mathfrak{F}, \mathfrak{G}) \models \mathsf{M} \\ &\Leftrightarrow \mathfrak{G} \models \varphi \text{ for all } (\mathfrak{F}, \mathfrak{G}) \models \mathsf{M} \\ &\Leftrightarrow \varphi \in \mathsf{Log}\left(\{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \mathsf{M}\}\right). \end{aligned}$$

Therefore, $\rho_2(\mathsf{M}) = \mathsf{Log}\left(\{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \mathsf{M}\}\right)$, and so $\rho_2(\mathsf{M})$ is the lax fragment of M. □

Remark 6.9. An algebraic reformulation of Lemma 6.8 is as follows:

(i) $\rho_1(\mathsf{M}) = \mathsf{Log}\left(\{A \mid (A, j) \models \mathsf{M} \text{ for some nucleus } j \text{ on } A\}\right)$.

(ii) $\rho_2(\mathsf{M}) = \mathsf{Log}\left(\{A_j \mid (A, j) \models \mathsf{M}\}\right)$.

Definition 6.10. For a si-logic L, we define
$$\sigma_1(\mathsf{L}) = \mathsf{PLL} + \{\varphi \mid \varphi \in \mathsf{L}\},$$
$$\sigma_2(\mathsf{L}) = \mathsf{PLL} + \{\tau(\varphi) \mid \varphi \in \mathsf{L}\}.$$

Lemma 6.11. *Let L be a si-logic.*

(i) $\sigma_1(\mathsf{L}) = \mathsf{Log}\left(\{(\mathfrak{F}, \mathfrak{G}) \mid \mathfrak{F} \models \mathsf{L}\}\right)$.

(ii) $\sigma_2(\mathsf{L}) = \mathsf{Log}\left(\{(\mathfrak{F}, \mathfrak{G}) \mid \mathfrak{G} \models \mathsf{L}\}\right)$.

Proof. We only show (ii) as (i) is proved similarly but uses Lemma 6.4(i) instead of Lemma 6.4(ii). Suppose $(\mathfrak{F}, \mathfrak{G})$ is an S-frame. By Lemma 6.4(ii), $\mathfrak{G} \models \mathsf{L}$ iff $(\mathfrak{F}, \mathfrak{G}) \models \{\tau(\varphi) \mid \varphi \in \mathsf{L}\}$. Thus, $\sigma_2(\mathsf{L}) = \mathsf{Log}\left(\{(\mathfrak{F}, \mathfrak{G}) \mid \mathfrak{G} \models \mathsf{L}\}\right)$. □

Remark 6.12. In algebraic terms, Lemma 6.11 can be expressed as follows:

(i) $\sigma_1(\mathsf{L}) = \mathsf{Log}\left(\{(A, j) \mid A \models \mathsf{L}\}\right)$.

(ii) $\sigma_2(\mathsf{L}) = \mathsf{Log}\left(\{(A, j) \mid A_j \models \mathsf{L}\}\right)$.

Lemma 6.13. *Let L be a si-logic.*

(i) *$\mathsf{L} = \rho_1\sigma_1(\mathsf{L})$. In fact, $\sigma_1(\mathsf{L})$ is the least element of $\rho_1^{-1}(\mathsf{L})$.*

(ii) *$\mathsf{L} = \rho_2\sigma_2(\mathsf{L})$. In fact, $\sigma_2(\mathsf{L})$ is the least element of $\rho_2^{-1}(\mathsf{L})$.*

Proof. (i). Let $\varphi \in \mathcal{L}_{\mathsf{IPC}}$. Then $\varphi \in \mathsf{L}$ implies $\varphi \in \sigma_1(\mathsf{L})$, which implies $\varphi \in \rho_1\sigma_1(\mathsf{L})$. Therefore, $\mathsf{L} \subseteq \rho_1\sigma_1(\mathsf{L})$. If $\varphi \notin \mathsf{L}$, then there is an L-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. Consider the S-frame $(\mathfrak{F}, \mathfrak{F})$. By Lemma 6.11(i), $(\mathfrak{F}, \mathfrak{F}) \models \sigma_1(\mathsf{L})$, and by Lemma 6.4(i), $(\mathfrak{F}, \mathfrak{F}) \not\models \varphi$. Thus, $\varphi \notin \sigma_1(\mathsf{L})$, and so by Lemma 6.8(i), $\varphi \notin \rho_1\sigma_1(\mathsf{L})$. This shows that $\mathsf{L} = \rho_1\sigma_1(\mathsf{L})$. If $\mathsf{M} \in \rho_1^{-1}(\mathsf{L})$, then for every $\varphi \in \mathcal{L}_{\mathsf{IPC}}$, we have $\varphi \in \mathsf{L}$ iff $\varphi \in \mathsf{M}$. Consequently, $\sigma_1(\mathsf{L}) \subseteq \mathsf{M}$, and hence $\sigma_1(\mathsf{L})$ is the least element of $\rho_1^{-1}(\mathsf{L})$.

(ii). Let $\varphi \in \mathcal{L}_{\mathsf{IPC}}$. Then $\varphi \in \mathsf{L}$ implies $\tau(\varphi) \in \sigma_2(\mathsf{L})$, which implies $\varphi \in \rho_2\sigma_2(\mathsf{L})$. Therefore, $\mathsf{L} \subseteq \rho_2\sigma_2(\mathsf{L})$. If $\varphi \notin \mathsf{L}$, then there is an L-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. By Lemma 6.11(ii), the S-frame $(\mathfrak{F}, \mathfrak{F})$ is a $\sigma_2(\mathsf{L})$-frame, and by Lemma 6.4(ii), $(\mathfrak{F}, \mathfrak{F}) \not\models \tau(\varphi)$. Thus, $\varphi \notin \sigma_2(\mathsf{L})$, and so by Lemma 6.8(ii), $\varphi \notin \rho_2\sigma_2(\mathsf{L})$. This shows that $\mathsf{L} = \rho_2\sigma_2(\mathsf{L})$. If $\mathsf{M} \in \rho_2^{-1}(\mathsf{L})$, then for every $\varphi \in \mathcal{L}_{\mathsf{IPC}}$, we have $\varphi \in \mathsf{L}$ iff $\tau(\varphi) \in \mathsf{M}$. Consequently, $\sigma_2(\mathsf{L}) \subseteq \mathsf{M}$, and hence $\sigma_2(\mathsf{L})$ is the least element of $\rho_2^{-1}(\mathsf{L})$. □

As follows from Lemma 6.13, for a si-logic L, both $\rho_1^{-1}(\mathsf{L})$ and $\rho_2^{-1}(\mathsf{L})$ have least elements, but they may not have largest elements. To see this we require the following lemmas.

Lemma 6.14. *Let $(\mathfrak{F}, \mathfrak{G})$ be an S-frame.*

(i) $(\mathfrak{F}, \mathfrak{G}) \models \bigcirc p \leftrightarrow p$ *iff* $\mathfrak{F} = \mathfrak{G}$.
(ii) $(\mathfrak{F}, \mathfrak{G}) \models \bigcirc p$ *iff* $\mathfrak{G} = \varnothing$.

Proof. (i). First suppose that $\mathfrak{F} = \mathfrak{G}$. Then it is clear that $(\mathfrak{F}, \mathfrak{G}) \models \bigcirc p \leftrightarrow p$. Next suppose that $\mathfrak{F} \neq \mathfrak{G}$. Let $x \in X \setminus S$. Then $x \notin {\uparrow}x \cap S$, so $x \notin {\uparrow}({\uparrow}x \cap S)$. Therefore, since ${\uparrow}({\uparrow}x \cap S)$ is a closed upset of X, there is a clopen upset U of X with ${\uparrow}({\uparrow}x \cap S) \subseteq U$ and $x \notin U$. Let v be a valuation on $(\mathfrak{F}, \mathfrak{G})$ such that $v(p) = U$. Clearly $x \not\models_v p$. On the other hand, $x \models_v \bigcirc p$ by (7). Thus, $(\mathfrak{F}, \mathfrak{G}) \not\models \bigcirc p \leftrightarrow p$.

(ii). If $\mathfrak{G} = \varnothing$, then it is clear that $(\mathfrak{F}, \mathfrak{G}) \models \bigcirc p$. If $\mathfrak{G} \neq \varnothing$, then let v be a valuation on $(\mathfrak{F}, \mathfrak{G})$ such that $v(p) = \varnothing$. For $x \in S$, we then have $x \not\models_v \bigcirc p$, so $(\mathfrak{F}, \mathfrak{G}) \not\models \bigcirc p$. □

For $\psi \in \mathcal{L}_{\mathsf{PLL}}$, let ψ^- be the formula obtained from ψ by deleting all occurrences of the $\bigcirc$ modality and let ψ^* be the formula obtained from ψ by replacing all subformulas of the form $\bigcirc\chi$ with $\top$. Clearly $\psi^-, \psi^* \in \mathcal{L}_{\mathsf{IPC}}$. Both ψ^- and ψ^* were considered in [17, Sec. 2].

Lemma 6.15. *Let M be a lax logic.*

(i) *If $\bigcirc p \leftrightarrow p \in \mathsf{M}$, then $\psi \in \mathsf{M}$ iff $\psi^- \in \mathsf{M}$ for every formula $\psi \in \mathcal{L}_{\mathsf{PLL}}$.*
(ii) *If $\bigcirc p \in \mathsf{M}$, then $\psi \in \mathsf{M}$ iff $\psi^* \in \mathsf{M}$ for every formula $\psi \in \mathcal{L}_{\mathsf{PLL}}$.*

Proof. (i). Suppose that $\bigcirc p \leftrightarrow p \in \mathsf{M}$ and let $\psi \in \mathcal{L}_{\mathsf{PLL}}$. By Lemma 6.14(i), M is the logic of the class of S-frames of the shape $(\mathfrak{F}, \mathfrak{F})$. For $(\mathfrak{F}, \mathfrak{F})$, a valuation v on $\mathfrak{F}$, and $x \in \mathfrak{F}$, we have $x \models_v \bigcirc\varphi$ iff $x \models_v \varphi$. Therefore, induction on ψ yields $(\mathfrak{F}, \mathfrak{F}) \models \psi$ iff $(\mathfrak{F}, \mathfrak{F}) \models \psi^-$. Thus, $\psi \in \mathsf{M}$ iff $\psi^- \in \mathsf{M}$.

(ii). Suppose $\bigcirc p \in \mathsf{M}$ and let $\psi \in \mathcal{L}_{\mathsf{PLL}}$. By Lemma 6.14(ii), M is the logic of the class of S-frames of the shape $(\mathfrak{F}, \varnothing)$. For $(\mathfrak{F}, \varnothing)$, a valuation v on $\mathfrak{F}$, and $x \in \mathfrak{F}$, we have $x \models_v \bigcirc\varphi$. Therefore, induction on ψ yields $(\mathfrak{F}, \varnothing) \models \psi$ iff $(\mathfrak{F}, \varnothing) \models \psi^*$. Thus, $\psi \in \mathsf{M}$ iff $\psi^* \in \mathsf{M}$. □

Lemma 6.16. *Let L be a si-logic.*

(i) *$\sigma_1(\mathsf{L}) + \bigcirc p \leftrightarrow p$ is a maximal element of both $\rho_1^{-1}(\mathsf{L})$ and $\rho_2^{-1}(\mathsf{L})$.*
(ii) *$\sigma_1(\mathsf{L}) + \bigcirc p$ is a maximal element of $\rho_1^{-1}(\mathsf{L})$.*

Proof. (i). Let $\mathsf{M} = \sigma_1(\mathsf{L}) + \bigcirc p \leftrightarrow p$. First we show that M is a maximal element of $\rho_1^{-1}(\mathsf{L})$. By Lemma 6.14(i), an S-frame $(\mathfrak{F}, \mathfrak{G})$ validates M iff $\mathfrak{F}$ is an L-frame and $\mathfrak{F} = \mathfrak{G}$.

Therefore, by Lemma 6.8(i), $\rho_1(\mathsf{M}) = \mathsf{L}$, so $\mathsf{M} \in \rho_1^{-1}(\mathsf{L})$. To see that M is maximal in $\rho_1^{-1}(\mathsf{L})$, suppose that $\mathsf{M} \subseteq \mathsf{M}' \in \rho_1^{-1}(\mathsf{L})$. We show that $\mathsf{M} = \mathsf{M}'$. Let $\psi \in \mathcal{L}_{\mathsf{PLL}}$. If $\psi \notin \mathsf{M}$, then by Lemma 6.15(i), $\psi^- \notin \mathsf{M}$, and so $\psi^- \notin \mathsf{L}$ as $\psi^- \in \mathcal{L}_{\mathsf{IPC}}$. Since $\rho_1(\mathsf{M}') = \mathsf{L}$, we see that $\psi^- \notin \mathsf{M}'$. Because $\mathsf{M} \subseteq \mathsf{M}'$, we have $\bigcirc p \leftrightarrow p \in \mathsf{M}'$, so $\psi \notin \mathsf{M}'$ by Lemma 6.15(i). Thus, $\mathsf{M} = \mathsf{M}'$, and hence M is maximal in $\rho_1^{-1}(\mathsf{L})$.

Next we show that M is a maximal element of $\rho_2^{-1}(\mathsf{L})$. By Lemma 6.8(ii), $\rho_2(\mathsf{M}) = \mathsf{L}$, so $\mathsf{M} \in \rho_2^{-1}(\mathsf{L})$. Suppose $\mathsf{M} \subseteq \mathsf{M}' \in \rho_2^{-1}(\mathsf{L})$. We show that $\mathsf{M} = \mathsf{M}'$. Let $\psi \in \mathcal{L}_{\mathsf{PLL}}$. If $\psi \notin \mathsf{M}$, then $\psi^- \notin \mathsf{M}$ by Lemma 6.15(i). Therefore, $\tau(\psi^-) \notin \mathsf{M}$ because $(\tau(\psi^-))^- = \psi^-$. Thus, $\psi^- \notin \mathsf{L}$, and so $\tau(\psi^-) \notin \mathsf{M}'$. Since $\mathsf{M} \subseteq \mathsf{M}'$, we have $\bigcirc p \leftrightarrow p \in \mathsf{M}'$, and hence $\psi^- = (\tau(\psi^-))^- \notin \mathsf{M}'$ by Lemma 6.15(i). Consequently, $\psi \notin \mathsf{M}'$, and so $\mathsf{M} = \mathsf{M}'$, which yields that M is maximal in $\rho_2^{-1}(\mathsf{L})$.

(ii). Let $\mathsf{M} = \sigma_1(\mathsf{L}) + \bigcirc p$. By Lemma 6.14(ii), an S-frame $(\mathfrak{F}, \mathfrak{G})$ validates M iff $\mathfrak{F}$ is an L-frame and $\mathfrak{G} = \varnothing$. Therefore, by Lemma 6.8(i), $\rho_1(\mathsf{M}) = \mathsf{L}$, so $\mathsf{M} \in \rho_1^{-1}(\mathsf{L})$. To see that M is maximal in $\rho_1^{-1}(\mathsf{L})$, suppose that $\mathsf{M} \subseteq \mathsf{M}' \in \rho_1^{-1}(\mathsf{L})$. We show that $\mathsf{M} = \mathsf{M}'$. Let $\psi \in \mathcal{L}_{\mathsf{PLL}}$. If $\psi \notin \mathsf{M}$, then by Lemma 6.15(ii), $\psi^* \notin \mathsf{M}$, and so $\psi^* \notin \mathsf{L}$ as $\psi^* \in \mathcal{L}_{\mathsf{IPC}}$. Since $\rho_1(\mathsf{M}') = \mathsf{L}$, we see that $\psi^* \notin \mathsf{M}'$. Because $\mathsf{M} \subseteq \mathsf{M}'$, we have $\bigcirc p \in \mathsf{M}'$, so $\psi \notin \mathsf{M}'$ by Lemma 6.15(ii). Thus, $\mathsf{M} = \mathsf{M}'$, and hence M is maximal in $\rho_1^{-1}(\mathsf{L})$. □

Proposition 6.17.

(i) *If* L *is a consistent si-logic, then* $\rho_1^{-1}(\mathsf{L})$ *does not have a largest element.*
(ii) *If* $\mathsf{KC} \not\subseteq \mathsf{L}$*, then* $\rho_2^{-1}(\mathsf{L})$ *does not have a largest element.*

Proof. (i). Let L be a consistent si-logic. Then a singleton frame $\{x\}$ is an L-frame. Therefore, $\sigma_1(\mathsf{L}) + \bigcirc p \leftrightarrow p$ and $\sigma_1(\mathsf{L}) + \bigcirc p$ are different since the S-frame $(\{x\}, \varnothing)$ validates $\sigma_1(\mathsf{L}) + \bigcirc p$ but refutes $\sigma_1(\mathsf{L}) + \bigcirc p \leftrightarrow p$. Thus, by Lemma 6.16, $\rho_1^{-1}(\mathsf{L})$ has at least two maximal elements, and hence does not have a largest element.

(ii). Let L be a si-logic with $\mathsf{KC} \not\subseteq \mathsf{L}$. Suppose that $\mathsf{L} = \mathsf{IPC} + \Gamma$, and set $\mathsf{M} = \sigma_1(\mathsf{KC}) + \{\tau(\gamma) \mid \gamma \in \Gamma\}$. By Lemmas 6.11(i) and 6.4(ii), an S-frame $(\mathfrak{F}, \mathfrak{G})$ validates M iff $\mathfrak{F}$ is a KC-frame and $\mathfrak{G}$ is an L-frame. Therefore, by Lemma 6.8(ii), $\mathsf{L} \subseteq \rho_2(\mathsf{M})$. To see the reverse inclusion, suppose that $\varphi \notin \mathsf{L}$. Then there is an L-frame $\mathfrak{G}$ with $\mathfrak{G} \not\models \varphi$. Let $\mathfrak{F}$ be the Esakia frame obtained from $\mathfrak{G}$ by adding a new isolated top node t. Algebraically this corresponds to adding a new bottom element to the Heyting algebra of clopen upsets of $\mathfrak{G}$. Clearly $\mathfrak{F}$ is a KC-frame. Since $\mathfrak{G}$ is clopen in $\mathfrak{F}$, we have that $\mathfrak{G}$ is a subframe of $\mathfrak{F}$. Therefore, $(\mathfrak{F}, \mathfrak{G})$ is an S-frame which validates M but refutes $\tau(\varphi)$ by Lemma 6.4(ii). Thus, $\varphi \notin \rho_2(\mathsf{M})$. Consequently, $\mathsf{L} = \rho_2(\mathsf{M})$, and so $\mathsf{M} \in \rho_2^{-1}(\mathsf{L})$. On the other hand, since $\mathsf{KC} \not\subseteq \mathsf{L}$, there is an L-frame $\mathfrak{H}$ that is not a KC-frame. By Lemmas 6.4 and 6.14, $(\mathfrak{H}, \mathfrak{H}) \models \sigma_1(\mathsf{L}) + \bigcirc p \leftrightarrow p$ but $(\mathfrak{H}, \mathfrak{H}) \not\models \mathsf{M}$. This shows that $\mathsf{M} \not\subseteq \sigma_1(\mathsf{L}) + \bigcirc p \leftrightarrow p$. If $\rho_2^{-1}(\mathsf{L})$ were to have a largest element, it would have to be $\sigma_1(\mathsf{L}) + \bigcirc p \leftrightarrow p$ since this is a maximal element of $\rho_2^{-1}(\mathsf{L})$ by Lemma 6.16(i). Because M is not contained in $\sigma_1(\mathsf{L}) + \bigcirc p \leftrightarrow p$, we conclude that $\rho_2^{-1}(\mathsf{L})$ does not have a largest element. □

Figure 2 illustrates the mappings σ_1 and ρ_1, where CPC is the classical propositional calculus and Fml is the inconsistent logic. The picture is similar for σ_2 and ρ_2. We next give a new characterization of subframe si-logics.

Theorem 6.18. *For a si-logic* L*, the following are equivalent:*

(i) L *is a subframe logic.*
(ii) $\sigma_2(\mathsf{L}) \subseteq \sigma_1(\mathsf{L})$.

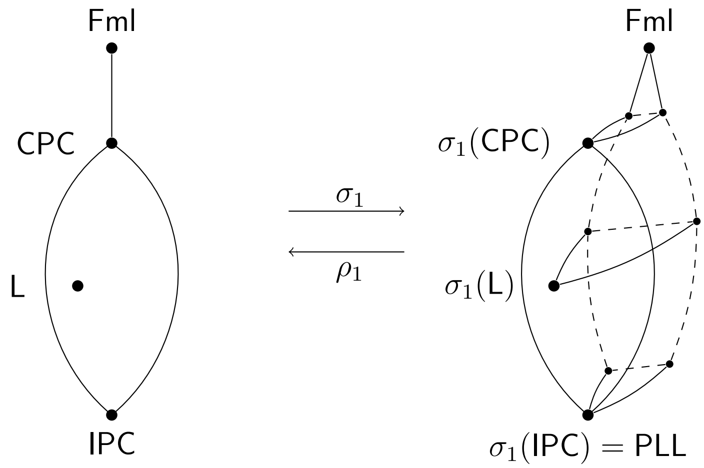


FIGURE 3

(iii) $\sigma_2(\mathsf{L}) + \{\varphi \mid \varphi \in \mathsf{L}\} = \sigma_1(\mathsf{L})$.
(iv) $\rho_2\sigma_1(\mathsf{L}) = \mathsf{L}$.
(v) $\sigma_1(\mathsf{L})$ *is closed under the rule* $\varphi/\tau(\varphi)$ *for every* $\varphi \in \mathcal{L}_{\mathsf{IPC}}$.

Proof. (i)⇒(ii). Suppose $(\mathfrak{F}, \mathfrak{G})$ is an S-frame such that $(\mathfrak{F}, \mathfrak{G}) \models \sigma_1(\mathsf{L})$. By Lemma 6.11(i), $\mathfrak{F} \models \mathsf{L}$. Since L is a subframe logic, $\mathfrak{G} \models \mathsf{L}$. Therefore, by Lemma 6.11(ii), $(\mathfrak{F}, \mathfrak{G}) \models \sigma_2(\mathsf{L})$. Thus, $\sigma_2(\mathsf{L}) \subseteq \sigma_1(\mathsf{L})$.

(ii)⇒(iii). This is obvious.

(iii)⇒(iv). By Lemmas 6.13(ii) and 6.8(ii), $\mathsf{L} = \rho_2\sigma_2(\mathsf{L}) = \mathsf{Log}(\{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \sigma_2(\mathsf{L})\})$ and $\rho_2\sigma_1(\mathsf{L}) = \mathsf{Log}(\{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \sigma_1(\mathsf{L})\})$. Therefore, it is sufficient to show that $\{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \sigma_2(\mathsf{L})\} = \{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \sigma_1(\mathsf{L})\}$. The inclusion $\supseteq$ is immediate from (iii). For the reverse inclusion, suppose that $(\mathfrak{F}, \mathfrak{G}) \models \sigma_2(\mathsf{L})$. By Lemma 6.11(ii), $\mathfrak{G} \models \mathsf{L}$, so $(\mathfrak{G}, \mathfrak{G}) \models \sigma_1(\mathsf{L})$ by Lemma 6.11(i). Thus, $\mathfrak{G} \in \{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \sigma_1(\mathsf{L})\}$.

(iv)⇒(v). Suppose that there is $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ such that $\varphi \in \sigma_1(\mathsf{L})$ but $\tau(\varphi) \notin \sigma_1(\mathsf{L})$. Then there is an S-frame $(\mathfrak{F}, \mathfrak{G})$ with $(\mathfrak{F}, \mathfrak{G}) \models \sigma_1(\mathsf{L})$ and $(\mathfrak{F}, \mathfrak{G}) \not\models \tau(\varphi)$. By Lemma 6.8(ii), $(\mathfrak{F}, \mathfrak{G}) \models \sigma_1(\mathsf{L})$ implies $\mathfrak{G} \models \rho_2\sigma_1(\mathsf{L}) = \mathsf{L}$, and by Lemma 6.4(ii), $(\mathfrak{F}, \mathfrak{G}) \not\models \tau(\varphi)$ implies $\mathfrak{G} \not\models \varphi$. Therefore, $\varphi \notin \mathsf{L}$, contradicting $\varphi \in \sigma_1(\mathsf{L})$.

(v)⇒(i). Let $\mathfrak{F}$ be an L-frame and $\mathfrak{G}$ be a subframe of $\mathfrak{F}$. By Lemma 6.11(i), $(\mathfrak{F}, \mathfrak{G}) \models \sigma_1(\mathsf{L})$. By (v), $(\mathfrak{F}, \mathfrak{G}) \models \tau(\varphi)$ for each $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ such that $\varphi \in \sigma_1(\mathsf{L})$. Therefore, $(\mathfrak{F}, \mathfrak{G}) \models \tau(\varphi)$ for each $\varphi \in \mathsf{L}$. Thus, $\mathfrak{G} \models \mathsf{L}$ by Lemma 6.4(ii), and we conclude that L is a subframe logic. □

Remark 6.19. Theorem 6.18 resembles [31, Prop. 3.4], which characterizes subframe modal logics in terms of relativizations of modal formulas. Relativization is a syntactic operation on modal formulas, thus an operation definable within the modal language. Subframes in the intuitionistic case cannot be characterized via relativizations. We can, nevertheless, obtain an approximation of the relativization via the map τ, which moves us to the setting of the intuitionistic modal logic PLL.

Remark 6.20. In general, $\sigma_1(\mathsf{L}) \not\subseteq \sigma_2(\mathsf{L})$. In fact, for any consistent si-logic L, from $\sigma_1(\mathsf{L}) \subseteq \sigma_2(\mathsf{L})$ it follows that $\mathsf{L} = \mathsf{IPC}$. To see this, suppose $\mathsf{L} \neq \mathsf{IPC}$. Then there is a finite frame $\mathfrak{F}$ that refutes L. Pick a point in $\mathfrak{F}$ and let $\mathfrak{G}$ be the subframe of $\mathfrak{F}$ consisting of this point.

Clearly $\mathfrak{G}$ is an L-frame. Therefore, by Lemma 6.11(ii), $(\mathfrak{F}, \mathfrak{G}) \models \sigma_2(\mathsf{L})$. On the other hand, by Lemma 6.11(i), $(\mathfrak{F}, \mathfrak{G}) \not\models \sigma_1(\mathsf{L})$. Thus, $\sigma_1(\mathsf{L}) \not\subseteq \sigma_2(\mathsf{L})$.

As a consequence of Theorem 6.18, we obtain the following characterization of the downward subframization of a si-logic.

Theorem 6.21. *Let* L *be a si-logic. Then* $\mathsf{Subf}_{\downarrow}(\mathsf{L}) = \rho_2\sigma_1(\mathsf{L})$.

Proof. Let $\mathfrak{G}$ be an Esakia frame. By Lemma 6.8(ii), $\mathfrak{G} \models \rho_2\sigma_1(\mathsf{L})$ iff there is an Esakia frame $\mathfrak{F}$ such that $(\mathfrak{F}, \mathfrak{G}) \models \sigma_1(\mathsf{L})$. By Lemma 6.11(i), $(\mathfrak{F}, \mathfrak{G}) \models \sigma_1(\mathsf{L})$ iff $\mathfrak{F} \models \mathsf{L}$. Therefore, $\mathfrak{G} \models \rho_2\sigma_1(\mathsf{L})$ iff $\mathfrak{G}$ is a subframe of some $\mathfrak{F} \models \mathsf{L}$. Thus, by Proposition 4.3(i), $\rho_2\sigma_1(\mathsf{L}) = \mathsf{Subf}_{\downarrow}(\mathsf{L})$. □

Remark 6.22.

(i) Let L be a si-logic and $\varphi \in \mathcal{L}_{\mathsf{IPC}}$. By Theorem 6.21, $\varphi \in \mathsf{Subf}_{\downarrow}(\mathsf{L})$ iff $\tau(\varphi) \in \sigma_1(\mathsf{L})$. Therefore, if $\sigma_1(\mathsf{L})$ is decidable, then so is $\mathsf{Subf}_{\downarrow}(\mathsf{L})$.

(ii) In contrast to Theorem 6.21, for every si-logic L, we have $\rho_1\sigma_2(\mathsf{L}) = \mathsf{IPC}$. Indeed, suppose L is a si-logic and $\mathfrak{F}$ is an Esakia frame. By Lemma 6.8(i), $\mathfrak{F} \models \rho_1\sigma_2(\mathsf{L})$ iff there is a subframe $\mathfrak{G}$ of $\mathfrak{F}$ such that $(\mathfrak{F}, \mathfrak{G}) \models \sigma_2(\mathsf{L})$. By Lemma 6.11(ii), $(\mathfrak{F}, \mathfrak{G}) \models \sigma_2(\mathsf{L})$ iff $\mathfrak{G} \models \mathsf{L}$. Therefore, $\mathfrak{F} \models \rho_1\sigma_2(\mathsf{L})$ iff $\mathfrak{G} \models \mathsf{L}$ for some subframe $\mathfrak{G}$ of $\mathfrak{F}$. Now, every frame contains the empty frame as a subframe and since the empty frame is an L-frame, we conclude that every frame validates $\rho_1\sigma_2(\mathsf{L})$. Thus, $\rho_1\sigma_2(\mathsf{L}) = \mathsf{IPC}$.

Remark 6.23. We recall that a subframe $\mathfrak{G}$ of an Esakia frame $\mathfrak{F}$ is *cofinal* provided it contains the maximum of $\mathfrak{F}$. Cofinal subframes of an Esakia frame $\mathfrak{F}$ correspond to *dense* nuclei on the Heyting algebra of clopen upsets of $\mathfrak{F}$, where we recall that a nucleus j is dense if $j0 = 0$. Since being a dense nucleus can be expressed by adding $\bigcirc\neg\bot$ to PLL, the correspondence between subframe logics and extensions of PLL discussed in this section extends to the correspondence between cofinal subframe logics and extensions of $\mathsf{PLL}+\bigcirc\neg\bot$.

7. Stabilization

In this section we aim to mirror the results of Sections 4 and 5 to the setting of stable logics. We will define the concept of stabilization, which is an analogue of subframization for stable logics. As we discussed in Section 3.2, when working with stable logics, we work with rooted Esakia frames. This is in contrast with subframe logics, and requires slight modifications of some of the characterizations by restricting to rooted Esakia frames only. This will have no major effect on the proofs.

Let Λ_{Stab} be the class of all stable logics. By Theorem 3.7, Λ_{Stab} is a complete sublattice of the lattice of all si-logics.

Definition 7.1. For a si-logic L, define the *downward stabilization of* L as

$$\mathsf{Stab}_{\downarrow}(\mathsf{L}) := \bigvee\{\mathsf{L}' \in \Lambda_{\mathsf{Stab}} \mid \mathsf{L}' \subseteq \mathsf{L}\}$$

and the *upward stabilization of* L as

$$\mathsf{Stab}_{\uparrow}(\mathsf{L}) := \bigwedge\{\mathsf{L}' \in \Lambda_{\mathsf{Stab}} \mid \mathsf{L} \subseteq \mathsf{L}'\}.$$

The following lemma is obvious.

Lemma 7.2. $\mathsf{Stab}_{\downarrow}$ *is an interior operator and* $\mathrm{Stab}_{\uparrow}$ *is a closure operator on the lattice of all si-logics.*

We next give a semantic characterization of downward and upward stabilizations (cf. Proposition 4.3).

Proposition 7.3. *Let* L *be a si-logic such that* $\mathsf{L} = \mathsf{Log}(K)$ *for some class* K *of rooted Esakia frames. Then*

(i) $\mathsf{Stab}_{\downarrow}(\mathsf{L}) = \mathsf{Log}\left(\{\mathfrak{G} \mid \mathfrak{G} \textit{ is a stable image of } \mathfrak{F} \in K\}\right)$.

(ii) $\mathsf{Stab}_{\uparrow}(\mathsf{L}) = \mathsf{Log}\left(\{\mathfrak{F} \mid \mathfrak{F} \textit{ is rooted and } \mathfrak{G} \models \mathsf{L} \textit{ for every stable image } \mathfrak{G} \textit{ of } \mathfrak{F}\ \}\right)$

$= \mathsf{Log}\left(\{\mathfrak{F} \mid \mathfrak{F} \textit{ is finite rooted and } \mathfrak{G} \models \mathsf{L} \textit{ for every stable image } \mathfrak{G} \textit{ of } \mathfrak{F}\ \}\right)$.

Proof. (i). Let $K' = \{\mathfrak{G} \mid \mathfrak{G}$ is a stable image of $\mathfrak{F} \in K\}$. Then K' is a class of rooted Esakia frames closed under stable images, so $\mathsf{Log}(K')$ is a stable logic by Theorem 3.7. Since $K \subseteq K'$, we have $\mathsf{Log}(K') \subseteq \mathsf{L}$. Let L' be a stable logic contained in L. Then the class K'' of rooted L'-frames contains K and is closed under stable images. Therefore, $K' \subseteq K''$, and so $\mathsf{L}' \subseteq \mathsf{Log}(K')$. Thus, $\mathsf{Log}(K')$ is the largest stable logic contained in L.

(ii). Let $K' = \{\mathfrak{F} \mid \mathfrak{F}$ is rooted and $\mathfrak{G} \models \mathsf{L}$ for every stable image $\mathfrak{G}$ of $\mathfrak{F}\ \}$ and $K'' = \{\mathfrak{F} \mid \mathfrak{F}$ is finite rooted and $\mathfrak{G} \models \mathsf{L}$ for every stable image $\mathfrak{G}$ of $\mathfrak{F}\ \}$. We show that $\mathsf{Stab}_{\uparrow}(\mathsf{L}) = \mathsf{Log}(K') = \mathsf{Log}(K'')$. It is clear that $K'' \subseteq K'$, so $\mathsf{Log}(K') \subseteq \mathsf{Log}(K'')$. It is obvious that both K' and K'' are closed under stable images, so $\mathsf{Log}(K'), \mathsf{Log}(K'')$ are stable logics by Theorem 3.7. Since K', K'' are contained in the class of rooted L-frames, $\mathsf{L} \subseteq \mathsf{Log}(K'), \mathsf{Log}(K'')$. Therefore, $\mathsf{Stab}_{\uparrow}(\mathsf{L}) \subseteq \mathsf{Log}(K'), \mathsf{Log}(K'')$. It is left to prove that $\mathsf{Log}(K'') \subseteq \mathsf{Stab}_{\uparrow}(\mathsf{L})$.

Let L' be a stable logic extending L, and let $\mathfrak{F}$ be a finite rooted L'-frame. Since L' is stable, all stable images of $\mathfrak{F}$ are L'-frames, and hence also L-frames. Therefore, $\mathfrak{F} \in K''$. Since L' is stable, L' is the logic of its finite rooted frames. Thus, $\mathsf{Log}(K'') \subseteq \mathsf{L}'$, so $\mathsf{Log}(K'')$ is the least stable extension of L, and hence $\mathsf{Stab}_{\uparrow}(\mathsf{L}) = \mathsf{Log}(K'')$. □

For an axiomatization of $\mathsf{Stab}_{\downarrow}(\mathsf{L})$ and $\mathsf{Stab}_{\uparrow}(\mathsf{L})$, we use stable canonical formulas (cf. Theorem 4.4).

Theorem 7.4. *Let* $\mathsf{L} = \mathsf{IPC} + \{\gamma(\mathfrak{H}_i, \mathfrak{D}_i) \mid i \in I\}$ *be a si-logic.*

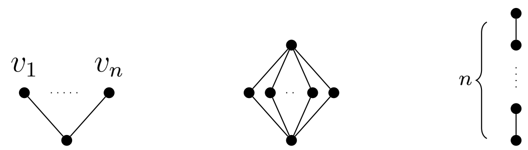
(i) $\mathsf{Stab}_{\downarrow}(\mathsf{L}) = \mathsf{IPC} + \{\gamma(\mathfrak{H}) \mid \mathsf{L} \vdash \gamma(\mathfrak{H})\}$.

(ii) $\mathsf{Stab}_{\uparrow}(\mathsf{L}) = \mathsf{IPC} + \{\gamma(\mathfrak{H}_i) \mid i \in I\}$.

Proof. (i). By [6, Thm. 6.11], $\mathsf{IPC} + \{\gamma(\mathfrak{H}) \mid \mathsf{L} \vdash \gamma(\mathfrak{H})\}$ is a stable logic, and clearly it is the largest stable logic contained in L. Therefore, $\mathsf{Stab}_{\downarrow}(\mathsf{L}) = \mathsf{IPC} + \{\gamma(\mathfrak{H}) \mid \mathsf{L} \vdash \gamma(\mathfrak{H})\}$.

(ii). Let $\mathsf{M} = \mathsf{IPC} + \{\gamma(\mathfrak{H}_i) \mid i \in I\}$ and let $\mathfrak{G}$ be a rooted M-frame. Then $\mathfrak{G} \models \gamma(\mathfrak{H}_i)$ for all $i \in I$. Therefore, by the semantic criterion of Section 3.2, $\mathfrak{G} \models \gamma(\mathfrak{H}_i, \mathfrak{D}_i)$ for all $i \in I$. Thus, $\mathfrak{G}$ is an L-frame, and so $\mathsf{L} \subseteq \mathsf{M}$. Since M is axiomatized by stable formulas, M is a stable logic. Suppose L' is a stable extension of L, and $\mathfrak{G}$ is a rooted L'-frame. If $\mathfrak{G} \not\models \gamma(\mathfrak{H}_i)$ for some $i \in I$, then $\mathfrak{H}_i$ is a stable image of some point-generated subframe $\mathfrak{H}$ of $\mathfrak{G}$. Therefore, $\mathfrak{H}_i$ is an L'-frame. But $\mathfrak{H}_i$ is not an L-frame, which contradicts to L' being an extension of L. Thus, $\mathfrak{G} \models \gamma(\mathfrak{H}_i)$ for all $i \in I$, and so $\mathsf{M} \subseteq \mathsf{L}'$. Consequently, M is the least stable extension of L, and hence $\mathsf{Stab}_{\uparrow}(\mathsf{L}) = \mathsf{M}$. □

Remark 7.5. If a si-logic L is axiomatized by a set of formulas Γ, then $\mathsf{Stab}_{\uparrow}(\mathsf{L})$ can be calculated effectively from Γ as follows: First use [6, Cor. 3.9] to transform Γ into an equivalent set of stable canonical formulas; then delete the additional parameters $\mathfrak{D}_i$ in the resulting canonical formulas; and finally apply Theorem 7.4(ii). On the other hand, applying Theorem 7.4(i) does not provide an effective axiomatization of $\mathsf{Stab}_{\downarrow}(\mathsf{L})$. We will come back to this issue at the end of Section 8.



$\mathfrak{F}_n$ n-fork $\qquad$ $\mathfrak{G}_n$ n-fork with top $\qquad$ $\mathfrak{C}_n$ n-chain

FIGURE 4

Remark 7.6. By restricting Wolter's describable operations (cf. Remark 4.6) to the class of rooted Esakia frames, we can obtain an alternative proof of Theorem 7.4. For a rooted Esakia frame $\mathfrak{G}$, let $\mathbf{C}(\mathfrak{G}) = \{\mathfrak{H} \mid \mathfrak{H} \text{ is a stable image of } \mathfrak{G}\}$. We show that

$$\mathfrak{G} \models \gamma(\mathfrak{F}) \text{ iff } \mathbf{C}(\mathfrak{G}) \models \gamma(\mathfrak{F}, \mathfrak{D}). \tag{10}$$

For the left to right direction, suppose $\mathbf{C}(\mathfrak{G}) \not\models \gamma(\mathfrak{F}, \mathfrak{D})$. Then there is a stable image $\mathfrak{H}$ of $\mathfrak{G}$ such that $\mathfrak{H} \not\models \gamma(\mathfrak{F}, \mathfrak{D})$. Then there is a finite stable image $\mathfrak{H}'$ of $\mathfrak{H}$ such that $\mathfrak{H}' \not\models \gamma(\mathfrak{F}, \mathfrak{D})$ (see, e.g., [6, Lem. 3.6]). By (3) of Section 3, there is a point-generated upset $\mathfrak{H}''$ of $\mathfrak{H}'$ and a stable map from $\mathfrak{H}''$ onto $\mathfrak{F}$ satisfying SCDC for $\mathfrak{D}$. Since $\mathfrak{H}'$ is finite, $\mathfrak{F}$ is a stable image of $\mathfrak{H}'$ (see [6, Lem. 6.1]). So $\mathfrak{F}$ is a stable image of $\mathfrak{G}$. As $\mathfrak{G}$ is rooted, this implies that $\mathfrak{G} \not\models \gamma(\mathfrak{F})$ by (4) of Section 3.

For the right to left direction, suppose $\mathfrak{G} \not\models \gamma(\mathfrak{F})$. Since $\mathfrak{G}$ is rooted, it follows from [9, Prop. 5.1] that $\mathfrak{F}$ is a stable image of $\mathfrak{G}$. Therefore, $\mathfrak{F} \in \mathbf{C}(\mathfrak{G})$. Thus, since $\mathfrak{F} \not\models \gamma(\mathfrak{F}, \mathfrak{D})$, we conclude that $\mathbf{C}(\mathfrak{G}) \not\models \gamma(\mathfrak{F}, \mathfrak{D})$.

Set $(\gamma(\mathfrak{F}, \mathfrak{D}))^c = \gamma(\mathfrak{F})$. Because every si-logic is characterized by its rooted Esakia frames, Wolter's result applied to Proposition 7.3 yields an alternative proof of Theorem 7.4.

We conclude this section by giving several examples of downward and upward stabilizations of si-logics. In addition to the si-logics from Section 4, we consider the following si-logics.

- The logics BD_n of finite rooted frames of depth $\leq n$.
- The logics BC_n of finite rooted frames of cardinality $\leq n$.

Proposition 7.7.

(i) $\mathsf{Stab}_\downarrow(\mathsf{BD}_n) = \mathsf{IPC}$ *and* $\mathsf{Stab}_\uparrow(\mathsf{BD}_n) = \mathsf{BC}_n$ *for all* $n \geq 2$.
(ii) *If* L *is consistent and has the disjunction property, then* $\mathsf{Stab}_\downarrow(\mathsf{L}) = \mathsf{IPC}$.
(iii) $\mathsf{Stab}_\downarrow(\mathsf{T}_n) = \mathsf{IPC}$ *and* $\mathsf{Stab}_\uparrow(\mathsf{T}_n) = \mathsf{BW}_n$ *for all* $n \geq 2$.

Proof. (i). First we show that $\mathsf{Stab}_\downarrow(\mathsf{BD}_n) = \mathsf{IPC}$ for all $n \geq 2$. Since $\mathsf{BD}_n \subseteq \mathsf{BD}_2$ for all $n \geq 2$, it suffices to show that $\mathsf{Stab}_\downarrow(\mathsf{BD}_2) = \mathsf{IPC}$. Let $\mathfrak{F}$ be a finite rooted frame. Suppose $\mathfrak{F}$ has at most $n+1$ elements, and $\mathfrak{F}_n$ is the n-fork shown in Figure 3. Mapping the root of $\mathfrak{F}_n$ to the root of $\mathfrak{F}$ and the top nodes of $\mathfrak{F}_n$ surjectively onto the other nodes of $\mathfrak{F}$ defines a stable map from $\mathfrak{F}_n$ onto $\mathfrak{F}$. Since $\mathfrak{F}_n$ is a BD_2-frame, by Proposition 7.3(i), $\mathfrak{F} \models \mathsf{Stab}_\downarrow(\mathsf{BD}_2)$ for every finite rooted frame $\mathfrak{F}$. Thus, $\mathsf{Stab}_\downarrow(\mathsf{BD}_2) = \mathsf{IPC}$.

Next we show that $\mathsf{Stab}_\uparrow(\mathsf{BD}_n) = \mathsf{BC}_n$ for all $n \geq 2$. Suppose $\mathfrak{F}$ is a finite rooted frame. If $\mathfrak{F}$ has no more than n elements, then every stable image of $\mathfrak{F}$ also has no more than n elements. Therefore, every stable image of $\mathfrak{F}$ is a BC_n-frame. On the other hand, if $\mathfrak{F}$ has at least $n+1$ elements, then we can define a stable map from $\mathfrak{F}$ on the $(n+1)$-chain $\mathfrak{C}_{n+1}$ (see Figure 3) as follows: Map the root r of $\mathfrak{F}$ to the root of $\mathfrak{C}_{n+1}$; map the immediate successors of r on top of each other; continue this process with the immediate successors of the immediate successors of r, and so on; if you run out of points in $\mathfrak{C}_{n+1}$, then map the remaining points to the top node of $\mathfrak{C}_{n+1}$. Since $\mathfrak{C}_{n+1}$ is not a BD_n-frame, $\mathfrak{F}$ has a stable image refuting BD_n. Thus, by Proposition 7.3(ii), $\mathsf{Stab}_{\uparrow}(\mathsf{BD}_n) = \mathsf{BC}_n$.

(ii). Suppose L is consistent and has the disjunction property. By [14, Thm. 15.5], if $\mathfrak{F}_1, \mathfrak{F}_2$ are rooted L-frames, then their disjoint union $\mathfrak{F}_1 \sqcup \mathfrak{F}_2$ is a generated subframe of some rooted L-frame. This implies that for every n, there is a rooted L-frame $\mathfrak{F}$ containing at least n maximal points. To see this, since L is consistent, the one-point frame $\mathfrak{F}_1$ is an L-frame. Therefore, $\mathfrak{F}_1 \sqcup \mathfrak{F}_1$ is a generated subframe of some rooted L-frame $\mathfrak{F}_2$. Clearly $\mathfrak{F}_2$ has at least 2 maximal points. By the same argument, $\mathfrak{F}_2 \sqcup \mathfrak{F}_2$ is a generated subframe of some rooted L-frame $\mathfrak{F}_3$ that has at least 4 maximal points. Continuing this process yields a rooted L-frame $\mathfrak{F}$ with at least n maximal points, say $\{x_1, x_2, \dots x_n\}$. We show that the n-fork $\mathfrak{F}_n$ is a stable image of $\mathfrak{F}$. Separate $x_1, \dots, x_n$ by disjoint clopen upsets $U_1, \dots, U_n$ with $x_i \in U_i$ for $1 \le i \le n$, and define a map $f : \mathfrak{F} \to \mathfrak{F}_n$ by

$$f(x) = \begin{cases} x_i & \text{if } x \in U_i \text{ for some } i \in I, \\ r & \text{otherwise,} \end{cases}$$

where r is the root of $\mathfrak{F}_n$. It is straightforward to see that f is an onto stable map. Thus, $\mathsf{Stab}_{\downarrow}(\mathsf{L}) \subseteq \mathsf{BD}_2$. Now apply (i) to conclude that $\mathsf{Stab}_{\downarrow}(\mathsf{L}) = \mathsf{IPC}$.

(iii). Since T_n is consistent and has the disjunction property for all $n \ge 2$, by (ii), $\mathsf{Stab}_{\downarrow}(\mathsf{T}_n) = \mathsf{IPC}$ for all $n \ge 2$.

Next we show that $\mathsf{Stab}_{\uparrow}(\mathsf{T}_n) = \mathsf{BW}_n$ for all $n \ge 2$. Let $K = \{\mathfrak{F} \mid \mathfrak{F}$ is finite rooted and $\mathfrak{G} \models \mathsf{T}_n$ for every stable image $\mathfrak{G}$ of $\mathfrak{F}\}$. By Proposition 7.3(ii), $\mathsf{Stab}_{\uparrow}(\mathsf{T}_n) = \mathsf{Log}(K)$. Let K' be the class of finite rooted frames of width $\le n$. We show that $K = K'$. Let $\mathfrak{F}$ be finite and rooted. If $\mathfrak{F} \in K'$, i.e. $\mathfrak{F}$ is of width $\le n$, then so are all its stable images (see [6, Thm. 7.3(2)]). Since width $\le n$ implies branching $\le n$, every stable image of $\mathfrak{F}$ vaildates T_n, so $\mathfrak{F} \in K$. Therefore, $K' \subseteq K$. Conversely, if $\mathfrak{F}$ has width greater than n, then by [6, Thm. 7.5(3)], either the $(n+1)$-fork or the $(n+1)$-fork with top (see Figure 3) is a stable image of $\mathfrak{F}$. Since neither of these is a T_n-frame, $\mathfrak{F} \notin K$. Thus, $K = K'$, and as BW_n is the logic of K', we conclude that $\mathsf{Stab}_{\uparrow}(\mathsf{T}_n) = \mathsf{BW}_n$. □

8. Stable logics and intuitionistic S4

As we saw in Section 6, there is a close connection between subframe logics and extensions of the lax logic PLL. In this section we show that there is a close connection between stable logics and extensions of $\mathsf{IS4}$. More precisely, we show that there are two natural embeddings of si-logics into multiple-conclusion consequence relations extending $\mathsf{IS4} + p \vee q/p, q$, which yield a new characterization of stable logics.

Recall that in Section 6, we embedded si-logics into logics extending PLL. In this section we work with multi-conclusion consequence relations as opposed to logics. This is due to the fact that stability of a si-logic ensures that the class of its *rooted* frames is stable and being a rooted frame, while cannot be captured by formulas, is captured by the multiple-conclusion disjunction rule $p \vee q/p, q$ meaning that a frame is rooted exactly when it validates the disjunction rule.

Definition 8.1. ([27]) *Intuitionistic* $\mathsf{S4}$ ($\mathsf{IS4}$) is the least set of formulas of the propositional modal language containing IPC, the axioms $\Box p \to p$, $\Box p \to \Box\Box p$, $\Box(p \to q) \to (\Box p \to \Box q)$, and closed under substitution, modus ponens, and necessitation.

Algebraic semantics of $\mathsf{IS4}$ is provided by interior Heyting algebras.

Definition 8.2. ([27]) *An interior Heyting algebra* is a pair $(A, \Box)$, where A is a Heyting algebra and $\Box$ is an *interior operator* on A; that is, $\Box$ is a unary function on A satisfying $\Box a \le a$, $\Box a \le \Box\Box a$, $\Box(a \wedge b) = \Box a \wedge \Box b$, and $\Box 1 = 1$.

Like the fixpoints of a nuclear algebra, the $\Box$-fixpoints $A_\Box := \{a \in A \mid \Box a = a\}$ of an interior Heyting algebra $(A, \Box)$ also form a Heyting algebra. But unlike the fixpoints of a nuclear algebra, the $\Box$-fixpoints form a bounded sublattice of A, so the join, meet, 0, and 1 in $A_\Box$ are the same as in A.

The implication in $A_\Box$ is calculated as $a \to_\Box b = \Box(a \to b)$. In fact, interior Heyting algebras correspond to pairs (A, A_0) of Heyting algebras such that A_0 is a bounded sublattice of A and the embedding $A_0 \rightarrowtail A$ has a right adjoint. Similarly to the case of nuclear algebras, this correspondence is obtained as follows: If $(A, \Box)$ is an interior Heyting algebra, then $A_\Box$ is a Heyting algebra and the embedding $A_\Box \hookrightarrow A$ has $\Box$ as a right adjoint; conversely, given such a pair (A, A_0), we have that the right adjoint to the inclusion $A_0 \hookrightarrow A$ is an interior operator on A; and these correspondences are inverse to each other (see, e.g., [13, Thm. 2.7]).

Given such a pair (A, A_0), let $\mathfrak{F} = (X, \le)$ be the Esakia frame of A and $\mathfrak{G} = (Y, \le)$ the Esakia frame of A_0. Since the embedding $A_0 \rightarrowtail A$ is a bounded lattice morphism, the dual map $\pi : X \to Y$ is an onto stable map. Moreover, the right adjoint $\Box : A \to A_0$ of the embedding $A_0 \rightarrowtail A$ is dually described as follows: if U is a clopen upset of $\mathfrak{F}$, then $\Box U = Y \setminus {\downarrow}\pi(X \setminus U)$. Therefore, for each clopen U in X, we have that ${\downarrow}\pi(U)$ is a clopen subset of Y. Thus, interior Heyting algebras correspond to pairs of Esakia frames $(\mathfrak{F}, \mathfrak{G})$ and an onto stable map between them satisfying ${\downarrow}\pi(U)$ is clopen in $\mathfrak{G}$ for each clopen U in $\mathfrak{F}$ (for monadic Heyting algebras this correspondence is discussed in detail in [4], and can be adjusted easily to our case as in [24, Thm. 6.6.4]). This yields the following definition.

Definition 8.3. An *St-frame* (stable frame) is a pair $(\mathfrak{F}, \mathfrak{G})$ such that $\mathfrak{F} = (X, \le)$ and $\mathfrak{G} = (Y, \le)$ are Esakia frames and $\pi : X \to Y$ is an onto stable map satisfying ${\downarrow}\pi(U)$ is clopen in Y for each clopen U in X.

The correspondence between interior Heyting algebras and St-frames allows us to interpret formulas of $\mathsf{IS4}$ in St-frames. Let $(\mathfrak{F}, \mathfrak{G})$ be an St-frame, where $\mathfrak{F} = (X, \le)$ and $\mathfrak{G} = (Y, \le)$. We interpret propositional letters as clopen upsets of $\mathfrak{F}$ and intuitionistic connectives as the corresponding operations in the Heyting algebra of clopen upsets of $\mathfrak{F}$. In addition, $\Box$ is interpreted as the corresponding unary function on the clopen upsets of $\mathfrak{F}$; that is, $\Box U = \pi^{-1}(Y \setminus {\downarrow}\pi(X \setminus U))$. Therefore, if v is a valuation on $(\mathfrak{F}, \mathfrak{G})$ and $x \in X$, then $x \notin \Box v(\varphi)$ iff $\pi(x) \in {\downarrow}\pi(X \setminus v(\varphi))$, which happens iff there is $z \in X \setminus v(\varphi)$ with $\pi(x) \le \pi(z)$. Thus,

$$x \models_v \Box\varphi \text{ iff } z \models_v \varphi \text{ for all } z \in X \text{ with } \pi(x) \le \pi(z).$$

We utilize the Gödel-McKinsey-Tarski translation to translate a formula φ of IPC into the formula $t(\varphi)$ of $\mathsf{IS4}$ as follows:

- $t(p) = \Box p$ for a propositional letter p,
- $t(\bot) = \Box\bot$,
- $t(\varphi \wedge \psi) = t(\varphi) \wedge t(\psi)$,
- $t(\varphi \vee \psi) = t(\varphi) \vee t(\psi)$,
- $t(\varphi \to \psi) = \Box(t(\varphi) \to t(\psi))$.

A straightforward induction shows that for every $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ and every interior Heyting algebra $(A, \Box)$, we have:

(i) $(A, \Box) \models \varphi$ iff $A \models \varphi$.
(ii) $(A, \Box) \models t(\varphi)$ iff $A_\Box \models \varphi$.

Translating this into dual terms yields the following lemma.

Lemma 8.4. *For every $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ and every St-frame $(\mathfrak{F}, \mathfrak{G})$:*

(i) *$(\mathfrak{F}, \mathfrak{G}) \models \varphi$ iff $\mathfrak{F} \models \varphi$.*
(ii) *$(\mathfrak{F}, \mathfrak{G}) \models t(\varphi)$ iff $\mathfrak{G} \models \varphi$.*

We recall (see, e.g., [23, 22, 9]) that a *multiple-conclusion rule* is an expression of the form Γ/Δ, where Γ and Δ are finite sets of formulas. A *multiple-conclusion consequence relation over* $\mathsf{IS4}$ is a set $\mathcal{S}$ of multiple-conclusion rules such that

- $\varphi/\varphi \in \mathcal{S}$.
- $\varphi, \varphi \to \psi/\psi \in \mathcal{S}$.
- $\varphi/\Box\varphi \in \mathcal{S}$.
- $/\varphi \in \mathcal{S}$ for each theorem φ of $\mathsf{IS4}$.
- If $\Gamma/\Delta \in \mathcal{S}$, then $\Gamma, \Gamma'/\Delta, \Delta' \in \mathcal{S}$.
- If $\Gamma/\Delta, \varphi \in \mathcal{S}$ and $\Gamma, \varphi/\Delta \in \mathcal{S}$, then $\Gamma/\Delta \in \mathcal{S}$.
- If $\Gamma/\Delta \in \mathcal{S}$ and s is a substitution, then $s(\Gamma)/s(\Delta) \in \mathcal{S}$.

Let $\mathcal{S}_{\mathsf{IS4}}$ be the (multiple-conclusion) consequence relation over $\mathsf{IS4}$ that in addition contains the disjunction rule $p \vee q/p, q$. If Γ is a set of multiple-conclusion rules, by $\mathcal{S}_{\mathsf{IS4}} + \Gamma$ we denote the least consequence relation extending $\mathcal{S}_{\mathsf{IS4}}$ containing Γ.

Let $(A, \Box)$ be an interior Heyting algebra. A multiple-conclusion rule Γ/Δ is *valid* on $(A, \Box)$ (written: $(A, \Box) \models \Gamma/\Delta$) if for every valuation v on A, from $v(\gamma) = 1$ for every $\gamma \in \Gamma$ it follows that $v(\delta) = 1$ for some $\delta \in \Delta$. If K is a class of interior Heyting algebras, then we write $K \models \Gamma/\Delta$ if $(A, \Box) \models \Gamma/\Delta$ for each $(A, \Box) \in K$.

An interior Heyting algebra $(A, \Box)$ is called *well-connected* if the underlying Heyting algebra A is well-connected (that is, $a \vee b = 1$ implies $a = 1$ or $b = 1$ for all $a, b \in A$). It is easy to see that an interior Heyting algebra $(A, \Box)$ validates the disjunction rule $p \vee q/p, q$ iff it is well-connected.

Dually well-connected interior Heyting algebras correspond to rooted St-frames, where an St-frame $(\mathfrak{F}, \mathfrak{G})$ is *rooted* provided $\mathfrak{F}$ is a rooted Esakia frame (note that in that case $\mathfrak{G}$ is also rooted).

Every multiple-conclusion consequence relation $\mathcal{S}$ extending $\mathcal{S}_{\mathsf{IS4}}$ is sound and complete with respect to the class of well-connected interior Heyting algebras validating every rule in $\mathcal{S}$ (resp. with respect to rooted St-frames validating $\mathcal{S}$). For a class K of well-connected interior Heyting algebras (resp. rooted St-frames), let $\mathsf{Con}(K)$ be the set of multiple-conclusion rules that are valid in K. Then $\mathsf{Con}(K)$ is a multiple-conclusion consequence relation extending $\mathcal{S}_{\mathsf{IS4}}$.

Definition 8.5. Let L be a si-logic and $\mathcal{S}$ a multiple-conclusion consequence relation extending $\mathcal{S}_{\mathsf{IS4}}$.

(i) We say that L is the *intuitionistic fragment* of $\mathcal{S}$ if for all formulas $\varphi \in \mathcal{L}_{\mathsf{IPC}}$,

$$\varphi \in \mathsf{L} \text{ iff } /\varphi \in \mathcal{S}.$$

(ii) We say that L is the *stable fragment* of $\mathcal{S}$ if for all formulas $\varphi \in \mathcal{L}_{\mathsf{IPC}}$,

$$\varphi \in \mathsf{L} \text{ iff } /t(\varphi) \in \mathcal{S}.$$

For a multiple-conclusion consequence relation $\mathcal{S}$ extending $\mathcal{S}_{\mathsf{IS4}}$, we define

$$\zeta_1(\mathcal{S}) = \{\varphi \in \mathcal{L}_{\mathsf{IPC}} \mid /\varphi \in \mathcal{S}\},$$
$$\zeta_2(\mathcal{S}) = \{\varphi \in \mathcal{L}_{\mathsf{IPC}} \mid /t(\varphi) \in \mathcal{S}\}.$$

Lemma 8.6. *Let $\mathcal{S}$ be a multiple-conclusion consequence relation extending $\mathcal{S}_{\mathsf{IS4}}$.*

(i) *$\zeta_1(\mathcal{S})$ is the intuitionistic fragment of $\mathcal{S}$ and*

$$\zeta_1(\mathcal{S}) = \mathsf{Log}\left(\{\mathfrak{F} \mid \exists \mathfrak{G} : (\mathfrak{F}, \mathfrak{G}) \textit{ is an St-frame and } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S}\}\right).$$

(ii) *$\zeta_2(\mathcal{S})$ is the stable fragment of $\mathcal{S}$ and*

$$\zeta_2(\mathcal{S}) = \mathsf{Log}\left(\{\mathfrak{G} \mid \exists \mathfrak{F} : (\mathfrak{F}, \mathfrak{G}) \textit{ is an St-frame and } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S}\}\right).$$

Proof. (i). For $\varphi \in \mathcal{L}_{\mathsf{IPC}}$, we have

$$\begin{aligned} & \varphi \in \mathsf{Log}\left(\{\mathfrak{F} \mid \exists \mathfrak{G} : (\mathfrak{F}, \mathfrak{G}) \text{ is an St-frame and } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S}\}\right) \\ \Leftrightarrow\quad & \mathfrak{F} \models \varphi \text{ for all } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S} \\ \Leftrightarrow\quad & \mathfrak{F} \models /\varphi \text{ for all } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S} \\ \Leftrightarrow\quad & (\mathfrak{F}, \mathfrak{G}) \models /\varphi \text{ for all } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S} \\ \Leftrightarrow\quad & /\varphi \in \mathcal{S} \\ \Leftrightarrow\quad & \varphi \in \zeta_1(\mathcal{S}). \end{aligned}$$

Therefore, $\zeta_1(\mathcal{S}) = \mathsf{Log}\left(\{\mathfrak{F} \mid \exists \mathfrak{G} : (\mathfrak{F}, \mathfrak{G}) \text{ is an St-frame and } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S}\}\right)$. Thus, $\zeta_1(\mathcal{S})$ is a si-logic, and so it is the intuitionistic fragment of $\mathcal{S}$.

(ii). For $\varphi \in \mathcal{L}_{\mathsf{IPC}}$, we have

$$\begin{aligned} & \varphi \in \mathsf{Log}\left(\{\mathfrak{G} \mid \exists \mathfrak{F} : (\mathfrak{F}, \mathfrak{G}) \text{ is an St-frame and } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S}\}\right) \\ \Leftrightarrow\quad & \mathfrak{G} \models \varphi \text{ for all } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S} \\ \Leftrightarrow\quad & \mathfrak{G} \models /\varphi \text{ for all } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S} \\ \Leftrightarrow\quad & (\mathfrak{F}, \mathfrak{G}) \models /t(\varphi) \text{ for all } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S} \\ \Leftrightarrow\quad & /t(\varphi) \in \mathcal{S} \\ \Leftrightarrow\quad & \varphi \in \zeta_2(\mathcal{S}). \end{aligned}$$

Therefore, $\zeta_2(\mathcal{S}) = \mathsf{Log}\left(\{\mathfrak{G} \mid \exists \mathfrak{F} : (\mathfrak{F}, \mathfrak{G}) \text{ is an St-frame and } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S}\}\right)$. Thus, $\zeta_2(\mathcal{S})$ is a si-logic, and so it is the stable fragment of $\mathcal{S}$. □

Conversely, for a si-logic L, define:

$$\eta_1(\mathsf{L}) = \mathcal{S}_{\mathsf{IS4}} + \{/\varphi \mid \varphi \in \mathsf{L}\},$$
$$\eta_2(\mathsf{L}) = \mathcal{S}_{\mathsf{IS4}} + \{/t(\varphi) \mid \varphi \in \mathsf{L}\}.$$

Lemma 8.7. *For every si-logic L, we have:*

(i) $\eta_1(\mathsf{L}) = \mathsf{Con}\left(\{(\mathfrak{F}, \mathfrak{G}) \mid \mathfrak{F} \textit{ is a rooted } \mathsf{L}\textit{-frame}\}\right)$,
(ii) $\eta_2(\mathsf{L}) = \mathsf{Con}\left(\{(\mathfrak{F}, \mathfrak{G}) \mid \mathfrak{G} \textit{ is a rooted } \mathsf{L}\textit{-frame})\right\}$.

Proof. We prove (ii), the proof of (i) is similar. For an St-frame $(\mathfrak{F},\mathfrak{G})$ we have $\mathfrak{G} \models \mathsf{L}$ iff $(\mathfrak{F},\mathfrak{G}) \models \{t(\varphi) \mid \varphi \in \mathsf{L}\}$, which happens iff $(\mathfrak{F},\mathfrak{G}) \models \{/t(\varphi) \mid \varphi \in \mathsf{L}\}$. Thus, $\eta_2(\mathsf{L}) = \mathsf{Con}\,(\{(\mathfrak{F},\mathfrak{G}) \mid \mathfrak{G}$ is a rooted L-frame$\})$. □

Lemma 8.8. *Let* L *be a si-logic.*

(i) $\mathsf{L} = \zeta_1\eta_1(\mathsf{L})$*, and* $\eta_1(\mathsf{L})$ *is the least multiple-conclusion consequence relation in* $\zeta_1^{-1}(\mathsf{L})$*.*
(ii) $\mathsf{L} = \zeta_2\eta_2(\mathsf{L})$*, and* $\eta_2(\mathsf{L})$ *is the least multiple-conclusion consequence relation in* ${\zeta_2}^{-1}(\mathsf{L})$*.*

Proof. (i). Let $\varphi \in \mathcal{L}_{\mathsf{IPC}}$. Then $\varphi \in \mathsf{L}$ implies $/\varphi \in \eta_1(\mathsf{L})$, which implies $\varphi \in \zeta_1\eta_1(\mathsf{L})$. Therefore, $\mathsf{L} \subseteq \zeta_1\eta_1(\mathsf{L})$. If $\varphi \notin \mathsf{L}$, then there is a rooted L-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. Consider the St-frame $(\mathfrak{F},\mathfrak{F})$, where π is the identity map. Then $(\mathfrak{F},\mathfrak{F}) \not\models /\varphi$, and $(\mathfrak{F},\mathfrak{F}) \models \eta_1(\mathsf{L})$ by Lemma 8.7(i). Therefore, by Lemma 8.6(i), $\varphi \notin \zeta_1\eta_1(\mathsf{L})$. This shows that $\mathsf{L} = \zeta_1\eta_1(\mathsf{L})$. If $\mathcal{S} \in \zeta_1^{-1}(\mathsf{L})$, then for every $\varphi \in \mathcal{L}_{\mathsf{IPC}}$, we have $\varphi \in \mathsf{L}$ iff $/\varphi \in \mathcal{S}$. Thus, $\eta_1(\mathsf{L}) \subseteq \mathcal{S}$, and hence $\eta_1(\mathsf{L})$ is the least element of $\zeta_1^{-1}(\mathsf{L})$.

(ii). Let $\varphi \in \mathcal{L}_{\mathsf{IPC}}$. Then $\varphi \in \mathsf{L}$ implies $/t(\varphi) \in \eta_2(\mathsf{L})$, which implies $\varphi \in \zeta_2\eta_2(\mathsf{L})$. Therefore, $\mathsf{L} \subseteq \zeta_2\eta_2(\mathsf{L})$. If $\varphi \notin \mathsf{L}$, then there is a rooted L-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. Then $(\mathfrak{F},\mathfrak{F}) \not\models /t(\varphi)$, and $(\mathfrak{F},\mathfrak{F})$ is a $\eta_2(\mathsf{L})$-frame by Lemma 8.7(ii). Thus, by Lemma 8.6(ii), $\varphi \notin \zeta_2\eta_2(\mathsf{L})$. This shows that $\mathsf{L} = \zeta_2\eta_2(\mathsf{L})$. If $\mathcal{S} \in \zeta_2^{-1}(\mathsf{L})$, then for every $\varphi \in \mathcal{L}_{\mathsf{IPC}}$, we have $\varphi \in \mathsf{L}$ iff $/t(\varphi) \in \mathcal{S}$. Consequently, $\eta_2(\mathsf{L}) \subseteq \mathcal{S}$, and hence $\eta_2(\mathsf{L})$ is the least element of $\zeta_2^{-1}(\mathsf{L})$. □

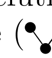
As follows from Lemma 8.8, for a si-logic L, both $\zeta_1^{-1}(\mathsf{L})$ and $\zeta_2^{-1}(\mathsf{L})$ have least elements, but they may not have largest elements. To see this we require the following lemma.

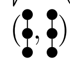
Lemma 8.9. *Let* $(\mathfrak{F},\mathfrak{G})$ *be an St-frame. Then* $(\mathfrak{F},\mathfrak{G}) \models /\Box p \leftrightarrow p$ *iff* π *is an isomorphism.*

Proof. Let $\mathfrak{F} = (X,\leq)$ and $\mathfrak{G} = (Y,\leq)$. First suppose that π is an isomorphism. Then it is clear that $(\mathfrak{F},\mathfrak{G}) \models \Box p \leftrightarrow p$. Next suppose that π is not an isomorphism. Then there are $x \not\leq y$ with $\pi(x) \leq \pi(y)$. Let U be a clopen upset of $\mathfrak{F}$, with $x \in U$ but $y \notin U$. Define a valuation v on $(\mathfrak{F},\mathfrak{G})$ with $v(p) = U$. Then $x \models_v p$ but $x \not\models_v \Box p$. Thus, $(\mathfrak{F},\mathfrak{G}) \not\models \Box p \leftrightarrow p$. □

For $\psi \in \mathcal{L}_{\mathsf{IS4}}$, let ψ^- be the formula obtained from ψ by deleting all occurrences of $\Box$. Similarly to Lemma 6.15, we can show that for every multiple-conclusion consequence relation $\mathcal{S}$ extending $\mathcal{S}_{\mathsf{IS4}}$, if $/\Box p \leftrightarrow p \in \mathcal{S}$, then $/\psi \in \mathcal{S}$ iff $/\psi^- \in \mathcal{S}$. From this we can infer, as in Lemma 6.16, that $\eta_1(\mathsf{L}) + /\Box p \leftrightarrow p$ is maximal in both $\zeta_1^{-1}(\mathsf{L})$ and $\zeta_2^{-1}(\mathsf{L})$. On the other hand, neither of $\zeta_1^{-1}(\mathsf{L})$ and $\zeta_2^{-1}(\mathsf{L})$ has to have a largest element, as the next example shows.

Example 8.10. Let γ abbreviate $(p \to q) \vee (q \to p)$ and let $\mathcal{S} = \eta_1(\mathsf{BD}_2) + /t(\gamma)$. By Lemma 8.4, an St-frame $(\mathfrak{F},\mathfrak{G})$ is an $\mathcal{S}$-frame iff $\mathfrak{F}$ is a BD_2-frame and $\mathfrak{G}$ is an LC-frame.

(i) We show that $\zeta_1(\mathcal{S}) = \mathsf{BD}_2$. By Lemma 8.6(i), $\mathsf{BD}_2 \subseteq \zeta_1(\mathcal{S})$. Conversely, suppose $\varphi \notin \mathsf{BD}_2$. Then there is a finite rooted BD_2-frame $\mathfrak{F}$ refuting φ. Let $n = |\mathfrak{F}|$ and let $\mathfrak{G}$ be the n-chain. As we saw in the proof of Proposition 7.7(i), $\mathfrak{G}$ is a stable image of $\mathfrak{F}$. Therefore, $(\mathfrak{F},\mathfrak{G})$ is an $\mathcal{S}$-frame refuting φ. Thus, $\zeta_1(\mathcal{S}) = \mathsf{BD}_2$. On the other hand, $\mathcal{S} \not\subseteq \eta_1(\mathsf{BD}_2) + /\Box p \leftrightarrow p$ because (, ) validates $\eta_1(\mathsf{BD}_2) + /\Box p \leftrightarrow p$ but refutes $\mathcal{S}$. Consequently, $\zeta_1^{-1}(\mathsf{BD}_2)$ does not have a largest element.
(ii) We show that $\zeta_2(\mathcal{S}) = \mathsf{LC}$. By Lemma 8.6(ii), $\mathsf{LC} \subseteq \zeta_2(\mathcal{S})$. Conversely, suppose $\varphi \notin \mathsf{LC}$. Then there is a finite chain $\mathfrak{G}$ refuting φ. Let $n = |\mathfrak{G}|$. As follows from the proof of Proposition 7.7(i), $\mathfrak{G}$ is a stable image of the $(n-1)$-fork $\mathfrak{F}$. Therefore,

$(\mathfrak{F}, \mathfrak{G})$ is an $\mathcal{S}$-frame and $(\mathfrak{F}, \mathfrak{G}) \not\models t(\varphi)$. Thus, $\varphi \notin \zeta_2(\mathcal{S})$. On the other hand, $\mathcal{S} \not\subseteq \eta_1(\mathsf{LC}) + \Box p \leftrightarrow p$ because  satisfies $\eta_1(\mathsf{LC}) + /\Box p \leftrightarrow p$ but refutes $\mathcal{S}$. Consequently, $\zeta_2^{-1}(\mathsf{LC})$ does not have a largest element.

We will use the above correspondence between si-logics and multiple-conclusion consequence relations extending $\mathcal{S}_{\mathsf{IS4}}$ to provide another characterization of stable logics.

Theorem 8.11. *For a si-logic* L*, the following are equivalent.*

(i) L *is a stable logic.*
(ii) $\eta_2(\mathsf{L}) \subseteq \eta_1(\mathsf{L})$.
(iii) $\eta_2(\mathsf{L}) + \{/\varphi \mid \varphi \in \mathsf{L}\} = \eta_1(\mathsf{L})$.
(iv) $\zeta_2\eta_1(\mathsf{L}) = \mathsf{L}$.
(v) *For every* $\varphi \in \mathcal{L}_{\mathsf{IPC}}$*, from* $/\varphi \in \eta_1(\mathsf{L})$ *it follows that* $/t(\varphi) \in \eta_1(\mathsf{L})$.

Proof. (i)⇒(ii). Suppose that $(\mathfrak{F}, \mathfrak{G}) \models \eta_1(\mathsf{L})$. By Lemma 8.7(i), $\mathfrak{F} \models \mathsf{L}$. Since L is a stable logic, $\mathfrak{G} \models \mathsf{L}$. Therefore, by Lemma 8.7(ii), $(\mathfrak{F}, \mathfrak{G}) \models \eta_2(\mathsf{L})$. Thus, $\eta_2(\mathsf{L}) \subseteq \eta_1(\mathsf{L})$.

(ii)⇒(iii). This is obvious.

(iii)⇒(iv). By Lemmas 8.8(ii) and 8.6(ii), $\mathsf{L} = \zeta_2\eta_2(\mathsf{L}) = \mathsf{Log}(\{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \eta_2(\mathsf{L})\})$ and $\zeta_2\eta_1(\mathsf{L}) = \mathsf{Log}(\{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \eta_1(\mathsf{L})\})$. Therefore, it is sufficient to show that $\{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \eta_2(\mathsf{L})\} = \{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \eta_1(\mathsf{L})\}$. The inclusion $\supseteq$ is immediate from (iii). For the reverse inclusion, suppose that $(\mathfrak{F}, \mathfrak{G}) \models \eta_2(\mathsf{L})$. By Lemma 8.7(ii), $\mathfrak{G} \models \mathsf{L}$, so $(\mathfrak{G}, \mathfrak{G}) \models \eta_1(\mathsf{L})$ by Lemma 8.7(i). Thus, $\mathfrak{G} \in \{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \eta_1(\mathsf{L})\}$.

(iv)⇒(v). Suppose that there is $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ such that $/\varphi \in \eta_1(\mathsf{L})$ but $/t(\varphi) \notin \eta_1(\mathsf{L})$. Then there is an St-frame $(\mathfrak{F}, \mathfrak{G})$ with $(\mathfrak{F}, \mathfrak{G}) \models \eta_1(\mathsf{L})$ and $(\mathfrak{F}, \mathfrak{G}) \not\models t(\varphi)$. By Lemma 8.6(ii), $(\mathfrak{F}, \mathfrak{G}) \models \eta_1(\mathsf{L})$ implies $\mathfrak{G} \models \zeta_2\eta_1(\mathsf{L}) = \mathsf{L}$. Also, $(\mathfrak{F}, \mathfrak{G}) \not\models t(\varphi)$ implies $\mathfrak{G} \not\models \varphi$. Therefore, $\varphi \notin \mathsf{L}$, contradicting $/\varphi \in \eta_1(\mathsf{L})$.

(v)⇒(i). Suppose that $\mathfrak{F}$ is a rooted L-frame and $\mathfrak{G}$ is a stable image of $\mathfrak{F}$. Then $(\mathfrak{F}, \mathfrak{G})$ is an St-frame, and by Lemma 8.7(i), $(\mathfrak{F}, \mathfrak{G}) \models \eta_1(\mathsf{L})$. By (v), $(\mathfrak{F}, \mathfrak{G}) \models t(\varphi)$ for each $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ such that $/\varphi \in \eta_1(\mathsf{L})$. Therefore, $(\mathfrak{F}, \mathfrak{G}) \models t(\varphi)$ for each $\varphi \in \mathsf{L}$. Thus, $\mathfrak{G} \models \mathsf{L}$, and we conclude that L is a stable logic. □

Theorem 8.12. *Let* L *be a si-logic. Then* $\mathsf{Stab}_{\downarrow}(\mathsf{L}) = \zeta_2\eta_1(\mathsf{L})$.

Proof. By Lemma 8.6(ii),

$$\zeta_2\eta_1(\mathsf{L}) = \mathsf{Log}\left(\{\mathfrak{G} \mid \exists \mathfrak{F} : (\mathfrak{F}, \mathfrak{G}) \text{ is an St-frame and } \mathfrak{F} \models \mathsf{L}\}\right).$$

Let

$$\begin{aligned} K &= \{\mathfrak{G} \mid \exists \mathfrak{F} : (\mathfrak{F}, \mathfrak{G}) \text{ is an St-frame and } \mathfrak{F} \models \mathsf{L}\}, \\ K' &= \{\mathfrak{G} \mid \mathfrak{G} \text{ is a stable image of a rooted } \mathsf{L}\text{-frame } \mathfrak{F}\}. \end{aligned}$$

By Proposition 7.3(i), $\mathsf{Stab}_{\downarrow}(\mathsf{L}) = \mathsf{Log}(K')$. Clearly $K \subseteq K'$, so $\mathsf{Stab}_{\downarrow}(\mathsf{L}) = \mathsf{Log}(K') \subseteq \mathsf{Log}(K) = \zeta_2\eta_1(\mathsf{L})$. Suppose that $\varphi \notin \mathsf{Stab}_{\downarrow}(\mathsf{L})$. Then there is $\mathfrak{G} \in K'$ refuting φ. Therefore, there is an L-frame $\mathfrak{F}$ such that $\mathfrak{G}$ is a stable image of $\mathfrak{F}$. Applying [6, Lem. 3.6] yields a finite stable image $\mathfrak{G}'$ of $\mathfrak{G}$ refuting φ. Since $\mathfrak{G}'$ is finite, $(\mathfrak{F}, \mathfrak{G}')$ is an St-frame (because the topological condition of Definition 8.3 trivializes), so $\mathfrak{G}' \in K$. Thus, $\varphi \notin \zeta_2\eta_1(\mathsf{L})$. □

Remark 8.13.

(i) Let L be a si-logic and $\varphi \in \mathcal{L}_{\mathsf{IPC}}$. By Theorem 8.12, $\varphi \in \mathsf{Stab}_{\downarrow}(\mathsf{L})$ iff $t(\varphi) \in \mathcal{S}_{\mathsf{IS4}} + \{/\varphi \mid \varphi \in \mathsf{L}\}$. In particular, if $\mathcal{S}_{\mathsf{IS4}} + \{/\varphi \mid \varphi \in \mathsf{L}\}$ is decidable, then so is $\mathsf{Stab}_{\downarrow}(\mathsf{L})$.

(ii) In contrast to Theorem 8.12, if L is consistent, then $\zeta_1\eta_2(\mathsf{L}) = \mathsf{IPC}$. Indeed, suppose $\mathfrak{F}$ is a nonempty rooted Esakia frame. Let $\mathfrak{G}$ be the one-point frame. Then $(\mathfrak{F}, \mathfrak{G})$ is an St-frame. Since L is consistent, $\mathfrak{G}$ is an L-frame, so $(\mathfrak{F}, \mathfrak{G}) \models \eta_2(\mathsf{L})$ by Lemma 8.7(ii), and hence $\mathfrak{F} \models \zeta_1\eta_2(\mathsf{L})$ by Lemma 8.6(i). Thus, $\zeta_1\eta_2(\mathsf{L}) = \mathsf{IPC}$.

Remark 8.14. We recall [9] that a stable map $f : \mathfrak{F} \to \mathfrak{G}$ between Esakia frames is *cofinal stable* provided $\max{\uparrow} f(x) = f(\max{\uparrow} x)$, where $\max U$ is the set of maximal points of U. A si-logic L is *cofinal stable* provided its rooted frames are closed under cofinal stable images (that is, if $\mathfrak{F}$ is a rooted L-frame, then so is every cofinal stable image of $\mathfrak{F}$). It follows from [9] that cofinal stable images of an Esakia frame $\mathfrak{F}$ correspond to pseudocomplemented sublattices (that is, bounded sublattices preserving $\neg$) of the dual Heyting algebra A of $\mathfrak{F}$. Since being a pseudocomplemented sublattice is expressed by adding $\Box\neg\Box p \leftrightarrow \neg\Box p$ to $\mathcal{S}_{\mathsf{IS4}}$, the correspondence between stable logics and multiple-conclusion consequence relations extending $\mathcal{S}_{\mathsf{IS4}}$ discussed in this section extends to the correspondence between cofinal stable logics and multiple-conclusion consequence relations extending $\mathcal{S}_{\mathsf{IS4}} + /\Box\neg\Box p \leftrightarrow \neg\Box p$.

9. Conclusions

In this paper we aimed to highlight and strenghten the parallels between the classes of subframe and stable si-logics. The most notable known parallels between the two classes can be summarized as follows. For a subframe si-logic L the class of its Esakia frames is closed under (not necessarily generated) subframes, while a stable si-logic has the property that the class of its *rooted* Esakia frames is closed under stable (not necessarily p-morphic) images. Algebraically subframes and stable images correspond to nuclei and interior operators on Heyting algebras, respectively. Subframe logics admit an axiomatization via subframe formulas, whereas stable logics via stable formulas.

To this we add a characterization of the upward subframization via Zakharyaschev's canonical formulas, and a characterization of upward stabilization via stable canonical formulas. We also characterize subframe logics and subframizations via embedding si-logics into the extensions of the propositional lax logic PLL, and we characterize stable logics and stabilizations via embedding si-logics into the extensions of the intuitionistic $\mathsf{S4}$.

For an overview and precise references to the above correspondences we refer the reader to Table 1 in the introduction of the paper. Whereas there are many parallels between the classes of subframe logics and stable logics, we once more emphasize that there are also subtle differences in the behavior of the classes. These differences are mostly due to the fact that subframe logics have the property that the class of *all* their frames is closed under subframes, whereas for stable logics, only the *rooted* ones are closed under stable images. Accordingly, Table 1 should be read with appropriate care.

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