STABLE MODAL LOGICS

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Abstract. Stable logics are modal logics characterized by a class of frames closed under relation preserving images. These logics admit all filtrations. Since many basic modal systems such as K4 and S4 are not stable, we introduce the more general concept of an M-stable logic, where M is an arbitrary normal modal logic that admits some filtration. Of course, M can be chosen to be K4 or S4. We give several characterizations of M-stable logics. We prove that there are continuum many S4-stable logics and continuum many K4-stable logics between K4 and S4. We axiomatize K4-stable and S4-stable logics by means of stable formulas, and discuss the connection between S4-stable logics and stable superintuitionistic logics. We conclude the paper with many examples (and non-examples) of stable, K4-stable, and S4-stable logics and provide their axiomatization in terms of stable rules and formulas.

1. Introduction

One direction in the study of modal logic has been to identify classes of modal logics that are finitely axiomatizable, have the finite model property (fmp), and are decidable. To give a few examples:

(i) Bull [10] and Fine [17] proved that every extension of S4.3 has the fmp, is finitely axiomatizable, and hence decidable;
(ii) Segerberg [32] showed that every logic above K4 of finite depth has the fmp;
(iii) Fine [18] proved that every subframe logic above K4 has the fmp; and
(iv) Zakharyaschev [37] showed that the same holds for cofinal subframe logics above K4.

One of the most standard techniques for proving the fmp in modal logic is the method of filtration, which gives rise to yet another important class of modal logics enjoying the fmp. If a model $\mathfrak{N}$ is a filtration of a model $\mathfrak{M}$, then $\mathfrak{N}$ is an image of $\mathfrak{M}$ under a relation preserving map. We call such maps *stable maps*[1]. Thus, if a modal logic is characterized by a class of frames closed under images of stable maps, its fmp can be proved via filtration. Such logics were called stable in [3].

Examples of stable logics are the basic modal logic K, the logic T of all reflexive frames, the logic D of all serial frames, the epistemic logic S5, the logic KMT of the frames where each point sees a reflexive point, etc. Stable logics enjoy the following strong property: they admit all filtrations.

There are modal logics that are not stable but still admit particular filtrations. For example, the well-known modal systems K4 and S4 admit transitive filtrations, but they do not admit all filtrations, hence are not stable. This generates a problem of how to deal with logics that only admit some filtrations. As a solution, we weaken the notion of stability by parametrizing it over a ground logic. If a modal logic M admits a filtration, we define M-stable logics as logics that are stable over M (meaning that they are characterized by a class of frames closed under those stable

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[1] In model theory such maps are called homomorphisms, but we reserve the term "homomorphism" for operation preserving maps between modal algebras. Ghilardi [20] calls such maps continuous, but we reserve the term "continuous" for structure preserving maps between topological spaces. Thus, we follow [3] in calling such maps "stable."

images that validate M). A stable logic is then simply a K-stable logic. It is our goal to develop the theory of M-stable modal logics.

In many ways stable logics parallel subframe logics. The defining property of subframe logics is that their classes of frames are closed under subframes. Transitive subframe logics admit selective filtration, and hence have the fmp. They also admit a uniform axiomatization via the so-called subframe formulas [18]. Subframe formulas are obtained from Zakharyaschev's canonical formulas [36, 37] and subframe rules are obtained from Jeřábek's canonical rules [22] by dropping the extra parameter $\mathfrak{D}$ of closed domains. Similarly, stable rules are obtained from the stable canonical rules of [3] by dropping the extra parameter $\mathfrak{D}$ of stable closed domains. Consequently, every stable logic is axiomatizable by stable rules. Stable rules are best described by their semantic property. The stable rule of a finite frame $\mathfrak{F}$ is refuted on a frame $\mathfrak{G}$ iff $\mathfrak{F}$ is an image of $\mathfrak{G}$ via a stable map. Thus, if a logic L is axiomatized by the stable rules of finite frames $\{\mathfrak{F}_i \mid i \in I\}$, then it is characterized by the class of finite frames omitting (not having as stable images) every $\mathfrak{F}_i$. This gives a geometric intuition in analogy with that for subframe formulas (see, e.g., [38]).

Another analogy between (elementary) subframe logics and stable logics arises from the model-theoretic perspective. It is a well known result of Łoś and Tarski that a first-order sentence is preserved by submodels iff it is equivalent to a set of universal sentences (see, e.g., [14, Thm. 3.2.2]). Consequently, if a modal logic L is characterized by a class of frames that is definable by universal sentences, then L is a subframe logic. On the other hand, by Lyndon's theorem, a first-order sentence is preserved by surjective homomorphisms (stable maps) iff it is equivalent to a set of positive sentences (see, e.g., [14, Thm. 3.2.4]). As a result, if a modal logic L is characterized by a class of frames that is definable by positive sentences, then L is stable. We will use this characterization to show that many well-known logics above $\mathsf{K4}$ and $\mathsf{S4}$ are $\mathsf{K4}$-stable and $\mathsf{S4}$-stable, respectively.

There are also essential differences between non-transitive subframe logics and stable logics. Since the method of filtration works well in the non-transitive case, every stable logic has the fmp, which in general is not true for subframe logics (see, e.g., [13, Exa. 11.32]). There even exists a transfinite chain of Kripke-incomplete subframe logics [35]. Stable logics form a well-behaved class also from a proof-theoretic perspective as every stable logic enjoys the so-called bounded proof property (the bpp) [7]. Whether all subframe logics enjoy the bpp is still an open problem.

Our main results include several characterizations of M-stable modal logics. Since logics above $\mathsf{K4}$ and $\mathsf{S4}$ play an important role in modal logic, we pay special attention to $\mathsf{K4}$-stable and $\mathsf{S4}$-stable logics. For logics above $\mathsf{K4}$, we can turn every stable rule $\rho(\mathfrak{F})$ of a rooted frame $\mathfrak{F}$ into a stable formula $\gamma(\mathfrak{F})$, which behaves similarly to $\rho(\mathfrak{F})$ on rooted frames. As a consequence, every $\mathsf{K4}$-stable logic is axiomatizable by stable formulas. The converse is not true for logics above $\mathsf{K4}$, but we prove that it is true for logics above $\mathsf{S4}$; that is, every logic axomatized by $\mathsf{S4}$-stable formulas is $\mathsf{S4}$-stable.

We also investigate the connection between $\mathsf{S4}$-stable logics and stable superintuitionistic logics (si-logics) studied in [2, 4]. We prove that the intuitionstic fragment $\rho\mathsf{M}$ of every $\mathsf{S4}$-stable logic M is a stable si-logic. In fact, given an axiomatization of M via stable formulas of finite rooted $\mathsf{S4}$-frames $\{\mathfrak{F}_i \mid i \in I\}$, we can obtain an axiomatization of $\rho\mathsf{M}$ by the stable intuitionistic formulas of the intuitionistic frames $\{\overline{\mathfrak{F}_\mathrm{i}} \mid i \in I\}$, where $\overline{\mathfrak{F}_i}$ is obtained from $\mathfrak{F}_i$ by unfolding each cluster into a chain. Conversely, stability is preserved by the least modal companion of a si-logic, and if the stable formulas of $\{\mathfrak{G}_\mathrm{i} \mid i \in I\}$ axiomatize a stable si-logic, then the $\mathsf{S4}$-stable formulas of $\{\mathfrak{G}_\mathrm{i} \mid i \in I\}$ axiomatize its least modal companion. However, stability is not preserved by the greatest modal companion of a si-logic. This is in contrast with subframe logics, where both the least and greatest companions of a subframe si-logic are subframe logics, and the intuitionistic fragment of every subframe logic above $\mathsf{S4}$ is a subframe si-logic (see, e.g., [13, Sec. 9.6]). We explicitly use these connections between $\mathsf{S4}$-stable logics and stable si-logics to give concrete axiomatizations of many well-known $\mathsf{K4}$-stable and $\mathsf{S4}$-stable logics via stable formulas.

The paper is organized as follows. In the next section we recall the necessary background and central notions from [3]. In Section 3 we lay out the general theory of M-stable logics and show that there are continuum many stable logics. In Section 4, we turn to more specific cases and discuss M-stable logics, where M is a normal extension of K4. In Section 5 we discuss the connection between S4-stable logics and stable si-logics. In the final section we present many examples (and non-examples) of stable, K4-stable, and S4-stable logics and provide their axiomatizations in terms of stable rules and formulas.

2. Preliminaries

We assume the reader is familiar with modal logic. We use [13, 23, 8, 34] as our main references for modal logic, [11] for universal algebra, [30, 24] for modal consequence relations, and [22, 3] for multi-conclusion modal consequence relations.

We recall that a *modal algebra* is a pair $\mathfrak{A} = (A, \Diamond)$ where A is a Boolean algebra and $\Diamond$ is a unary function on A preserving all finite joins. We also recall that a *modal space* (aka a descriptive frame) is a pair $\mathfrak{X} = (X, R)$ where X is a Stone space (compact Hausdorff zero-dimensional space) and R is a binary relation on X satisfying $R[x] := \{y \in X \mid xRy\}$ is closed for every $x \in X$ and $R^{-1}[U] := \{x \in X \mid xRy \text{ for some } y \in U\}$ is clopen for every clopen U of X. If $\mathfrak{X}$ is a finite modal space, then the topology is discrete, and we view $\mathfrak{X}$ as a finite Kripke frame.

We will often use the duality between modal algebras and modal spaces. The dual modal space of a modal algebra $\mathfrak{A} = (A, \Diamond)$ is $\mathfrak{X} = (X, R)$, where X is the Stone space of A (that is, the points of X are the ultrafilters of A and the topology on X is generated by the basic open sets $\varphi(a) = \{x \in X \mid a \in x\}$ for all $a \in A$) and xRy iff $(\forall a \in A)(a \in y \Rightarrow \Diamond a \in x)$. If $\mathfrak{X} = (X, R)$ is a modal space, then its dual modal algebra is $\mathfrak{A} = (A, \Diamond)$, where A is the Boolean algebra of clopen subsets of X and $\Diamond a = R^{-1}[a]$ for all $a \in A$.

Morphisms between modal algebras are modal algebra homomorphisms, morphisms between modal spaces are continuous p-morphisms, and the duality extends to morphisms by taking preimages of the morphisms in question.

We recall (see, e.g., [23, p. 174]) that an element a of a modal algebra $\mathfrak{A}$ is an *opremum* if $a \neq 1$ and for each $b \neq 1$ there is $n \in \omega$ with $\blacksquare_n b \leq a$, where $\Box^0 b = b$, $\Box^{n+1} b = \Box\Box^n b$, and $\blacksquare_n b = \bigwedge_{k \leq n} \Box^k b$. A modal algebra $\mathfrak{A}$ is subdirectly irreducible iff it has an opremum.

An element x of a modal space $\mathfrak{X} = (X, R)$ is a *root* if $X = R^\omega[x]$ and a *topo-root* if $R^\omega[x]$ is dense in X, where $R^0[x] = \{x\}$, $R^{n+1}[x] = \{y \in X \mid zRy \text{ for some } z \in R^n[x]\}$, and $R^\omega[x] = \bigcup_{n \in \omega} R^n[x]$. We call $\mathfrak{X}$ *rooted* if it has a root, and *topo-rooted* if the set of topo-roots is not co-dense (the interior is nonempty). By [33, Thm. 2], a modal algebra $\mathfrak{A}$ is subdirectly irreducible iff its dual modal space $\mathfrak{X}$ is topo-rooted. Therefore, if $\mathfrak{A}$ is finite, then $\mathfrak{A}$ is subdirectly irreducible iff $\mathfrak{X}$ is rooted [31, Thm. 3.1].

In this paper we will often be interested in maps between modal algebras that are not full modal algebra homomorphisms, but preserve $\Diamond$ only "half-way." Such maps were studied in [5] under the name of semi-homomorphisms and in [20] under the name of continuous morphisms. We follow [3] in calling them stable homomorphisms.

Definition 2.1. Let $\mathfrak{A} = (A, \Diamond)$ and $\mathfrak{B} = (B, \Diamond)$ be modal algebras.

1. A Boolean homomorphism $h : A \to B$ is *stable* provided $\Diamond h(a) \leq h(\Diamond a)$ for all $a \in A$.
2. We call $\mathfrak{A}$ a *stable subalgebra* of $\mathfrak{B}$ if A is a Boolean subalgebra of B and the inclusion $A \hookrightarrow B$ is a stable homomorphism.

Dually stable homomorphisms correspond to continuous relation preserving maps (see [3, Lem. 3.3]).

Definition 2.2.

(1) Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, R)$ be modal spaces. A map $f : X \to Y$ is called *stable* provided it is continuous and xRy implies $f(x)Rf(y)$.
(2) We call $\mathfrak{Y}$ a *stable image* of $\mathfrak{X}$ if there is an onto stable map $f : X \to Y$.

A *multi-conclusion rule* is an expression of the form Γ/Δ, where Γ and Δ are finite sets of formulas. A modal algebra $\mathfrak{A} = (A, \Diamond)$ *validates* a rule Γ/Δ (in symbols: $\mathfrak{A} \models \Gamma/\Delta$) if for every valuation $V : \mathsf{Prop} \to A$, from $V(\gamma) = 1$ for all $\gamma \in \Gamma$ it follows that $V(\delta) = 1$ some $\delta \in \Delta$. Just as formulas correspond to equations, multi-conclusion rules correspond to universal clauses, namely the rule Γ/Δ corresponds to the universal clause $\forall\bar{x} \bigwedge_{\gamma\in\Gamma} \gamma(\bar{x}) \to \bigvee_{\delta\in\Delta} \delta(\bar{x})$, where $\bar{x}$ is a set of variables containing a variable for each propositional letter used in the formulas from Γ and Δ.

We recall the *stable rules* of [3, Sec. 7]. Let $\mathfrak{A} = (A, \Diamond)$ be a finite modal algebra. For every $a \in A$, let p_a be a propositional letter such that $a \neq b$ implies $p_a \neq p_b$. The *stable (multi-conclusion) rule* $\rho(\mathfrak{A})$ is defined as Γ/Δ, where

$$\begin{aligned}\Gamma = &\{p_{a\vee b} \leftrightarrow p_a \vee p_b \mid a, b \in A\} \cup \\ &\{p_{\neg a} \leftrightarrow \neg p_a \mid a \in A\} \cup \\ &\{\Diamond p_a \to p_{\Diamond a} \mid a \in A\}\end{aligned}$$

and

$$\Delta = \{p_a \mid a \in A, a \neq 1\}.$$

Stable rules generalize the Jankov rules of [22], which in model theory correspond to diagrams of finite modal algebras [14, p. 68]. Recall that satisfying the diagram of a structure is equivalent to the structure being isomorphically embeddable [14, Prop. 2.1.8]. On the other hand, refutation of the stable rule of a finite modal algebra $\mathfrak{A}$ is equivalent to $\mathfrak{A}$ being stably embeddable:

Proposition 2.3. [3, Prop. 7.1] *Let $\mathfrak{A}, \mathfrak{B}$ be modal algebras with $\mathfrak{A}$ finite. Then $\mathfrak{B} \not\models \rho(\mathfrak{A})$ iff there is a stable embedding $h : \mathfrak{A} \rightarrowtail \mathfrak{B}$.*

Recall that varieties are classes of algebras closed under the operations of taking homomorphic images $\mathbf{H}$, subalgebras $\mathbf{S}$, and products $\mathbf{P}$. There is a one-to-one correspondence between normal modal logics and varieties of modal algebras. If Γ is set of formulas, then we denote by $\mathcal{V}(\Gamma)$ the variety corresponding to the logic axiomatized by Γ. Just as formulas axiomatize varieties of algebras, multi-conclusion rules axiomatize *universal classes* of algebras. These are classes of algebras closed under the operations of taking isomorphic copies $\mathbf{I}$, subalgebras $\mathbf{S}$, and ultraproducts $\mathbf{P_U}$. If $\mathcal{K}$ is a class of modal algebras, then we denote by $\mathcal{V}(\mathcal{K})$ the variety generated by $\mathcal{K}$, and by $\mathcal{U}(\mathcal{K})$ the universal class generated by $\mathcal{K}$. It is well known that $\mathcal{V}(\mathcal{K}) = \mathbf{HSP}(\mathcal{K})$ and $\mathcal{U}(\mathcal{K}) = \mathbf{ISP_U}(\mathcal{K})$. Note that $\mathcal{U}(\mathcal{K})$ is contained in $\mathcal{V}(\mathcal{K})$, but in general the inclusion is proper.

Universal classes of modal algebras correspond to normal modal multi-conclusion consequence relations. A *normal modal multi-conclusion consequence relation* is a set $\mathcal{S}$ of rules such that

$$\begin{aligned}&\varphi/\varphi \in \mathcal{S};\\ &\varphi, \varphi \to \psi/\psi \in \mathcal{S};\\ &\varphi/\Box\varphi \in \mathcal{S};\\ &/\varphi \in \mathcal{S} \text{ for each theorem } \varphi \in \mathsf{K};\\ &\text{if } \Gamma/\Delta \in \mathcal{S}, \text{ then } \Gamma, \Gamma'/\Delta, \Delta' \in \mathcal{S};\\ &\text{if } \Gamma/\Delta, \varphi \in \mathcal{S} \text{ and } \Gamma, \varphi/\Delta \in \mathcal{S}, \text{ then } \Gamma/\Delta \in \mathcal{S};\\ &\text{if } \Gamma/\Delta \in \mathcal{S} \text{ and } s \text{ is a substitution, then } s(\Gamma)/s(\Delta) \in \mathcal{S}.\end{aligned}$$

If $\mathcal{S}$ is a normal modal multi-conclusion consequence relation, then we denote by $\mathcal{U}(\mathcal{S})$ the universal class corresponding to $\mathcal{S}$. As shown in [22, Thm. 2.2], $\mathcal{S}$ is complete with respect to $\mathcal{U}(\mathcal{S})$. If $\mathcal{K}$ is a class of modal algebras, then $\mathcal{S}_{\mathcal{K}} = \{\Gamma/\Delta \mid \mathfrak{A} \models \Gamma/\Delta \text{ for every } \mathfrak{A} \in \mathcal{K}\}$ is a normal modal multi-conclusion consequence relation. If $\mathcal{R}$ is a set of rules, then we denote by $\mathsf{CR}(\mathcal{R})$ the least normal modal multi-conclusion consequence relation containing $\mathcal{R}$. If $\mathcal{S} = \mathsf{CR}(\mathcal{R})$, then we say that $\mathcal{R}$ *axiomatizes* $\mathcal{S}$.

For a normal modal logic L, we denote by $\mathcal{S}_\mathsf{L}$ the normal modal multi-conclusion consequence relation axiomatized by $\{/\varphi \mid \varphi \in \mathsf{L}\}$. A set of rules $\mathcal{R}$ gives rise to the logic $\mathsf{Log}(\mathcal{R}) = \{\varphi \mid /\varphi \in \mathsf{CR}(\mathcal{R})\}$. If $\mathsf{L} = \mathsf{Log}(\mathcal{R})$, then we say that L *is axiomatized by* $\mathcal{R}$. More generally, if $\mathcal{R}$ is a set of rules and M is a normal modal logic, then we say that the logic $\mathsf{L} = \mathsf{M} + \{\varphi \mid /\varphi \in \mathsf{CR}(\mathcal{R})\}$ is *axiomatized by* $\mathcal{R}$ *over* M. We have $\mathcal{V}(\mathsf{Log}(\mathcal{R})) = \mathcal{V}(\mathcal{U}(\mathsf{CR}(\mathcal{R})))$ and $\mathcal{V}(\mathsf{M} + \{\varphi \mid /\varphi \in \mathsf{CR}(\mathcal{R})\}) = \mathcal{V}(\mathcal{U}(\mathcal{S}_\mathsf{M} + \mathcal{R}))$.

3. M-stable modal logics

Stable modal logics are modal logics axiomatized by stable rules [3, Sec. 7]. As we pointed out in the introduction, they admit all filtrations (where admitting filtration is meant in the weak sense, see Definition 3.1(2)). Many logics that admit filtration do not admit all filtrations—e.g., $\mathsf{K4}$ only admits filtrations that produce transitive frames—and such logics are not stable. We therefore relativize the concept of a stable logic to that of an M-stable logic, where M is a normal modal logic admitting filtration (in the strong sense, see Definition 3.1(3)). Thus, M-stable logics are logics above M that admit all M-filtrations (in the weak sense). To facilitate the study of M-stable logics, we give several equivalent descriptions of M-stability. We also collect several observations on how M-stable logics lie in the lattice of all modal logics. We conclude the section by showing that there are continuum many (weakly transitive) stable logics.

We recall that an algebraic account of filtrations in modal logic was first given in [27, 28] (see also [25, 26]). For a more recent discussion of filtrations algebraically we refer to [20, 15, 3]. Here we follow the construction discussed in [3, Sec. 4].

Definition 3.1.

(1) Suppose $\mathfrak{A} = (A, \Diamond)$ is a modal algebra, V is a valuation on $\mathfrak{A}$, and Σ is a finite set of formulas closed under subformulas. Let A' be the Boolean subalgebra of A generated by $V(\Sigma)$. Then A' is finite because Σ is finite. Set $D = \{V(\varphi) \mid \Diamond\varphi \in \Sigma\}$. Let $\Diamond'$ be a modal operator on A' and V' be a valuation on $\mathfrak{A}' = (A', \Diamond')$ satisfying
   - The inclusion $\mathfrak{A}' \hookrightarrow \mathfrak{A}$ is a stable homomorphism;
   - $V'(p) = V(p)$ for all propositional letters $p \in \Sigma$;
   - $\Diamond' a = \Diamond a$ for all $a \in D$.

   Then $(\mathfrak{A}', V')$ is called a *filtration of* $(\mathfrak{A}, V)$ *through* Σ.

(2) A normal modal logic M *admits filtration (in the weak sense)* if for every non-theorem φ of M, there is a counter-model $(\mathfrak{A}, V)$ of φ and a filtration $(\mathfrak{A}', V')$ of $(\mathfrak{A}, V)$ through some finite set Σ of formulas containing φ and closed under subformulas such that $\mathfrak{A}'$ is an M-algebra.

(3) A normal modal logic M *admits filtration (in the strong sense)* if for every M-algebra $\mathfrak{A}$, every valuation V on $\mathfrak{A}$, and every finite set Σ of formulas closed under subformulas, there is a filtration $(\mathfrak{A}', V')$ of $(\mathfrak{A}, V)$ through Σ such that $\mathfrak{A}'$ is an M-algebra.

Our definition of admitting filtration in the weak sense follows [13, p. 142], and admitting filtration in the strong sense follows [20, p. 201]. Clearly the latter is stronger than the former, but the former is sufficient for proving the fmp. Indeed, by the Filtration Theorem (see, e.g., [13, Thm. 5.23]), if $(\mathfrak{A}', V')$ is a filtration of $(\mathfrak{A}, V)$ through some Σ, then $V(\varphi) = V'(\varphi)$ for all $\varphi \in \Sigma$. It follows that if a normal modal logic M admits filtration in the weak sense, then M has the fmp. On the other hand, admitting filtration in the strong sense ensures the finite embeddability property (see Remark 3.4).

Definition 3.2. Let M be a normal modal logic and let L be a normal extension of M.

(1) Suppose $\mathcal{K}$ and $\mathcal{V}$ are two classes of modal algebras with $\mathcal{K} \subseteq \mathcal{V}$. We say that $\mathcal{K}$ is $\mathcal{V}$*-stable* provided for $\mathfrak{A}, \mathfrak{B} \in \mathcal{V}$, if $\mathfrak{B} \in \mathcal{K}$ and there is a stable embedding $\mathfrak{A} \rightarrowtail \mathfrak{B}$, then $\mathfrak{A} \in \mathcal{K}$.

(2) Let $\mathcal{K}$ be a class of M-algebras. We say that $\mathcal{K}$ is M-*stable* if $\mathcal{K}$ is $\mathcal{V}(\mathsf{M})$-stable. We say that $\mathcal{K}$ is *finitely* M-*stable* provided for every finite M-algebra $\mathfrak{A}$ and any $\mathfrak{B} \in \mathcal{K}$, whenever there is a stable embedding $\mathfrak{A} \rightarrowtail \mathfrak{B}$, then $\mathfrak{A} \in \mathcal{K}$.

(3) We say that L is M-*stable* if the variety $\mathcal{V}(\mathsf{L})$ is generated by an M-stable class.

Proposition 3.3. *If* M *is a normal modal logic that admits filtration in the strong sense, then every* M*-stable logic admits filtration in the weak sense, and hence has the fmp.*

Proof. Let L be M-stable. Then $\mathcal{V}(\mathsf{L})$ is generated by an M-stable class $\mathcal{K}$. If $\mathsf{L} \nvdash \varphi$, then there is $\mathfrak{A} \in \mathcal{K}$ and a valuation V on $\mathfrak{A}$ such that $\mathfrak{A} \not\models \varphi$. Let $\mathsf{Sub}(\varphi)$ be the set of subformulas of φ. Since M admits filtration in the strong sense, there is a finite M-algebra $\mathfrak{A}'$ and a valuation V' on $\mathfrak{A}'$ such that $(\mathfrak{A}', V')$ is a filtration of $(\mathfrak{A}, V)$ through $\mathsf{Sub}(\varphi)$. Because $\mathcal{K}$ is M-stable, $\mathfrak{A}' \in \mathcal{K}$. Thus, L admits filtration in the weak sense, and hence L has the fmp. □

Roughly speaking, whenever L is M-stable and M admits filtration in the strong sense, the fmp of L can be shown with the "same proof" as the fmp for M.

Remark 3.4. We briefly discuss connection between M-stability and the notion of the *finite embeddability property* (fep for short) [19, Sec. 6.5]. The fep is equivalent to the finite model property for quasi-equations [16, 9], so it is a slightly stronger notion than the fmp. If a normal modal logic M admits filtration in the strong sense, then the corresponding variety of modal algebras has the fep. Every M-stable class of algebras has the fep, but in general we do not know whether the variety $\mathcal{V}(\mathsf{L})$ corresponding to an M-stable logic L has the fep since by Definition 3.2(3), $\mathcal{V}(\mathsf{L})$ is only generated by an M-stable class and may itself not be an M-stable class. However, if L is a normal extension of $\mathsf{K4}$, then it follows from [22, Lem. 3.23] that the notions of fmp and fmp for quasi-equations coincide. As the fmp for quasi-equations is equivalent to the fep, we conclude that the notions of fmp and fep coincide for normal extensions of $\mathsf{K4}$. Therefore, Proposition 3.3 yields that if L is $\mathsf{K4}$-stable, then $\mathcal{V}(\mathsf{L})$ has the fep.

In what follows, we will mainly be interested in admitting filtration in the strong sense, and will simply refer to this condition as admitting filtration.

Lemma 3.5. *Let* M *be a normal modal logic that admits filtration and let* $\mathcal{K}$ *be a finitely* M*-stable class of* M*-algebras.*

(1) $\mathcal{S}_{\mathcal{K}}$ *is axiomatized over* $\mathcal{S}_{\mathsf{M}}$ *by the stable rules of finite* M*-algebras.*

(2) $\mathcal{U}(\mathcal{K}) = \mathcal{U}(\mathcal{K}_{\mathsf{fin}})$*, where* $\mathcal{K}_{\mathsf{fin}}$ *is the class of finite members of* $\mathcal{K}$*.*

Proof. (1). Suppose that $\mathcal{K}$ is finitely M-stable. Let $\mathcal{A}$ be the set of finite non-isomorphic M-algebras that do not belong to $\mathcal{K}$ and let $\Psi = \{\rho(\mathfrak{A}) \mid \mathfrak{A} \in \mathcal{A}\}$. We show that $\mathcal{S}_{\mathcal{K}}$ is axiomatized over $\mathcal{S}_{\mathsf{M}}$ by Ψ. For this it is sufficient to show that $\mathcal{U}(\mathcal{K})$ is exactly the class of M-algebras satisfying Ψ. First we show that each member of $\mathcal{K}$ satisfies Ψ. If there are $\mathfrak{B} \in \mathcal{K}$ and $\mathfrak{A} \in \mathcal{A}$ such that $\mathfrak{B} \not\models \rho(\mathfrak{A})$, then by Proposition 2.3, there is a stable embedding $\mathfrak{A} \rightarrowtail \mathfrak{B}$. Since $\mathcal{K}$ is finitely M-stable and $\mathfrak{A}$ is finite, $\mathfrak{A} \in \mathcal{K}$, a contradiction. Because $\mathcal{U}(\mathcal{K})$ is generated by $\mathcal{K}$, it follows that each member of $\mathcal{U}(\mathcal{K})$ satisfies Ψ. Conversely, suppose that an M-algebra $\mathfrak{B}$ satisfies $\rho(\mathfrak{A})$ for each $\mathfrak{A} \in \mathcal{A}$. If $\mathfrak{B} \notin \mathcal{U}(\mathcal{K})$, then there is a multi-conclusion rule Γ/Δ such that $\mathcal{K} \models \Gamma/\Delta$ but $\mathfrak{B} \not\models \Gamma/\Delta$. Let $\mathfrak{B}'$ be an M-filtration of $\mathfrak{B}$ through $\mathsf{Sub}(\Gamma \cup \Delta)$ with $\mathfrak{B}' \not\models \Gamma/\Delta$. Since $\mathfrak{B}'$ is a stable subalgebra of $\mathfrak{B}$, we have $\mathfrak{B} \not\models \rho(\mathfrak{B}')$ by Proposition 2.3. As $\mathfrak{B}$ satisfies $\rho(\mathfrak{A})$ for each $\mathfrak{A} \in \mathcal{A}$, we see that $\mathfrak{B}' \in \mathcal{K}$, so $\mathfrak{B}' \in \mathcal{U}(\mathcal{K})$. But this contradicts to $\mathfrak{B}' \not\models \Gamma/\Delta$. Therefore, $\mathfrak{B} \in \mathcal{U}(\mathcal{K})$.

(2). The inclusion $\mathcal{U}(\mathcal{K}_{\mathsf{fin}}) \subseteq \mathcal{U}(\mathcal{K})$ is obvious. To see the reverse inclusion, let Γ/Δ be a multi-conclusion rule that is refuted in $\mathcal{U}(\mathcal{K})$. Then there is $\mathfrak{A} \in \mathcal{K}$ that refutes Γ/Δ. Let $\mathfrak{A}'$ be an M-filtration of $\mathfrak{A}$ through $\mathsf{Sub}(\Gamma \cup \Delta)$. Then $\mathfrak{A}'$ refutes Γ/Δ and $\mathfrak{A}' \in \mathcal{K}$ since $\mathfrak{A}'$ is finite and $\mathcal{K}$ is finitely M-stable. Thus, $\mathfrak{A}' \in \mathcal{K}_{\mathsf{fin}}$, and so $\mathcal{U}(\mathcal{K}_{\mathsf{fin}})$ refutes Γ/Δ. □

Definition 3.6. Let $\mathfrak{F} = (W, R)$ be a finite Kripke frame. We call $r \in W$ a *strong root of* $\mathfrak{F}$ if rRw for all $w \in W$.

Note that if r is a strong root, then it is reflexive. In algebraic terms, a strong root corresponds to an atom a of a finite modal algebra $\mathfrak{A}$ such that $a \leq \Diamond b$ for all $0 \neq b \in \mathfrak{A}$.

Definition 3.7.

(1) Let $\mathfrak{F} = (W, R)$ be a finite Kripke frame and let $r \notin W$. We set $\mathfrak{F}^r = (W', R')$ where $W' = W \cup \{r\}$ and $R' = R \cup \{(r, w) \mid w \in W'\}$. Figuratively speaking, $\mathfrak{F}^r$ is obtained by adding a strong root beneath $\mathfrak{F}$.

(2) We say that a normal modal logic M has the $(*)$-*property* if for each finite M-frame $\mathfrak{F}$ we have that $\mathfrak{F}^r$ is also an M-frame.

If $\mathfrak{A} = (A, \Diamond)$ is the dual algebra of $\mathfrak{F}$, then the dual algebra of $\mathfrak{F}^r$ is the algebra $\mathfrak{A}' = (A', \Diamond')$, where A' is the Boolean algebra generated by A and a fresh atom a with $\Diamond' a = a$ and $\Diamond' b = \Diamond b \vee a$ for every atom $b \in A$. Consequently, a normal modal logic M has the $(*)$-property if for every finite M-algebra $\mathfrak{A} = (A, \Diamond)$, the algebra $\mathfrak{A}' = (A', \Diamond')$ is an M-algebra. Examples of normal modal logics satisfying the $(*)$-property are K, D, T, $\mathsf{K4}$, and $\mathsf{S4}$. On the other hand, the logics KB, $\mathsf{S5}$, and GL do not satisfy the $(*)$-property.

For a class $\mathcal{K}$, we let $\mathcal{K}_{\mathsf{si}}$ be the class of subdirectly irreducible members of $\mathcal{K}$.

Theorem 3.8. *Suppose* M *is a normal modal logic that admits filtration and* L *is a normal extension of* M. *The following are equivalent.*

(1) L *is* M*-stable.*

(2) $\mathcal{V}(\mathsf{L})$ *is generated by a finitely* M*-stable class.*

(3) $\mathcal{V}(\mathsf{L})$ *is generated by an* M*-stable class of finite modal algebras.*

(4) L *is axiomatizable over* M *by stable rules of finite* M*-algebras.*

(5) $\mathcal{V}(\mathsf{L})$ *is generated by an* M*-stable universal class of modal algebras.*

Moreover, if M *has the* $(*)$*-property, then the above conditions are equivalent to the following ones:*

(6) $\mathcal{V}(\mathsf{L})$ *is generated by an* M*-stable class of finite subdirectly irreducible algebras.*

(7) $\mathcal{V}(\mathsf{L})$ *is generated by a* $\mathcal{V}(\mathsf{M})_{\mathsf{si}}$*-stable class.*

(8) $\mathcal{V}(\mathsf{L})$ *is generated by a finitely* M*-stable class of subdirectly irreducible algebras.*

Proof. The proof is similar to [3, Thm. 7.6]. The implication (1) $\Rightarrow$ (2) is trivial since every M-stable class is finitely M-stable. For the implication (2) $\Rightarrow$ (3), suppose that $\mathcal{V}(\mathsf{L})$ is generated by a finitely M-stable class $\mathcal{K}$. By Lemma 3.5(2), $\mathcal{K}$ and $\mathcal{K}_{\mathsf{fin}}$ generate the same universal class, and hence they generate the same variety. Thus, $\mathcal{V}(\mathsf{L})$ is generated by the M-stable class $\mathcal{K}_{\mathsf{fin}}$ of finite modal algebras. The implication (3) $\Rightarrow$ (2) is obvious. For the implication (2) $\Rightarrow$ (4), suppose that $\mathcal{V}(\mathsf{L})$ is generated by a finitely M-stable class $\mathcal{K}$. By Lemma 3.5(1), $\mathcal{S}_{\mathcal{K}}$ is axiomatized over $\mathcal{S}_{\mathsf{M}}$ by the stable rules of finite M-algebras. Since the variety $\mathcal{V}(\mathsf{L})$ is generated by $\mathcal{K}$, the same rules axiomatize L over M. For the implication (4) $\Rightarrow$ (5), suppose that L is axiomatized over M by the set Ψ of stable rules. As validity of stable rules is preserved by stable embeddings, the universal class $\mathcal{U}(\mathcal{S}_{\mathsf{M}} + \Psi)$ is M-stable. Since $\mathsf{L} = \mathsf{M} + \{\varphi \mid /\varphi \in \mathsf{CR}(\mathcal{R})\}$, we have $\mathcal{V}(\mathsf{L}) = \mathcal{V}(\mathsf{M} + \{\varphi \mid /\varphi \in \mathsf{CR}(\mathcal{R})\}) = \mathcal{V}(\mathcal{U}(\mathcal{S}_{\mathsf{M}} + \Psi))$. Because $\mathcal{U}(\mathcal{S}_{\mathsf{M}} + \Psi)$ is an M-stable universal class, we conclude that $\mathcal{V}(\mathsf{L})$ is generated by an M-stable universal class. The implication (5) $\Rightarrow$ (1) is obvious.

Finally, suppose that M has the $(*)$-property. Obviously (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (2). Therefore, it is sufficient to prove that (3) implies (6). Suppose $\mathcal{K}$ is a stable class of finite M-algebras that generates $\mathcal{V}(\mathsf{L})$. It is sufficient to show that $\mathcal{K}_{\mathsf{si}}$ generates $\mathcal{V}(\mathsf{L})$, and for this it is sufficient to show that $\mathcal{K}$ is contained in the variety generated by $\mathcal{K}_{\mathsf{si}}$. Suppose $\mathfrak{A} \in \mathcal{K}$. If $\mathfrak{A}$ is subdirectly irreducible, then $\mathfrak{A} \in \mathcal{K}_{\mathsf{si}}$, and there is nothing to prove. Otherwise $\mathfrak{A}$ is a subdirect product of its subdirectly irreducible homomorphic images. Therefore, to conclude that $\mathfrak{A}$ is in the variety generated by $\mathcal{K}_{\mathsf{si}}$, it is sufficient to see that every subdirectly irreducible homomorphic image $\mathfrak{B}$ of $\mathfrak{A}$ belongs to this variety. Let $\mathfrak{B}$ be a subdirectly irreducible homomorphic image of $\mathfrak{A}$. Since $\mathfrak{A}$ is finite, so is $\mathfrak{B}$. Let $\mathfrak{X} = (X, R)$ be the dual of $\mathfrak{A}$ and $\mathfrak{Y} = (Y, R)$ the dual of $\mathfrak{B}$. Since $\mathfrak{B}$ is finite and subdirectly irreducible, $\mathfrak{Y}$ is a finite rooted M-frame. Consider $\mathfrak{Y}^r = (Y', R')$ (see Definition 3.7(1)). Because

M has the $(*)$-property, $\mathfrak{Y}^r$ is an M-frame. Since $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$, $\mathfrak{Y}$ is a generated subframe of $\mathfrak{X}$. As $\mathfrak{A}$ is not subdirectly irreducible but $\mathfrak{B}$ is, $\mathfrak{X}$ is not rooted but $\mathfrak{Y}$ is. So $\mathfrak{Y} \neq \mathfrak{X}$. Define $f : X \to Y'$ by mapping the points of Y to themselves and the remaining points of X to r. It is easy to see that f is an onto stable map. Therefore, there is a stable embedding from the dual algebra $\mathfrak{B}'$ of $\mathfrak{Y}^r$ to $\mathfrak{A}$. Since $\mathfrak{A} \in \mathcal{K}$ and $\mathcal{K}$ is M-stable, we conclude that $\mathfrak{B}' \in \mathcal{K}$. As $\mathfrak{Y}^r$ is finite and rooted, $\mathfrak{B}'$ is subdirectly irreducible, and hence $\mathfrak{B}' \in \mathcal{K}_{\mathsf{si}}$. Now, $\mathfrak{Y}$ is a generated subframe of $\mathfrak{Y}^r$, so $\mathfrak{B}$ is a homomorphic image of $\mathfrak{B}'$, and hence $\mathfrak{B}$ belongs to the variety generated by $\mathcal{K}_{\mathsf{si}}$, as desired. □

Remark 3.9. The definition of a normal modal multi-conclusion consequence relation $\mathcal{M}$ admitting filtration, the proof that such $\mathcal{M}$ has the fmp, the definition of $\mathcal{M}$-stable multi-conclusion consequence relations and an analogue of Theorem 3.8 are proved similarly, so we skip the details. $\mathcal{M}$-stable multi-conclusion consequence relations generalize the stable multi-conclusion consequence relations studied in [3].

For a normal modal logic M, we denote by NExtM the sublattice of the lattice of all normal modal logics consisting of normal extensions of M.

Proposition 3.10. *Suppose* $\mathsf{M}, \mathsf{L}, \mathsf{N}$ *are normal modal logics with* $\mathsf{M} \subseteq \mathsf{L} \subseteq \mathsf{N}$.

(1) *If* N *is* M*-stable, then* N *is* L*-stable.*
(2) *The converse of* (1) *is not true in general, i.e. if* N *is* L*-stable, then* N *may not be* M*-stable.*
(3) *If* $\mathcal{V}(\mathsf{L})$ *is a* $\mathcal{V}(\mathsf{M})$*-stable class, then* N *is* L*-stable iff* N *is* M*-stable.*
(4) *The* M*-stable logics form a* $\bigwedge$*-subsemilattice of* NExtM.

Proof. (1). Since N is M-stable, $\mathcal{V}(\mathsf{N})$ is generated by an M-stable class $\mathcal{K}$. As $\mathcal{K}$ is M-stable, it is obviously L-stable. Thus, N is L-stable.

(2). We will see in Section 6 that taking $\mathsf{M} = \mathsf{K}$, $\mathsf{L} = \mathsf{K4}$, and $\mathsf{N} = \mathsf{S4}$ provides the desired example.

(3). One implication follows from (1). For the other, suppose that N is L-stable. Then $\mathcal{V}(\mathsf{N})$ is generated by an L-stable class $\mathcal{K}$. Since $\mathcal{V}(\mathsf{L})$ is $\mathcal{V}(\mathsf{M})$-stable, $\mathcal{K}$ is also $\mathcal{V}(\mathsf{M})$-stable. Therefore, N is M-stable.

(4). Suppose $\{\mathsf{L}_i \mid i \in I\}$ is a family of M-stable logics. Then every L_i is generated by some M-stable class $\mathcal{K}_i$. Clearly the class $\bigcup\{\mathcal{K}_i \mid i \in I\}$ is also M-stable, and generates $\mathcal{V}(\bigwedge\{\mathsf{L}_i \mid i \in I\})$. □

Problem 1. Suppose M is a normal modal logic that admits filtration. Do the M-stable logics form a complete sublattice of NExtM? In particular, do the stable logics form a complete sublattice of NExtK?

Remark 3.11. For a normal modal multi-conclusion consequence relation $\mathcal{M}$, let $\mathsf{NExt}\mathcal{M}$ be the sublattice of the lattice of all normal modal multi-conclusion consequence relations consisting of normal extensions of $\mathcal{M}$. If $\mathcal{M}$ admits filtration, then the $\mathcal{M}$-stable multi-conclusion consequence relations do form a complete sublattice of $\mathsf{NExt}\mathcal{M}$. To see this, let $\{\mathcal{S}_i \mid i \in I\}$ be a family of $\mathcal{M}$-stable multi-conclusion consequence relations. Then each $\mathcal{S}_i$ is axiomatized over $\mathcal{M}$ by a set Σ_i of stable rules of finite $\mathcal{M}$-algebras. But then $\bigvee\{\mathcal{S}_i \mid i \in I\}$ is axiomatized by $\bigcup\{\Sigma_i \mid i \in I\}$, and hence $\bigvee\{\mathcal{S}_i \mid i \in I\}$ is $\mathcal{M}$-stable. That $\bigwedge\{\mathcal{S}_i \mid i \in I\}$ is $\mathcal{M}$-stable is proved as in Proposition 3.10(4).

The reason that the same argument does not work for M-stable logics is that if each logic L_i is axiomatizable above M by Σ_i, it is unclear whether $\bigvee\{\mathsf{L}_i \mid i \in I\}$ is axiomatizable by $\bigcup\{\Sigma_i \mid i \in I\}$. In algebraic terms, if $\mathcal{V}_i$ is the variety corresponding to L_i and $\mathcal{U}_i$ is the universal class of M-algebras validating Σ_i, then $\mathcal{V}_i$ is generated by $\mathcal{U}_i$. But it is unclear whether $\bigcap\{\mathcal{V}_i \mid i \in I\}$ is generated by $\bigcap\{\mathcal{U}_i \mid i \in I\}$.

As we will see in the next section, if M is a normal extension of $\mathsf{K4}$ that admits filtration and has the $(*)$-property, then the M-stable logics do form a complete sublattice of NExtM.

While it is unclear whether the M-stable logics form a $\bigvee$-subsemilattice of NExtM, we will show that the tabular M-stable logics do form a $\bigvee$-subsemilattice of NExtM. For a variety $\mathcal{V}$, let $\mathcal{V}_{\mathsf{si}}$ be the class of subdirectly irreducible members of $\mathcal{V}$.

Proposition 3.12. *Let* M *be a normal modal logic admitting filtration and satisfying the* $(*)$*-property.*

(1) *If* L *is a tabular* M*-stable normal extension of* M*, then* $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ *is* M*-stable.*
(2) *The tabular* M*-stable logics form a* $\bigvee$*-subsemilattice of* NExtM*.*

Proof. (1). Since L is M-stable, by Theorem 3.8, there is an M-stable class $\mathcal{K}$ of subdirectly irreducible algebras that generates $\mathcal{V}(\mathsf{L})$. Since L is tabular, we may assume that $\mathcal{K}$ is a finite class of finite subdirectly irreducible algebras. Let $\mathfrak{B} \in \mathcal{V}(\mathsf{L})_{\mathsf{si}}$ and let $\mathfrak{C}$ be a stable subalgebra of $\mathfrak{B}$. By Jónsson's Lemma, $\mathfrak{B} \in \mathbf{HS}(\mathcal{K})$, so there is $\mathfrak{A}$ in $\mathbf{S}(\mathcal{K})$ such that $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$. Since $\mathfrak{C}$ is finite, it is subdirectly irreducible by [3, Prop. 6.4]. Therefore, it is sufficient to show that $\mathfrak{C}$ is an L-algebra. Let $\mathfrak{X}$ be the dual of $\mathfrak{A}$, let $\mathfrak{Y}$ be the dual of $\mathfrak{B}$, and let $\mathfrak{Z}$ be the dual of $\mathfrak{C}$. Then $\mathfrak{Y}$ is a generated subframe of $\mathfrak{X}$ and $\mathfrak{Z}$ is a stable image of $\mathfrak{Y}$. Since $\mathcal{K}$ is M-stable, so is $\mathbf{S}(\mathcal{K})$. Thus, all stable images of $\mathfrak{X}$ are L-frames. If $\mathfrak{X} = \mathfrak{Y}$, then $\mathfrak{Z}$ is a stable image of $\mathfrak{X}$, and so $\mathfrak{Z}$ is an L-frame. If $\mathfrak{X} \neq \mathfrak{Y}$, then by the $(*)$-property, we may add a new strong root to $\mathfrak{Z}$ to obtain an M-frame $\mathfrak{Z}'$. As we observed in the proof of Theorem 3.8, $\mathfrak{Z}'$ is a stable image of $\mathfrak{X}$. Therefore, $\mathfrak{Z}'$ is an L-frame, and hence so is $\mathfrak{Z}$. Thus, $\mathfrak{C} \in \mathcal{V}(\mathsf{L})_{\mathsf{si}}$.

(2). Suppose $\{\mathsf{L}_i \mid i \in I\}$ is a family of tabular M-stable logics. By (1), $\mathcal{V}(\mathsf{L}_i)_{\mathsf{si}}$ is M-stable for all $i \in I$. Therefore, $\mathcal{V}(\bigvee\{\mathsf{L}_i \mid i \in I\})_{\mathsf{si}} = \bigcap\{\mathcal{V}(\mathsf{L}_i)_{\mathsf{si}} \mid i \in I\}$ is M-stable. Thus, $\bigvee\{\mathsf{L}_i \mid i \in I\}$ is M-stable, and it is clearly tabular. □

Remark 3.13. The proof of Proposition 3.12 uses essentially that subdirectly irreducible L-algebras are finite, and does not extend directly to non-tabular logics.

As we already pointed out, stable logics are simply the K-stable logics. In [3, Def. 7.5], Condition (5) of Theorem 3.8 was used as a definition of stable logics. Theorem 3.8 then extends the characterization of stable logics given in [3, Prop. 7.6]. The next theorem shows that there are infinitely many stable logics.

Theorem 3.14.

(1) *For a finite modal algebra* $\mathfrak{A}$*, let* $\mathbf{Stable}(\mathfrak{A})$ *be the class of modal algebras that are isomorphic to stable subalgebras of* $\mathfrak{A}$*, and let* $\mathsf{Log}(\mathbf{Stable}(\mathfrak{A}))$ *be the logic of* $\mathbf{Stable}(\mathfrak{A})$*. Then* $\mathsf{Log}(\mathbf{Stable}(\mathfrak{A}))$ *is a stable modal logic.*
(2) *Every extension of* $\mathsf{S5}$ *is a stable modal logic.*

Proof. (1). Clearly $\mathbf{Stable}(\mathfrak{A})$ is a stable class of finite modal algebras. Now apply Theorem 3.8.

(2). It is well known that an $\mathsf{S5}$-algebra is subdirectly irreducible iff its dual is a cluster. It is easy to see that the class of finite clusters is a stable class. Since $\mathsf{S5}$ is the logic of this class, $\mathsf{S5}$ is a stable logic by Theorem 3.8. It is also well known that for every extension L of $\mathsf{S5}$ there is n such that L is the logic of m-clusters for $m \leq n$. This class is stable by the same reasoning. Thus, every extension of $\mathsf{S5}$ is stable. □

We conclude this section by showing that there are continuum many stable logics. In fact, we will show that there are continuum many stable logics above the logic $\mathsf{wK4}$ of weakly transitive frames, where a frame $\mathfrak{F} = (X, R)$ is *weakly transitive* provided xRy, yRz, and $x \neq z$ imply xRz for all $x, y, z \in X$. For our proof we will make use of Jankov formulas for finite $\mathsf{wK4}$-algebras (see [29] or [1, Sec. 7.2]). For a finite subdirectly irreducible $\mathsf{wK4}$-algebra $\mathfrak{A}$, let $\chi(\mathfrak{A})$ be the Jankov formula of $\mathfrak{A}$. Then for a $\mathsf{wK4}$-algebra $\mathfrak{B}$, we have:

$$\mathfrak{B} \not\models \chi(\mathfrak{A}) \text{ iff } \mathfrak{A} \text{ is a subalgebra of a homomorphic image of } \mathfrak{B} \text{ (see, e.g., [1, Prop. 7.5]).}$$

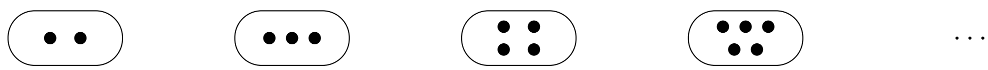
Dually, if $\mathfrak{F}$ is a finite rooted weakly transitive frame and $\mathfrak{X}$ is an arbitrary weakly transitive space, then we have:

$$\mathfrak{X} \not\models \chi(\mathfrak{F}) \text{ iff } \mathfrak{F} \text{ is a p-morphic image of a generated subframe of } \mathfrak{X}.$$

We will often not distinguish between modal algebras and their duals. If $\mathfrak{A}$ is a finite modal algebra and $\mathfrak{F}$ is its dual, then we often write $\rho(\mathfrak{F})$ instead of $\rho(\mathfrak{A})$. As usual, we denote a reflexive point by $\circ$ and an irreflexive point by $\bullet$.

Theorem 3.15. *There is a continuum of weakly transitive non-transitive stable modal logics.*

Proof. For $n \geq 2$ let $\mathfrak{C}_n = (X_n, R_n)$ be the irreflexive n-point cluster depicted in Figure 1; that is, $X_n = \{x_1, \dots x_n\}$ and $R_n = \{(x_i, x_j) \in X_n \times X_n \mid i \neq j\}$.



$\mathfrak{C}_2$ $\mathfrak{C}_3$ $\mathfrak{C}_4$ $\mathfrak{C}_5$

Figure 1

Let $\mathbb{N}_{\geq 2} = \{n \in \mathbb{N} \mid n \geq 2\}$. For $I \subseteq \mathbb{N}_{\geq 2}$ set

$$\mathcal{K}_I = \{\mathfrak{X} \mid \exists n \in I \text{ such that } \mathfrak{X} \text{ is a stable image of } \mathfrak{C}_n\}.$$

It is clear that $\mathcal{K}_I$ is a stable class of modal spaces. Let L_I be the logic of $\mathcal{K}_I$. Since $\mathcal{K}_I$ is stable, L_I is a stable modal logic. We show that if $I \neq J$, then $\mathsf{L}_I \neq \mathsf{L}_J$. For this we first show that $n \in I$ iff $\chi(\mathfrak{C}_n) \notin \mathsf{L}_I$. If $n \in I$, then $\mathfrak{C}_n \in \mathcal{K}_I$, so $\mathfrak{C}_n \models \mathsf{L}_I$. Clearly $\mathfrak{C}_n \not\models \chi(\mathfrak{C}_n)$, which implies that $\chi(\mathfrak{C}_n) \notin \mathsf{L}_I$. Conversely, suppose that $\chi(\mathfrak{C}_n) \notin \mathsf{L}_I$. Since L_I is the logic of $\mathcal{K}_I$, there is $\mathfrak{X} \in \mathcal{K}_I$ such that $\mathfrak{X} \not\models \chi(\mathfrak{C}_n)$. Therefore, $\mathfrak{C}_n$ is a p-morphic image of a generated subframe of $\mathfrak{X}$. But the only generated subframe of $\mathfrak{X}$ is $\mathfrak{X}$, so $\mathcal{K}_I$ is closed under generated subframes. Also a p-morphic image of $\mathfrak{X}$ is a stable image of $\mathfrak{X}$, and $\mathcal{K}_I$ is closed under stable images. Thus, $\mathfrak{C}_n \in \mathcal{K}_I$. If $n \notin I$, then there is $m \in I$ and an onto stable map $f : \mathfrak{C}_m \twoheadrightarrow \mathfrak{C}_n$. Since $m = |\mathfrak{C}_m| > |\mathfrak{C}_n| = n$, we see that f must identify at least two points of $\mathfrak{C}_m$. Therefore, there are distinct $x, y \in \mathfrak{C}_m$ with $f(x) = f(y)$. Thus, $xR_m y$ and $f(x) \not R_n f(y)$, which is a contradiction because f is stable. Consequently, $n \in I$, and so $n \in I$ iff $\chi(\mathfrak{C}_n) \notin \mathsf{L}_I$. Now, if $I \neq J$, then without loss of generality we may assume that there is $n \in I \setminus J$. Therefore, $\chi(\mathfrak{C}_n) \in \mathsf{L}_J \setminus \mathsf{L}_I$, and hence $\mathsf{L}_I \neq \mathsf{L}_J$. Since each $\mathfrak{C}_n$ is weakly transitive and non-transitive, we conclude that $\{\mathsf{L}_I \mid I \subseteq \mathbb{N}_{\geq 2}\}$ is a continual family of weakly transitive non-transitive stable logics. $\square$

4. Transitive M-stable logics

We next study M-stability when M is a normal extension of $\mathsf{K4}$ that admits filtration and has the $(*)$-property. In this case we will show that M-stable logics are axiomatizable by stable formulas. As a corollary we derive that the M-stable logics form a complete sublattice of NextM. If in addition M is a normal extension of $\mathsf{S4}$, then the converse is also true, and the M-stable logics are exactly the normal extensions of M axiomatizable by stable formulas. At the end of the section we point out that the results of this section can be further generalized by replacing $\mathsf{K4}$ with a normal modal logic that has a master modality, admits filtration, and satisfies the $(*)$-property.

Let $\mathfrak{A} = (A, \Diamond)$ be a $\mathsf{K4}$-algebra. As usual, for $a \in A$, we set $\Diamond^+ a = a \vee \Diamond a$ and $\Box^+ a = a \wedge \Box a$. Then $\mathfrak{A}^+ = (A, \Diamond^+)$ is an $\mathsf{S4}$-algebra. Following [28, Def. 1.10], we call $\mathfrak{A}$ *well-connected* if $\Diamond^+ a \wedge \Diamond^+ b = 0$ implies $a = 0$ or $b = 0$. Equivalently, $\mathfrak{A}$ is well-connected if $\Box^+ a \vee \Box^+ b = 1$ implies $a = 1$ or $b = 1$. Each subdirectly irreducible $\mathsf{K4}$-algebra is well-connected. To see this, suppose $\mathfrak{A}$ is subdirectly irreducible and $\Box^+ a \vee \Box^+ b = 1$. If $a, b \neq 1$, then since $\mathfrak{A}$ is subdirectly irreducible, it has an opremum $c \neq 1$, so $a, b \neq 1$ implies $\Box^+ a, \Box^+ b \leq c$, so $\Box^+ a \vee \Box^+ b \leq c \neq 1$, a contradiction. Therefore, $a = 1$ or $b = 1$, and hence $\mathfrak{A}$ is well-connected. While the converse is not true in general, it is true for finite $\mathsf{K4}$-algebras.

For a class $\mathcal{K}$ of $\mathsf{K4}$-algebras, we use the following notation:

- $\mathcal{K}_{\mathsf{si}}$ denotes the subdirectly irreducible members of $\mathcal{K}$;
- $\mathcal{K}_{\mathsf{fsi}}$ denotes the finite subdirectly irreducible members of $\mathcal{K}$;
- $\mathcal{K}_{\mathsf{wc}}$ denotes the well-connected members of $\mathcal{K}$.

For a $\mathsf{K4}$-space $\mathfrak{X} = (X, R)$, let R^+ be the reflexive closure of R. Then $\mathfrak{X}^+ = (X, R^+)$ is an $\mathsf{S4}$-space. Since in a $\mathsf{K4}$-space $R^\omega = R^+$, we see that a $\mathsf{K4}$-space is rooted iff there is $x \in X$ such that $X = R^+[x]$. It is well known that a $\mathsf{K4}$-algebra is well-connected iff its dual $\mathsf{K4}$-space is rooted.

Lemma 4.1. *Suppose $\mathfrak{A} = (A, \Diamond_A)$ and $\mathfrak{B} = (B, \Diamond_B)$ are $\mathsf{K4}$-algebras. If $\mathfrak{B}$ is well-connected and there is a stable embedding $h : \mathfrak{A} \to \mathfrak{B}$, then $\mathfrak{A}$ is well-connected.*

Proof. Since h is stable, we see that $\Diamond_B h(a) \leq h(\Diamond_A a)$ for all $a \in A$. Therefore, $\Diamond_B^+ h(a) \leq h(\Diamond_A^+ a)$ for all $a \in A$. Now, let $a, b \in A$ with $\Diamond_A^+ a \wedge \Diamond_A^+ b = 0$. Then $\Diamond_B^+ h(a) \wedge \Diamond_B^+ h(b) = 0$. As $\mathfrak{B}$ is well-connected, $h(a) = 0$ or $h(b) = 0$. Since h is an embedding, $a = 0$ or $b = 0$. Thus, $\mathfrak{A}$ is well-connected. □

As was shown in [3, Sec. 6.2], if $\mathfrak{A}$ is a finite subdirectly irreducible $\mathsf{K4}$-algebra, then the stable rule $\rho(\mathfrak{A}) = \Gamma/\Delta$ can be rewritten as a formula.

Definition 4.2. The *stable formula* of a finite subdirectly irreducible $\mathsf{K4}$-algebra $\mathfrak{A}$ is defined as

$$\gamma(\mathfrak{A}) := \bigwedge\{\Box^+\gamma \mid \gamma \in \Gamma\} \to \bigvee\{\Box^+\delta \mid \delta \in \Delta\}.$$

If $\mathfrak{F}$ is a finite rooted $\mathsf{K4}$-frame, then we write $\gamma(\mathfrak{F})$ for the stable formula of the dual algebra of $\mathfrak{F}$.

As follows from [3, Thm. 6.8], for every $\mathsf{K4}$-algebra $\mathfrak{B}$, we have $\mathfrak{B} \not\models \gamma(\mathfrak{A})$ iff there is a subdirectly irreducible homomorphic image $\mathfrak{C}$ of $\mathfrak{B}$ such that $\mathfrak{A}$ is isomorphic to a stable subalgebra of $\mathfrak{C}$. If $\mathfrak{B}$ is well-connected, then one implication of this equivalence can be strengthened.

Lemma 4.3. *Suppose $\mathfrak{A}$ is a finite subdirectly irreducible $\mathsf{K4}$-algebra and $\mathfrak{B}$ is a well-connected $\mathsf{K4}$-algebra. If $h : \mathfrak{A} \rightarrowtail \mathfrak{B}$ is a stable embedding, then $\mathfrak{B} \not\models \gamma(\mathfrak{A})$.*

Proof. Let V be a valuation on $\mathfrak{A}$ such that $V(p_a) = a$, and let $V' = h \circ V$. As in the proof of [3, Thm. 6.8], we have that $V'(\Box^+\gamma) = 1$ for all $\gamma \in \Gamma$ and $V'(\Box^+\delta) \neq 1$ for all $\delta \in \Delta$. Therefore, $V'(\bigwedge\{\Box^+\gamma \mid \gamma \in \Gamma) = 1$, and since $\mathfrak{B}$ is well-connected, $V'(\bigvee\{\Box^+\delta \mid \delta \in \Delta) \neq 1$. Thus, V' witnesses that $\mathfrak{B} \not\models \gamma(\mathfrak{A})$. □

Example 4.4. The converse of Lemma 4.3 is not true in general. Let $\mathfrak{A}$ and $\mathfrak{B}$ be the $\mathsf{K4}$-algebras that are dual to the $\mathsf{K4}$-frames $\mathfrak{F}$ and $\mathfrak{G}$ shown below.

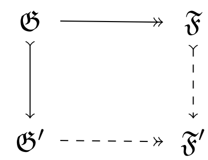
$\mathfrak{F}$ $\mathfrak{G}$

Clearly both $\mathfrak{F}, \mathfrak{G}$ are rooted and $\mathfrak{F}$ is a generated subframe of $\mathfrak{G}$. So $\mathfrak{A}$ is a subdirectly irreducible homomorphic image of $\mathfrak{B}$, and hence $\mathfrak{B} \not\models \gamma(\mathfrak{A})$. On the other hand, $\mathfrak{F}$ is not a stable image of $\mathfrak{G}$ since an onto stable map would send the root of $\mathfrak{G}$ to the root of $\mathfrak{F}$. But the root of $\mathfrak{G}$ is reflexive while the root of $\mathfrak{F}$ is irreflexive, a contradiction. Thus, there does not exist a stable embedding of $\mathfrak{A}$ into $\mathfrak{B}$.

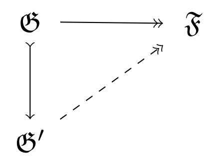
Of course, the key is that the root of $\mathfrak{F}$ is irreflexive. The next lemma shows that this is essential. Note that for finite $\mathsf{K4}$-frames, strong roots from Definition 3.7 are the same as reflexive roots.

Lemma 4.5. *Let $\mathfrak{F} = (X, R)$, $\mathfrak{G} = (Y, Q)$, and $\mathfrak{G}' = (Y', Q')$ be finite* $\mathsf{K4}$*-frames such that $\mathfrak{F}$ is a stable image of $\mathfrak{G}$ and $\mathfrak{G}$ is a generated subframe of $\mathfrak{G}'$.*

(1) *There is a finite* $\mathsf{K4}$*-frame $\mathfrak{F}' = (X', R')$ such that $\mathfrak{F}$ is a generated subframe of $\mathfrak{F}'$, $\mathfrak{F}'$ is a stable image of $\mathfrak{G}'$, and the following diagram commutes.*



(2) *If in addition $\mathfrak{F}$ has a strong root, then $\mathfrak{F}$ is a stable image of $\mathfrak{G}'$ and the following diagram commutes.*



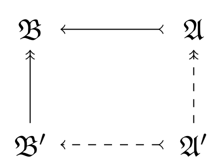
Proof. (1). If $\mathfrak{G} = \mathfrak{G}'$, then there is nothing to show as we can take $\mathfrak{F}'$ to be $\mathfrak{F}$. Otherwise we let $\mathfrak{F}'$ be obtained by adding a strong root r to $\mathfrak{F}$. It is easy to see that $\mathfrak{F}'$ is a $\mathsf{K4}$-frame and that $\mathfrak{F}$ is a generated subframe of $\mathfrak{F}'$. Moreover, the same argument as in the proof of Theorem 3.8 yields that $\mathfrak{F}'$ is a stable image of $\mathfrak{G}'$. Furthermore, it follows from the definition that the diagram commutes.

(2). Let $f : Y \to X$ be an onto stable map. Define $g : Y' \to X$ so that the restriction of g to Y is f and g maps $Y' \setminus Y$ to the reflexive root r of $\mathfrak{F}$ (provided $Y' \setminus Y \neq \varnothing$). Then it is easy to see that g is an onto stable map, and that the diagram commutes. □

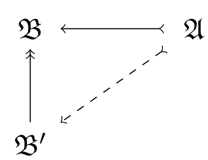
We can reformulate Lemma 4.5 in algebraic terms as follows.

Lemma 4.6. *Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{B}'$ be finite* $\mathsf{K4}$*-algebras such that there is a stable embedding of $\mathfrak{A}$ into $\mathfrak{B}$ and $\mathfrak{B}$ is a homomorphic image of $\mathfrak{B}'$.*

(1) *There is a finite* $\mathsf{K4}$*-algebra $\mathfrak{A}'$ such that $\mathfrak{A}$ is a homomorphic image of $\mathfrak{A}'$, $\mathfrak{A}'$ is isomorphic to a stable subalgebra of $\mathfrak{B}'$, and the following diagram commutes.*



(2) *If in addition $\mathfrak{A}$ has an atom a such that $a \leq \Diamond b$ for all $0 \neq b \in \mathfrak{A}$, then there is a stable embedding of $\mathfrak{A}$ into $\mathfrak{B}'$ and the following diagram commutes.*



We next build on Theorem 3.8 and obtain several more convenient characterizations of M-stability when M is a normal extension of $\mathsf{K4}$ that admits filtration and satisfies the $(*)$-property. For a class $\mathcal{K}$ of $\mathsf{K4}$-algebras, let $\mathcal{K}_{\mathsf{wc}}$ be the class of well-connected members of $\mathcal{K}$ and let $\mathcal{K}_{\mathsf{fsi}}$ be the class of finite subdirectly irreducible members of $\mathcal{K}$.

Theorem 4.7. *Let* M *be a normal extension of* $\mathsf{K4}$ *that admits filtration and has the* $(*)$*-property. For a normal extension* L *of* M*, the following are equivalent.*

(1) L *is* M*-stable.*
(2) $\mathcal{V}(\mathsf{L})_{\mathsf{wc}}$ *is* M*-stable.*
(3) $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ *is finitely* M*-stable.*
(4) $\mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$ *is* M*-stable and generates* $\mathcal{V}(\mathsf{L})$*.*

Moreover, each M*-stable logic is axiomatizable by stable formulas.*

Proof. For the implication (1) $\Rightarrow$ (2), assume that L is M-stable. By Theorem 3.8, $\mathcal{V}(\mathsf{L})$ is generated by an M-stable class $\mathcal{K}$ of finite M-algebras.

Claim 4.8. *For any finite subdirectly irreducible* M*-algebra* $\mathfrak{A}$*, if* $\mathfrak{A} \not\models \mathsf{L}$*, then* $\gamma(\mathfrak{A}) \in \mathsf{L}$*.*

Proof. It is sufficient to prove that $\gamma(\mathfrak{A}) \notin \mathsf{L}$ implies $\mathfrak{A} \models \mathsf{L}$. Suppose that $\gamma(\mathfrak{A}) \notin \mathsf{L}$. Since $\mathcal{K}$ generates $\mathcal{V}(\mathsf{L})$, there is $\mathfrak{B} \in \mathcal{K}$ such that $\mathfrak{B} \not\models \gamma(\mathfrak{A})$. By [3, Thm. 6.8], there is a subdirectly irreducible homomorphic image $\mathfrak{C}$ of $\mathfrak{B}$ and a stable embedding of $\mathfrak{A}$ into $\mathfrak{C}$. By Lemma 4.6(1), there is a finite $\mathsf{K4}$-algebra $\mathfrak{D}$ such that $\mathfrak{D}$ is isomorphic to a stable subalgebra of $\mathfrak{B}$ and $\mathfrak{A}$ is a homomorphic image of $\mathfrak{D}$. Since M has the $(*)$-property, it follows from the proof of Lemma 4.5(1) that $\mathfrak{D}$ is an M-algebra. As $\mathcal{K}$ is M-stable and $\mathfrak{B} \in \mathcal{K}$, we have that $\mathfrak{D} \in \mathcal{K}$. Because $\mathcal{V}(\mathsf{L})$ is closed under homomorphic images, $\mathfrak{A} \in \mathcal{V}(\mathsf{L})$. Therefore, $\mathfrak{A} \models \mathsf{L}$. □

Now suppose $\mathfrak{A}, \mathfrak{B}$ are M-algebras with $\mathfrak{B} \in \mathcal{V}(\mathsf{L})_{\mathsf{wc}}$ and there is a stable embedding of $\mathfrak{A}$ into $\mathfrak{B}$. Since $\mathfrak{B}$ is well-connected, so is $\mathfrak{A}$ by Lemma 4.1. If $\mathfrak{A} \not\models \mathsf{L}$, then $\mathfrak{A} \not\models \varphi$ for some $\varphi \in \mathsf{L}$. As M admits filtration, there is a finite M-algebra $\mathfrak{C}$ such that $\mathfrak{C}$ is a stable subalgebra of $\mathfrak{A}$ and $\mathfrak{C} \not\models \varphi$. But then there is a stable embedding of $\mathfrak{C}$ into $\mathfrak{B}$. Since $\mathfrak{C}$ is finite and well-connected, it is subdirectly irreducible. By Claim 4.8, $\gamma(\mathfrak{C}) \in \mathsf{L}$. Because there is a stable embedding of $\mathfrak{C}$ into $\mathfrak{B}$, it follows from Lemma 4.3 that $\mathfrak{B} \not\models \gamma(\mathfrak{C})$, which contradicts to $\mathfrak{B} \models \mathsf{L}$. Thus, $\mathfrak{A} \models \mathsf{L}$, so $\mathfrak{A} \in \mathcal{V}(\mathsf{L})_{\mathsf{wc}}$, and hence $\mathcal{V}(\mathsf{L})_{\mathsf{wc}}$ is M-stable.

The implication (2) $\Rightarrow$ (3) follows from the fact that every subdirectly irreducible $\mathsf{K4}$-algebra is well-connected. For the implication (3) $\Rightarrow$ (4), observe that if $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ is finitely M-stable, then $\mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$ is M-stable. By Lemma 3.5(2), $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ and $\mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$ generate the same universal class, and hence the same variety. Therefore, $\mathcal{V}(\mathsf{L})$ is generated by $\mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$. The implication (4) $\Rightarrow$ (1) is obvious.

Finally, we show that M-stable logics are axiomatizable by stable formulas. Suppose that L is M-stable. Let $\mathcal{A}$ be the set of finite non-isomorphic subdirectly irreducible M-algebras not belonging to $\mathcal{V}(\mathsf{L})$. We claim that $\mathsf{L} = \mathsf{M} + \{\gamma(\mathfrak{A}) \mid \mathfrak{A} \in \mathcal{A}\}$. The inclusion $\mathsf{M} + \{\gamma(\mathfrak{A}) : \mathfrak{A} \in \mathcal{A}\} \subseteq \mathsf{L}$ follows from Claim 4.8. For the reverse inclusion, let $\mathcal{V}$ be the variety corresponding to $\mathsf{M} + \{\gamma(\mathfrak{A}) : \mathfrak{A} \in \mathcal{A}\}$. As subdirectly irreducible members of $\mathcal{V}$ generate $\mathcal{V}$, it is sufficient to show that each subdirectly irreducible member of $\mathcal{V}$ belongs to $\mathcal{V}(\mathsf{L})$. Let $\mathfrak{B}$ be a subdirectly irreducible member of $\mathcal{V}$. If $\mathfrak{B} \not\models \mathsf{L}$, then since M admits filtration, there is a finite M-algebra $\mathfrak{B}'$ such that $\mathfrak{B}'$ is a stable subalgebra of $\mathfrak{B}$ and $\mathfrak{B}' \not\models \mathsf{L}$. Because $\mathfrak{B}$ is subdirectly irreducible, it is well-connected. Therefore, $\mathfrak{B}'$ is well-connected by Lemma 4.1. Thus, as $\mathfrak{B}'$ is finite, it is subdirectly irreducible. So $\mathfrak{B}' \in \mathcal{A}$. Now, $\mathfrak{B} \not\models \gamma(\mathfrak{B}')$ by Lemma 4.3. Consequently, $\mathfrak{B} \notin \mathcal{V}$, a contradiction. This yields that $\mathfrak{B} \models \mathsf{L}$, and hence $\mathsf{L} = \mathsf{M} + \{\gamma(\mathfrak{A}) \mid \mathfrak{A} \in \mathcal{A}\}$. □

Corollary 4.9. *If* M *is a normal extension of* $\mathsf{K4}$ *that admits filtration and has the* $(*)$*-property, then the* M*-stable logics form a complete sublattice of* NExtM*.*

Proof. Let $\{\mathsf{L}_i \mid i \in I\}$ be a family of M-stable logics. By Theorem 4.7, each L_i is axiomatized above M by a set Σ_i of stable formulas of finite subdirectly irreducible M-algebras. But then $\bigvee\{\mathsf{L}_i \mid i \in I\}$ is axiomatized by $\bigcup\{\Sigma_i \mid i \in I\}$, and hence $\bigvee\{\mathsf{L}_i \mid i \in I\}$ is M-stable. That $\bigwedge\{\mathsf{L}_i \mid i \in I\}$ is M-stable follows from Proposition 3.10(4). □

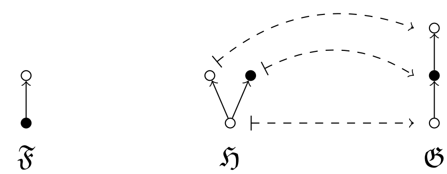
In particular, since $\mathsf{K4}$ admits filtration and has the $(*)$-property, we obtain:

Corollary 4.10. *Let* L *be a normal extension of* $\mathsf{K4}$*. The following are equivalent.*

(1) L *is* $\mathsf{K4}$*-stable.*
(2) $\mathcal{V}(\mathsf{L})_{\mathsf{wc}}$ *is* $\mathsf{K4}$*-stable.*
(3) $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ *is finitely* $\mathsf{K4}$*-stable.*
(4) $\mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$ *is* $\mathsf{K4}$*-stable and generates* $\mathcal{V}(\mathsf{L})$*.*

Moreover, each $\mathsf{K4}$*-stable logic is axiomatizable by stable formulas, and hence the stable* $\mathsf{K4}$*-logics form a complete sublattice of* $\mathsf{NExtK4}$*.*

Example 4.11. On the other hand, there exist logics above $\mathsf{K4}$ that are axiomatizable over $\mathsf{K4}$ by stable formulas, but are not $\mathsf{K4}$-stable logics. To see this, consider the $\mathsf{K4}$-frames $\mathfrak{F}$, $\mathfrak{G}$, and $\mathfrak{H}$ shown below.



We set $\mathsf{L} = \mathsf{K4} + \gamma(\mathfrak{F})$. Clearly $\mathfrak{H}$ is the only non-singleton rooted upset of $\mathfrak{H}$ and $\mathfrak{F}$ is not a stable image of $\mathfrak{H}$ since $\mathfrak{H}$ has a reflexive root and $\mathfrak{F}$ has an irreflexive root. Therefore, $\mathfrak{H} \models \gamma(\mathfrak{F})$, and so $\mathfrak{H} \models \mathsf{L}$. Next consider the map $\mathfrak{H} \to \mathfrak{G}$ indicated in the picture above. It is easy to see that it is a stable map from $\mathfrak{H}$ onto $\mathfrak{G}$. If L were $\mathsf{K4}$-stable, Theorem 4.7 would yield $\mathfrak{G} \models \gamma(\mathfrak{F})$. However, $\mathfrak{G} \not\models \gamma(\mathfrak{F})$ as we already discussed in Example 4.4. Thus, L is not $\mathsf{K4}$-stable.

Remark 4.12. It is of interest to study further the class of logics axiomatized by $\mathsf{K4}$-stable formulas over $\mathsf{K4}$. It is not even clear whether all such logics have the fmp, which we leave as an open question here.

In Example 4.11 it was essential that the root of $\mathfrak{G}$ was irreflexive. We next show that every logic that is axiomatizable over $\mathsf{K4}$ by stable formulas of finite $\mathsf{K4}$-frames with strong roots is $\mathsf{K4}$-stable. In algebraic terms we will show that a logic is $\mathsf{K4}$-stable if it is axiomatizable over $\mathsf{K4}$ by stable formulas of finite $\mathsf{K4}$-algebras that have an atom a such that $a \le \Diamond b$ for each $b \neq 0$. For convenience, we call such algebras *strongly subdirectly irreducible.*

Proposition 4.13.

(1) *Let* $\mathfrak{A}$ *be a finite strongly subdirectly irreducible* $\mathsf{K4}$*-algebra. For a well-connected* $\mathsf{K4}$*-algebra* $\mathfrak{B}$ *we have* $\mathfrak{B} \not\models \gamma(\mathfrak{A})$ *iff there is a stable embedding of* $\mathfrak{A}$ *into* $\mathfrak{B}$*.*
(2) *Suppose* $\mathsf{L} = \mathsf{K4} + \{\gamma(\mathfrak{A}_i) \mid i \in I\}$*, where each* $\mathfrak{A}_i$ *is a finite strongly subdirectly irreducible* $\mathsf{K4}$*-algebra. Then* L *is* $\mathsf{K4}$*-stable.*

Proof. (1). The right to left direction was already proven in Lemma 4.3. For the left to right direction, let $\mathfrak{B}$ be a $\mathsf{K4}$-algebra such that $\mathfrak{B} \not\models \gamma(\mathfrak{A})$. (Note that for this direction it is not needed that $\mathfrak{B}$ is well-connected.) Since $\mathsf{K4}$ admits filtration, there is a finite $\mathsf{K4}$-algebra $\mathfrak{C}$ that is a stable subalgebra of $\mathfrak{B}$ and $\mathfrak{C} \not\models \gamma(\mathfrak{A})$. By [3, Thm. 6.8], there is a subdirectly irreducible homomorphic image $\mathfrak{D}$ of $\mathfrak{C}$ and a stable embedding of $\mathfrak{A}$ into $\mathfrak{D}$. Since $\mathfrak{A}$ is strongly subdirectly irreducible, by Lemma 4.6(2), there is a stable embedding of $\mathfrak{A}$ into $\mathfrak{C}$, and hence a stable embedding of $\mathfrak{A}$ into $\mathfrak{B}$.

(2). It is immediate from (1) that the class of well-connected algebras of L is $\mathsf{K4}$-stable. Now apply Theorem 4.7. ☐

Since every finite subdirectly irreducible $\mathsf{S4}$-algebra is strongly subdirectly irreducible, Proposition 4.13 yields:

Corollary 4.14. *Let* $\mathfrak{A}$ *be a finite subdirectly irreducible* $\mathsf{S4}$*-algebra. For every well-connected* $\mathsf{S4}$*-algebra* $\mathfrak{B}$ *we have* $\mathfrak{B} \not\models \gamma(\mathfrak{A})$ *iff there is a stable embedding of* $\mathfrak{A}$ *into* $\mathfrak{B}$*.*

This immediately yields that if M is a normal extension of $\mathsf{S4}$ that admits filtration and has the $(*)$-property, then all logics axiomatizable over M by stable formulas of finite subdirectly irreducible M-algebras are M-stable. Thus, we obtain the following improvement of Theorem 4.7.

Corollary 4.15. *Let* M *be a normal extension of* $\mathsf{S4}$ *that admits filtration and has the* $(*)$*-property. For a normal extension* L *of* M*, the following are equivalent.*

(1) L *is* M*-stable.*
(2) L *is axiomatizable over* M *by stable rules of finite* M*-algebras.*
(3) L *is axiomatizable over* M *by stable formulas of finite subdirectly irreducible* M*-algebras.*
(4) $\mathcal{V}(\mathsf{L})$ *is generated by an* M*-stable class of* M*-algebras.*
(5) $\mathcal{V}(\mathsf{L})$ *is generated by an* M*-stable class of finite* M*-algebras.*
(6) $\mathcal{V}(\mathsf{L})_{\mathsf{wc}}$ *is* M*-stable.*
(7) $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ *is finitely* M*-stable.*
(8) $\mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$ *is* M*-stable and generates* $\mathcal{V}(\mathsf{L})$*.*

In particular, since $\mathsf{S4}$ admits filtration and has the $(*)$-property, Corollary 4.15 is true for $\mathsf{S4}$.

Remark 4.16. We recall (see, e.g., [34, Sec. 5]) that a normal modal logic M has a *master modality* if there is a compound-box $[\mathsf{m}]$ such that for every M-algebra $\mathfrak{A}$ and every $a \in A$ we have $[\mathsf{m}]a \leq \blacksquare a$ for each compound box $\blacksquare$. Such logics are also known under the name of *ω-transitive logics.* If M is an extension of $\mathsf{K4}$, then $\Box^+$ acts as a master modality. If $[\mathsf{m}]$ is a master modality of M, then an M-algebra $\mathfrak{A}$ is well-connected iff $[\mathsf{m}]a \vee [\mathsf{m}]b = 1$ implies $a = 1$ or $b = 1$ for all $a, b \in \mathfrak{A}$.

Let $\mathfrak{A}$ be a finite subdirectly irreducible M-algebra. Define the *stable formula* $\gamma(\mathfrak{A})$ of $\mathfrak{A}$ as

$$\gamma(\mathfrak{A}) := \bigwedge\{[\mathsf{m}]\gamma \mid \gamma \in \Gamma\} \to \bigvee\{[\mathsf{m}]\delta \mid \delta \in \Delta\},$$

where Γ/Δ is the stable rule of $\mathfrak{A}$.

The results of this section generalize to the following: Let M be a normal modal logic that has a master modality, admits filtration, and satisfies the $(*)$-property. For a normal extension L of M, the following are equivalent.

(1) L is M-stable.
(2) The well-connected L-algebras are an M-stable class.
(3) $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ is finitely M-stable.
(4) $\mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$ is M-stable and generates $\mathcal{V}(\mathsf{L})$.

Moreover, each M-stable logic is axiomatizable by stable formulas of finite subdirectly irreducible M-algebras. Furthermore, the same proof as in [22, Lem. 3.23] shows that M has the fmp iff it has the fmp for quasi-equations. Thus, as discussed in Remark 3.4, similarly to $\mathsf{K4}$-stable logics, every M-stable logic has the fep.

5. Connection with stable superintuitionistic logics

In this section we will study the relationship between $\mathsf{S4}$-stable logics and stable superintuitionistic logics (si-logics). We will show that the intuitionistic fragment of an $\mathsf{S4}$-stable logic is a stable si-logic, and that the least modal companion of a stable si-logic is $\mathsf{S4}$-stable. We also translate axiomatizations of stable si-logics to axiomatizations of $\mathsf{S4}$-stable logics and vice versa. We then discuss similar connections between $\mathsf{K4}$-stable logics and $\mathsf{S4}$-stable logics. We summarize our findings in Table 1. Since there are continuum many stable si-logics, our observations allow us to show that there are continuum many $\mathsf{S4}$-stable logics, and continuum many $\mathsf{K4}$-stable logics between $\mathsf{K4}$ and $\mathsf{S4}$.

From now on we will mainly work with frames instead of algebras to utilize their geometric intuition. We start by recalling a few facts about intuitionistic fragments of normal extensions of $\mathsf{S4}$ and modal companions of si-logics. We follow the notation of [13, Sec. 9.6]. Let M be a normal extension of $\mathsf{S4}$ and let L be a si-logic. The *intuitionistic fragment* of M is defined as

$\rho\mathsf{M} := \{\varphi \mid t(\varphi) \in \mathsf{M}\}$, where $t(\varphi)$ is the Gödel translation of φ. If $\mathsf{L} = \rho(\mathsf{M})$, then M is called a *modal companion* of L. It is well known that every si-logic L has a *least* modal companion that we denote by $\tau(\mathsf{L})$. For an $\mathsf{S4}$-frame $\mathfrak{F} = (X, R)$ its *skeleton* $\rho\mathfrak{F} = (\rho X, \rho R)$ is obtained by modding out the clusters of $\mathfrak{F}$. Clearly $\rho\mathfrak{F}$ is an intuitionistic frame. It is well known (see, e.g., [13, Lem. 9.67]) that for every $\mathsf{S4}$-frame $\mathfrak{F}$, we have $\mathfrak{F} \models \tau\mathsf{L}$ iff $\rho\mathfrak{F} \models \mathsf{L}$, and if $\mathfrak{F}$ is a partial order, then $\mathfrak{F} \models \mathsf{M}$ iff $\mathfrak{F} \models \rho\mathsf{M}$.

Next we recall some relevant facts from [2, Sec. 6] about stable si-logics. Suppose $\mathfrak{F}$ and $\mathfrak{G}$ are finite intuitionistic frames. We call $\mathfrak{F}$ a *stable image* of $\mathfrak{G}$ if there is an order preserving map from $\mathfrak{G}$ onto $\mathfrak{F}$. If $\mathfrak{F}$ is rooted, then we denote the *stable* (*intuitionistic*) *formula* of $\mathfrak{F}$ by $\gamma(\mathfrak{F})$.[2] We have $\mathfrak{G} \models \gamma(\mathfrak{F})$ iff $\mathfrak{F}$ is a stable image of $\mathfrak{G}$. A si-logic L is stable iff L is axiomatizable by stable formulas of some finite rooted frames.

The next theorem shows that stability is preserved by least modal companions, allowing us to translate axiomatizations of stable si-logics to axiomatizations of their least modal companions. We will use these results in Section 6 to axiomatize $\mathsf{S4}$-stable logics. We point out that the greatest modal companion of a stable si-logic is not necessarily $\mathsf{S4}$-stable. For instance, the Grzegorczyk logic $\mathsf{S4.Grz}$ is the greatest modal companion of IPC, and we will see in Section 6 that it is not $\mathsf{S4}$-stable.

Theorem 5.1.
(1) *Let $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, R)$ be finite rooted $\mathsf{S4}$-frames. If $\mathfrak{G}$ is a stable image of $\mathfrak{F}$, then $\rho\mathfrak{G}$ is a stable image of $\rho\mathfrak{F}$.*
(2) *If L is a stable si-logic, then $\tau\mathsf{L}$ is $\mathsf{S4}$-stable.*
(3) *If $\mathsf{L} = \mathsf{IPC} + \{\gamma(\mathfrak{G}_i) \mid i \in I\}$, then $\tau\mathsf{L} = \mathsf{S4} + \{\gamma(\mathfrak{G}_i) \mid i \in I\}$.*

Proof. (1). Let $f : X \to Y$ be an onto stable map. Since the quotient map $\pi_Y : Y \to \rho Y$ is an onto p-morphism, the composition $\pi_Y \circ f : X \to \rho Y$ is onto and stable. Define $g : \rho X \to \rho Y$ by $g(\pi_X(x)) = \pi_Y(f(x))$. Because $\pi_Y \circ f$ is stable, g is well defined, and it is clear that g is onto and stable. Therefore, $\rho\mathfrak{G}$ is a stable image of $\rho\mathfrak{F}$.

(2). Let L be a stable si-logic. By [2, Thm. 6.8], L has the fmp. Therefore, so does $\tau\mathsf{L}$ (see, e.g., [13, p. 328]). Thus, $\tau\mathsf{L}$ is the logic of its finite rooted frames. We show that this class is $\mathsf{S4}$-stable. Let $\mathfrak{F}$ be a finite rooted $\tau\mathsf{L}$-frame and $\mathfrak{G}$ be a finite rooted $\mathsf{S4}$-frame that is a stable image of $\mathfrak{F}$. Since $\mathfrak{F}$ is a $\tau\mathsf{L}$-frame, $\rho\mathfrak{F}$ is an L-frame. By (1), $\rho\mathfrak{G}$ is a stable image of $\rho\mathfrak{F}$. As L is stable, $\rho\mathfrak{G} \models \mathsf{L}$. Therefore, $\mathfrak{G} \models \tau\mathsf{L}$, and hence the class of finite rooted $\tau\mathsf{L}$-frames is $\mathsf{S4}$-stable. Thus, by Corollary 4.15, $\tau\mathsf{L}$ is an $\mathsf{S4}$-stable logic.

(3). Let $\mathsf{M} = \mathsf{S4} + \{\gamma(\mathfrak{G}_i) \mid i \in I\}$. By Corollary 4.15 and (2), both $\tau\mathsf{L}$ and M are $\mathsf{S4}$-stable. Therefore, to see that $\tau\mathsf{L} = \mathsf{M}$, it is sufficient to check that the two logics have the same finite rooted frames. Let $\mathfrak{F}$ be a finite rooted $\mathsf{S4}$-frame. If $\mathfrak{F} \not\models \tau\mathsf{L}$, then $\rho\mathfrak{F} \not\models \mathsf{L}$, so $\mathfrak{G}_i$ is a stable image of $\rho\mathfrak{F}$ for some $i \in I$. Since $\rho\mathfrak{F}$ is a stable image of $\mathfrak{F}$, we conclude that $\mathfrak{G}_i$ is a stable image of $\mathfrak{F}$. Thus, $\mathfrak{F} \not\models \gamma(\mathfrak{G}_i)$, and hence $\mathfrak{F} \not\models \mathsf{M}$. Conversely, if $\mathfrak{F} \not\models \mathsf{M}$, then $\mathfrak{G}_i$ is a stable image of $\mathfrak{F}$ for some $i \in I$. From (1) it follows that $\rho\mathfrak{G}_i$ is a stable image of $\rho\mathfrak{F}$. Since $\mathfrak{G}_i$ is partially ordered, $\mathfrak{G}_i \cong \rho\mathfrak{G}_i$, implying that $\mathfrak{G}_i$ is a stable image of $\rho\mathfrak{F}$. Thus, $\rho\mathfrak{F} \not\models \mathsf{L}$, and so $\mathfrak{F} \not\models \tau\mathsf{L}$. ☐

Next we will show that stability is preserved by intuitionistic fragments, which will allow us to translate axiomatizations of $\mathsf{S4}$-stable logics to axiomatizations of their intuitionistic fragments.

For a finite rooted $\mathsf{S4}$-frame $\mathfrak{F} = (X, R)$, let $\overline{\mathfrak{F}} = (X, \overline{R})$ be the partially ordered $\mathsf{S4}$-frame that is obtained from $\mathfrak{F}$ by unraveling each n-cluster into an n-chain (see Figure 2); that is, if $X = C_1 \cup \cdots \cup C_k$ is the division of $\mathfrak{F}$ into clusters, with $C_i = \{x^i_1, \ldots, x^i_{n_i}\}$, then for all $x = x^i_l$ and

[2]Stable formulas in the modal and intuitionistic case, while syntactically different, have similar semantic behavior. This justifies the same name and notation in both cases. It should always be clear from the context which formula we are working with.

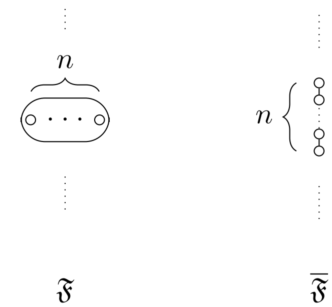


FIGURE 2

$y = x^j_m$, we have

$$x\overline{R}y \quad \text{iff} \begin{cases} i = j \text{ and } l \geq m \text{ or} \\ i \neq j \text{ and } xRy, \end{cases}$$

where $1 \leq i, j \leq k$ and $1 \leq l \leq n_i$, $1 \leq m \leq n_j$. Note that $x^i_{n_i}$ is the root of the chain C_i in $\overline{\mathfrak{F}}$.

Theorem 5.2.

(1) *Let $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, R)$ be finite rooted* S4*-frames, with $\mathfrak{G}$ being partially ordered. Then $\mathfrak{F}$ is a stable image of $\mathfrak{G}$ iff $\overline{\mathfrak{F}}$ is a stable image of $\mathfrak{G}$.*
(2) *If* M *is* S4*-stable, then* ρM *is stable.*
(3) *If* $\mathsf{M} = \mathsf{S4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$, *then* $\rho\mathsf{M} = \mathsf{IPC} + \{\gamma(\overline{\mathfrak{F}}_i) \mid i \in I\}$.

Proof. (1). Since $\mathfrak{F}$ is easily seen to be a stable image of $\overline{\mathfrak{F}}$, the implication from right to left is obvious. Conversely, suppose that $f : \mathfrak{G} \to \mathfrak{F}$ is an onto stable map. We transform f into a stable map $\overline{f} : \mathfrak{G} \to \overline{\mathfrak{F}}$ by shuffling the values of f belonging to some cluster of $\mathfrak{F}$. Let C_i be a cluster of $\mathfrak{F}$ and let $Y' = f^{-1}(C_i)$. We view Y' as a subframe of $\mathfrak{G}$, and define $\overline{f} : Y' \to C_i$ by induction on the depth of points in Y'. The idea is to map the points of the smallest depth injectively onto the first $n_i - 1$ points of C_i and all the other points of Y' to the root $x^i_{n_i}$. More precisely, suppose $\{y_1, \ldots, y_m\} \subseteq Y'$ are the points of depth d and we have mapped all the points of Y' of smaller depth injectively onto $\{x^i_1, \ldots, x^i_l\}$. If $m \leq n_i - l$, then set $\overline{f}(y_h) = x^i_{l+h}$ for all $1 \leq h \leq m$. If $m \not\leq n_i - l$, then define $\overline{f}$ as before for all y_l with $l \leq m - (n_i - l)$ and map all the other points of Y' to $x^i_{n_i}$. It is straightforward to check that $\overline{f}$ is stable.

(2). Since M is S4-stable, it has the fmp. Therefore, so does ρM (see, e.g., [13, p. 328]). It thus suffices to show that the finite rooted ρM-frames form a stable class. Suppose $\mathfrak{G}$ is a stable image of a finite rooted ρM-frame $\mathfrak{F}$. From $\mathfrak{F} \models \rho\mathsf{M}$ it follows that $\mathfrak{F} \models \mathsf{M}$. Since M is S4-stable, $\mathfrak{G} \models \mathsf{M}$. Consequently, $\mathfrak{G} \models \rho\mathsf{M}$.

(3). Since M is S4-stable, ρM is stable by (2). Let $\mathsf{L} = \mathsf{IPC} + \{\gamma(\overline{\mathfrak{F}}_i) \mid i \in I\}$. By [2, Thm. 6.11], L is stable. Therefore, both ρM and L have the fmp, and hence it suffices to show that the two logics have the same finite rooted frames. Suppose $\mathfrak{G}$ is a finite rooted partially ordered frame. If $\mathfrak{G} \not\models \mathsf{L}$, then there is $i \in I$ such that $\mathfrak{G} \not\models \gamma(\overline{\mathfrak{F}}_i)$. Therefore, $\overline{\mathfrak{F}}_i$ is a stable image of $\mathfrak{G}$. By (1), $\mathfrak{F}_i$ is a stable image of $\mathfrak{G}$. Thus, $\mathfrak{G} \not\models \gamma(\mathfrak{F}_i)$, and so $\mathfrak{G} \not\models \mathsf{M}$. Since $\mathfrak{G}$ is a partially ordered frame, we conclude that $\mathfrak{G} \not\models \rho\mathsf{M}$. Conversely, if $\mathfrak{G} \not\models \rho\mathsf{M}$, then $\mathfrak{G} \not\models \mathsf{M}$, and hence $\mathfrak{G} \not\models \gamma(\mathfrak{F}_i)$ for some $i \in I$. Therefore, $\mathfrak{F}_i$ is a stable image of $\mathfrak{G}$. By (1), $\overline{\mathfrak{F}_i}$ is a stable image of $\mathfrak{G}$. Thus, $\mathfrak{G} \not\models \gamma(\overline{\mathfrak{F}}_i)$, yielding that $\mathfrak{G} \not\models \mathsf{L}$. □

Corollary 5.3.

(1) *A si-logic* L *is stable iff* τL *is* S4*-stable.*

(2) *A* $\mathsf{S4}$*-stable logic is the least modal companion of a si-logic iff it can be axiomatized by stable formulas of finite rooted partially ordered* $\mathsf{S4}$*-frames.*

Proof. (1). It is well known that $\mathsf{L} = \rho\tau\mathsf{L}$ (see, e.g., [13, Thm. 9.57]). Now apply Theorems 5.1(3) and 5.2(3).

(2). Suppose M is the least modal companion of a si-logic L. Then $\mathsf{M} = \tau\mathsf{L}$, and so $\mathsf{L} = \rho\mathsf{M}$. Since M is $\mathsf{S4}$-stable, L is stable by Theorem 5.2(2). Therefore, by [2, Thm. 6.11], there are finite rooted partially ordered frames $\{\mathfrak{F}_i \mid i \in I\}$ such that $\mathsf{L} = \mathsf{IPC} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$. Thus, $\mathsf{M} = \mathsf{S4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$ by Theorem 5.1(3). Conversely, if $\mathsf{M} = \mathsf{S4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$ for some finite rooted partially ordered $\mathsf{S4}$-frames $\{\mathfrak{F}_i \mid i \in I\}$, then $\rho\mathsf{M} = \mathsf{IPC} + \{\gamma(\overline{\mathfrak{F}}_i) \mid i \in I\}$ by Theorem 5.2(3). Since $\overline{\mathfrak{F}}_i = \mathfrak{F}_i$ for all $i \in I$, we conclude that $\tau\rho\mathsf{M} = \mathsf{IPC} + \{\gamma(\overline{\mathfrak{F}}_i) \mid i \in I\} = \mathsf{M}$, and hence M is the least modal companion of $\rho\mathsf{M}$. □

Next we discuss connections between $\mathsf{S4}$-stable and $\mathsf{K4}$-stable logics. For a formula φ, let φ^+ be obtained from φ by replacing each subformula of φ of the form $\Box\psi$ by $\psi \wedge \Box\psi$. If $\mathsf{L} = \mathsf{S4} + \Gamma$ is a normal extension of $\mathsf{S4}$, let $\mathsf{L}^+ = \mathsf{K4} + \Gamma^+$, where $\Gamma^+ = \{\varphi^+ \mid \varphi \in \Gamma\}$. For a binary relation R on X, let $R^+ := R \cup \{(x,x) \mid x \in X\}$ be the *reflexive closure* of R. For a $\mathsf{K4}$-space $\mathfrak{F} = (X, R)$, define the *reflexivization* of $\mathfrak{F}$ as $\mathfrak{F}^+ = (X, R^+)$. Then $\mathfrak{F}^+$ is an $\mathsf{S4}$-space and $\mathfrak{F} \models \mathsf{L}^+$ iff $\mathfrak{F}^+ \models \mathsf{L}$. Therefore, L^+ is the logic of $\{\mathfrak{F} \mid \mathfrak{F}^+ \models \mathsf{L}\}$ (see, e.g., [13, Sec. 3.9]).

Lemma 5.4.

(1) *Let* $\mathfrak{F}$ *be a finite* $\mathsf{S4}$*-frame and let* $\mathfrak{G}$ *be a* $\mathsf{K4}$*-space. Then* $\mathfrak{F}$ *is a stable image of* $\mathfrak{G}$ *iff* $\mathfrak{F}$ *is a stable image of* $\mathfrak{G}^+$*.*

(2) *If* $\mathsf{L} = \mathsf{S4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$*, where the* $\mathfrak{F}_{\mathsf{i}}$ *are* $\mathsf{S4}$*-frames, then* $\mathsf{L}^+ = \mathsf{K4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$*.*

(3) *If* L *is* $\mathsf{S4}$*-stable, then* L^+ *is* $\mathsf{K4}$*-stable.*

Proof. (1). Immediate since $\mathfrak{F}$ is reflexive.

(2). By (1) and Corollary 4.14, if $\mathfrak{G}$ is a rooted $\mathsf{K4}$-space, then $\mathfrak{G} \models \gamma(\mathfrak{F}_i)$ iff $\mathfrak{G}^+ \models \gamma(\mathfrak{F}_i)$. Therefore, $\mathfrak{G} \models \mathsf{L}^+$ iff $\mathfrak{G}^+ \models \mathsf{L}$ iff $\mathfrak{G}^+ \models \{\gamma(\mathfrak{F}_i) \mid i \in I\}$ iff $\mathfrak{G} \models \{\gamma(\mathfrak{F}_i) \mid i \in I\}$. Thus, L^+ and $\mathsf{K4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$ have the same $\mathsf{K4}$-spaces, and hence the two logics coincide.

(3). If L is $\mathsf{S4}$-stable, then L is axiomatizable by stable formulas of $\mathsf{S4}$-frames. By (2), L^+ is axiomatized by the same stable formulas. In particular, L^+ is axiomatizable by stable formulas of frames with reflexive roots. Thus, L^+ is $\mathsf{K4}$-stable by Proposition 4.13. □

For two normal modal logics L and M, let $\mathsf{L} \vee \mathsf{M}$ denote the join of these logics in the lattice of normal modal logics.

Lemma 5.5. *Let* L *be a normal extension of* $\mathsf{K4}$*.*

(1) *If* $\mathsf{S4} \subseteq \mathsf{L}$*, then* L *is* $\mathsf{K4}$*-stable iff* L *is* $\mathsf{S4}$*-stable.*

(2) *If* L *is* $\mathsf{K4}$*-stable, then* $\mathsf{S4} \vee \mathsf{L}$ *is* $\mathsf{S4}$*-stable.*

(3) *If* $\mathsf{L} = \mathsf{K4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$*, then* $\mathsf{S4} \vee \mathsf{L} = \mathsf{S4} + \{\gamma(\mathfrak{F}_i) \mid \mathfrak{F}_i = \mathfrak{F}_i^+\}$*.*

(4) *If* $\mathsf{L} = \mathsf{K4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$*, then* $\mathsf{L} \subseteq \mathsf{S4}$ *iff each* $\mathfrak{F}_i$ *contains an irreflexive point.*

Proof. (1). Observe that $\mathcal{V}(\mathsf{S4})$ is a $\mathcal{V}(\mathsf{K4})$-stable class and apply Proposition 3.10(3).

(2). By Theorem 4.7, the rooted L-spaces are $\mathsf{K4}$-stable. Therefore, the rooted $(\mathsf{S4} \vee \mathsf{L})$-spaces are $\mathsf{S4}$-stable. Thus, $\mathsf{S4} \vee \mathsf{L}$ is $\mathsf{S4}$-stable by Corollary 4.15.

(3). Let $\mathfrak{G}$ be a rooted $\mathsf{S4}$-space. We have $\mathfrak{G} \models \mathsf{S4} \vee \mathsf{L}$ iff $\mathfrak{G} \models \mathsf{L}$ iff $\mathfrak{G} \models \gamma(\mathfrak{F}_i)$ for all $i \in I$. It is obvious that $\mathfrak{G} \models \gamma(\mathfrak{F}_i)$ for every $\mathfrak{F}_i$ that contains an irreflexive point because no such $\mathfrak{F}_i$ can be a stable image of a reflexive space. Therefore, $\mathfrak{G} \models \gamma(\mathfrak{F}_i)$ for all $i \in I$ is equivalent to $\mathfrak{G} \models \gamma(\mathfrak{F}_i)$ for all $\mathfrak{F}_i$ with $\mathfrak{F}_i = \mathfrak{F}_i^+$. Thus, $\mathsf{S4} \vee \mathsf{L} = \mathsf{S4} + \{\gamma(\mathfrak{F}_i) \mid \mathfrak{F}_i = \mathfrak{F}_i^+\}$.

(4). First suppose that each $\mathfrak{F}_i$ contains an irreflexive point. Then $\mathfrak{F}_i \neq \mathfrak{F}_i^+$ for all $i \in I$. Therefore, (3) implies that $\mathsf{S4} \vee \mathsf{L} = \mathsf{S4}$, and hence $\mathsf{L} \subseteq \mathsf{S4}$. Conversely, suppose that some $\mathfrak{F}_i$ is reflexive. Since $\mathfrak{F}_i \not\models \mathsf{L}$ and $\mathfrak{F}_i$ is an $\mathsf{S4}$-frame, we see that $\mathsf{L} \not\subseteq \mathsf{S4}$. □

In the following table we summarize the main results of this section.

	τ	ρ	$\mathsf{S4} \vee -$	$(-)^+$
preserves stability	✓	✓	✓	✓
reflects stability	✓	-	-	✓
$\mathsf{IPC} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$	$\mathsf{S4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$	$\times$	$\times$	$\times$
$\mathsf{S4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$	$\times$	$\mathsf{IPC} + \{\gamma(\overline{\mathfrak{F}}_i) \mid i \in I\}$	$\times$	$\mathsf{K4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$
$\mathsf{K4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$	$\times$	$\times$	$\mathsf{S4} + \{\gamma(\mathfrak{F}_i) \mid \mathfrak{F}_i = \mathfrak{F}_i^+\}$	$\times$

"✓" means yes; "-" means no; "$\times$" means not applicable.

TABLE 1

- That τ preserves and reflects stability is the content of Corollary 5.3(1).
- That ρ preserves stability follows from Theorem 5.2(3). That ρ does not reflect stability follows from the fact that IPC is stable, S4.Grz is not S4-stable (see the next section), and that $\rho(\mathsf{S4.Grz}) = \mathsf{IPC}$.
- That $\mathsf{S4} \vee -$ preserves stability follows from Lemma 5.5(2). It does not reflect stability because $\mathsf{GL} \vee \mathsf{S4}$ is the inconsistent logic, which is S4-stable, but as we will see in the next section, GL is not K4-stable.
- That $(-)^+$ preserves stability follows from Lemma 5.4(3). It also reflects stability because $\mathsf{S4} \vee -$ preserves stability and for every normal extension M of S4 we have $\mathsf{S4} \vee \mathsf{M}^+ = \mathsf{M}$.
- The axiomatization results follow from Theorems 5.1(3) and 5.2(3) and Lemmas 5.5(3) and 5.4(2).

We conclude this section by showing that there are continuum many K4-stable and S4-stable logics.

Theorem 5.6.
(1) *There are continuum many* K4*-stable logics above* S4*.*
(2) *There are continuum many* K4*-stable logics between* K4 *and* S4*.*

Proof. (1). By [2, Thm. 6.13], there are continuum many stable si-logics. Since $\mathsf{L} \neq \mathsf{L}'$ implies $\tau\mathsf{L} \neq \tau\mathsf{L}'$, this together with Lemma 5.1 yields continuum many S4-stable logics above S4. By Lemma 5.5(1), these logics are also K4-stable. Thus, there are continuum many K4-stable logics above S4.

(2). Consider the sequence $\{\mathfrak{F}_n \mid n \in \mathbb{N}_{\geq 1}\}$, shown in Figure 3, where $\mathbb{N}_{\geq 1} = \{n \in \mathbb{N} \mid n \geq 1\}$. By [2, Lem. 6.12], $\mathfrak{F}_n$ is not a stable image of $\mathfrak{F}_m$ for $n \neq m$. We slightly modify the sequence. For $n \in \mathbb{N}_{\geq 1}$, let $\mathfrak{G}_n$ be the K4-frame that is obtained from $\mathfrak{F}_n$ by making x_1 irreflexive. The proof of [2, Lem. 6.12] shows that $\mathfrak{G}_n$ is not a stable image of $\mathfrak{G}_m$ for $n \neq m$.

For $I \subseteq \mathbb{N}_{\geq 1}$ let $\mathsf{L}_I = \mathsf{K4} + \{\gamma(\mathfrak{G}_n) \mid n \in I\}$. Since each $\mathfrak{G}_n$ has a reflexive root, by Proposition 4.13, every L_I is K4-stable. As each $\mathfrak{G}_n$ has a an irreflexive point, by Lemma 5.5(4), $\mathsf{L}_I \subseteq \mathsf{S4}$ for every $I \subseteq \mathbb{N}_{\geq 1}$. Thus, every L_I is a K4-stable logic between K4 and S4. Finally, if $n \in I \setminus J$, then $\gamma(\mathfrak{G}_n) \in \mathsf{L}_J \setminus \mathsf{L}_I$, so the cardinality of $\{\mathsf{L}_I \mid I \subseteq \mathbb{N}_{\geq 1}\}$ is that of continuum, completing the proof. □

6. EXAMPLES OF STABLE, K4-STABLE, AND S4-STABLE LOGICS

In this final section we will give many examples (and non-examples) of stable, K4-stable, and S4-stable logics. Moreover, we will look at the concept of stability from the model-theoretic perspective, especially in relation with Lyndon's theorem.[3]

[3]We are grateful to one of the referees for sharing his/her observations about the connection between stable logics and Lyndon's theorem, which led to the results in the first part of this section.

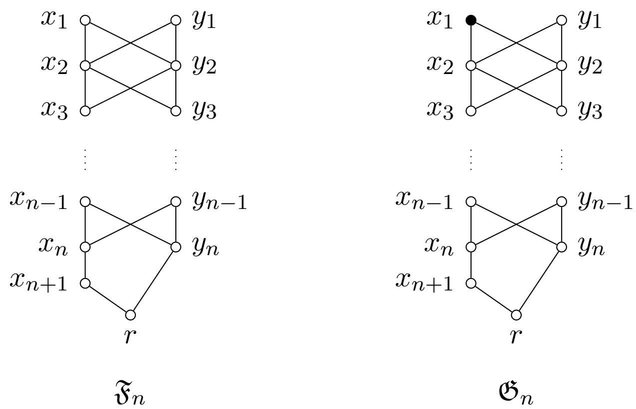


FIGURE 3

As we pointed out in the introduction, stable logics parallel subframe logics. It is well known (see, e.g., [13, Thm. 11.21]) that a normal extension of $\mathsf{K4}$ is a subframe logic iff it is the logic of a class of Kripke frames closed under subframes. We start by showing that a parallel result holds for stable logics, and more generally for M-stable logics when M admits filtration.

Proposition 6.1. *Let* L *and* M *be normal modal logics with* M *admitting filtration.*

(1) L *is stable iff* L *is the logic of a class of Kripke frames closed under stable images.*
(2) *If* $\mathsf{M} \subseteq \mathsf{L}$*, then* L *is* M*-stable iff* L *is the logic of an* M*-stable class of Kripke frames.*

Proof. We only show (1), the proof of (2) is an easy adaption. The left to right implication follows from Theorem 3.8. For the right to left implication, suppose L is the logic of a class $\mathcal{K}$ of Kripke frames closed under stable images. We show that the corresponding class $\mathbf{Cm}(\mathcal{K}) := \{\mathbf{Cm}(\mathfrak{F}) \mid \mathfrak{F} \in \mathcal{K}\}$ of complex algebras[4] is finitely stable. Let $\mathfrak{A} \in \mathbf{Cm}(\mathcal{K})$ and let $\mathfrak{B}$ be a finite stable subalgebra of $\mathfrak{A}$. Then $\mathfrak{A} = \mathbf{Cm}(\mathfrak{F})$ for some $\mathfrak{F} \in \mathcal{K}$ and $\mathfrak{B} = \mathbf{Cm}(\mathfrak{G})$ for some finite frame $\mathfrak{G}$. Since $\mathfrak{B}$ is a finite stable subalgebra of $\mathfrak{A}$, we see that $\mathfrak{G}$ is a finite stable image of $\mathfrak{F}$. As $\mathcal{K}$ is closed under stable images, $\mathfrak{G} \in \mathcal{K}$, and hence $\mathfrak{B} \in \mathbf{Cm}(\mathcal{K})$. Therefore, L is the logic of a finitely stable class of modal algebras. Thus, L is stable by Theorem 3.8. □

We recall that a first-order formula is *positive* if it is built from atomic formulas via the connectives $\wedge, \vee$ and quantifiers $\forall, \exists$. By Lyndon's theorem, a consistent first-order theory is preserved under homomorphisms iff it has a set of positive axioms (see, e.g., [14, Thm. 3.2.4]). For Kripke frames, homomorphisms correspond to stable maps. Therefore, from Lyndon's theorem and Proposition 6.1 we immediately obtain:

Corollary 6.2. *Suppose* L *and* M *are normal modal logics,* M *admits filtration and is characterized by a class* $\mathcal{C}$ *of Kripke frames.*

(1) *If* L *is the logic of a class of frames definable by positive formulas, then* L *is stable.*
(2) *If* L *is the logic of a class of frames definable by positive formulas within* $\mathcal{C}$*, then* L *is* M*-stable.*

Recall that a normal modal logic L is *elementary* if there is a set Ψ of first-order formulas such that L is the logic of the class of Kripke frames that validate all formulas in Ψ. It is known (see, e.g., [13, Thm. 11.26]) that a subframe logic L above $\mathsf{K4}$ is elementary iff L is the logic of a class of Kripke frames axiomatized by universal formulas.

[4] As usual, the *complex algebra* of a frame $\mathfrak{F} = (X, R)$ is the modal algebra $\mathbf{Cm}(\mathfrak{F}) = (\wp(X), R^{-1})$; see, e.g., [8, Def. 5.21].

Problem 2. Suppose L is a stable logic. Is L elementary iff L is the logic of a class of frames definable by positive formulas?

In relation to Problem 2, we do not even have an example of a stable logic (or a $\mathsf{K4}$-stable logic or an $\mathsf{S4}$-stable logic) which is not elementary. We point out that there are well-known examples of non-elementary subframe logics such as GL and $\mathsf{S4.Grz}$. As we will see in Theorem 6.9, these logics are not stable. Thus, we have the following open problem.

Problem 3. Is every stable logic ($\mathsf{K4}$-stable logic or $\mathsf{S4}$-stable logic) elementary?

Some examples of positive first-order formulas are:

reflexivity: $\forall x\,(xRx)$;
seriality: $\forall x \exists y\,(xRy)$;
universality: $\forall x \forall y\,(xRy)$;
every world sees a reflexive world: $\forall x \exists y\,(xRy \wedge yRy)$.

The logics of the corresponding classes of Kripke frames are:

$\mathsf{T} = \mathsf{K} + \Box p \to p$;
$\mathsf{D} = \mathsf{K} + \Box p \to \Diamond p$;
$\mathsf{S5} = \mathsf{T} + (\Box\Box p \to \Box p) + (p \to \Box\Diamond p)$;
$\mathsf{KMT} = \mathsf{K} + \{\Diamond\,((\Box p_1 \to p_1) \wedge \cdots \wedge (\Box p_n \to p_n)) \mid n \geq 1\}$.

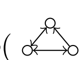
The logics T, D, and $\mathsf{S5}$ are well known, and KMT is discussed in [21]. Observe that all T-frames are reflexive and all D-frames are serial. In particular, both logics have the property that the class of all Kripke frames is first-order definable.[5] Since reflexivity and seriality are expressed by positive formulas, both T and D are stable logics by Corollary 6.2.

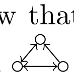
The case of $\mathsf{S5}$ is slightly different than that of T and D. On the one hand, having a universal relation is expressed by a positive first-order formula, so $\mathsf{S5}$ is the logic of a class of frames definable by a positive formula, and hence $\mathsf{S5}$ is stable. On the other hand, all $\mathsf{S5}$-frames do not form a stable class because equivalence relations are not preserved by stable images.

The logic KMT is yet of a different type. As shown in [21], KMT is the logic of the class of frames in which every world sees a reflexive world. However, not all KMT-frames satisfy this condition. In fact, it is shown in [21] that the class of all KMT-frames is not definable by any first-order formula. Still, it is proved in [21] that a Kripke frame is a KMT-frame iff the successors of any world form a non-finitely colorable subframe. This class is closed under stable images, and hence all KMT-frames form a stable class.

By [3, Thm. 8.3], T is axiomatized by the stable rules $\rho(\bullet)$ and $\rho(\bullet\!\leftrightarrow\!\circ)$, and D is axiomatized by the stable rules $\rho(\bullet)$ and $\rho(\bullet\!\leftrightarrow\!\circ)$. We next give axiomatizations of $\mathsf{S5}$ and KMT. As in the proof of Theorem 3.15, by $\mathfrak{C}_n$ we denote the irreflexive n-cluster, and by $\mathfrak{C}'_n$ the frame that arises by adding a strong root r_n to $\mathfrak{C}_n$ so that $x_i R r_n$ for all $2 \leq i \leq n$; in other words, the strong root r_n is seen by all elements of $\mathfrak{C}'_n$ except by x_1. Observe that x_1 does not see a reflexive world neither in $\mathfrak{C}_n$ nor in $\mathfrak{C}'_n$, and hence $\Psi := \forall x \exists y\,(xRy \wedge yRy)$ is refuted in both $\mathfrak{C}_n$ and $\mathfrak{C}'_n$.

Theorem 6.3.

(1) $\mathsf{S5}$ *is axiomatized by* $\Gamma := \{\rho(\bullet), \rho(\bullet\!\leftrightarrow\!\circ), \rho($  $), \rho($  $)\}$.
(2) KMT *is axiomatized by* $\Delta := \{\rho(\mathfrak{C}_n) \mid n \geq 1\} \cup \{\rho(\mathfrak{C}'_n) \mid n \geq 1\}$.

Proof. (1). First we show that a finite rooted frame validates Γ iff it is a cluster. Since none of the frames $\bullet$, $\bullet\!\leftrightarrow\!\circ$, , and  is a cluster, and hence neither is a stable image of a cluster, every finite cluster validates Γ. Conversely, suppose that $\mathfrak{F} = (X, R)$ is a finite rooted frame that is not a cluster. If $\mathfrak{F}$ is a singleton, then it must be irreflexive, so $\bullet$ is a stable image of $\mathfrak{F}$, and hence

[5]Logics axiomatizable by Sahlqvist formulas always have this property.

$\mathfrak{F} \not\models \rho(\bullet)$. Suppose that $\mathfrak{F}$ has at least two points. If $\mathfrak{F}$ contains an irreflexive point x, then $\bullet\!\leftrightarrow\!\circ$ is a stable image of $\mathfrak{F}$ as mapping x to the irreflexive point of $\bullet\!\leftrightarrow\!\circ$ and the rest to the reflexive point of $\bullet\!\leftrightarrow\!\circ$ is an onto stable map. Therefore, $\mathfrak{F} \not\models \rho(\bullet\!\leftrightarrow\!\circ)$. Suppose that $\mathfrak{F}$ is reflexive. If $\mathfrak{F}$ contains exactly two points x and y, then without loss of generality we may assume that xRy and $y\not R x$. Thus, mapping x to the root of $\substack{\circ\\ \uparrow\\ \circ}$ and y to the other point of $\substack{\circ\\ \uparrow\\ \circ}$ is stable and onto, and hence $\mathfrak{F} \not\models \rho(\substack{\circ\\ \uparrow\\ \circ})$. Suppose $\mathfrak{F}$ has at least three points. Since $\mathfrak{F}$ is not a cluster, without loss of generality we may assume that there are $x, y \in \mathfrak{F}$ with $x\not R y$. Then mapping x to the top node, y to the bottom right node, and all the other points to the bottom left node of $\overset{\circ}{\circ\!\leftrightarrow\!\circ}$ provides an onto stable map. This yields $\mathfrak{F} \not\models \rho(\overset{\circ}{\circ\!\leftrightarrow\!\circ})$.

Now, let L be the logic axiomatized over K by Γ. Since $\mathsf{S5}$ is the logic of finite clusters and each such validates Γ, we see that $\mathsf{L} \subseteq \mathsf{S5}$. Conversely, by Theorem 3.8, L is the logic of a stable class of finite rooted frames. Each such must be a cluster. Therefore, $\mathsf{S5} \subseteq \mathsf{L}$, and hence $\mathsf{S5}$ is axiomatized over K by Γ.

(2). First we show that a finite frame validates Δ iff it satisfies the positive formula Ψ. Suppose that the finite frame $\mathfrak{F}$ refutes Δ. Then there are $n \geq 1$ and a stable onto map $f : \mathfrak{F} \to \mathfrak{C}_n$ or a stable onto map $g : \mathfrak{F} \to \mathfrak{C}'_n$. Since $\mathfrak{C}_n$ and $\mathfrak{C}'_n$ refute Ψ, we conclude that $\mathfrak{F}$ refutes Ψ. For the converse, suppose $\mathfrak{F}$ refutes Ψ. Then $\mathfrak{F}$ has a node u_1 such that all successors of u_1 are irreflexive. Let $u_2, \dots, u_n$ be the successors of u_1. If $\mathfrak{F}$ consists only of $u_1, u_2, \dots u_n$, then define $f : \mathfrak{F} \to \mathfrak{C}_n$ by $f(u_i) = x_i$ for all $1 \leq i \leq n$. If $\mathfrak{F}$ contains at least one other node, then define $g : \mathfrak{F} \to \mathfrak{C}'_n$ by

$$g(x) = \begin{cases} x_i & \text{if } x = u_i \text{ for } 1 \leq i \leq n, \\ r_n & \text{otherwise.} \end{cases}$$

In both cases it is easy to see that the defined map is stable and onto. Thus, $\mathfrak{F}$ refutes Δ.

Let L be the normal modal logic axiomatized over K by Δ. It is shown in [21] that KMT has the fmp and a finite frame is a KMT-frame iff it satisfies Ψ. Therefore, a finite frame is a KMT-frame iff it validates Δ. Thus, since both KMT and L have the fmp and have the same finite frames, the two logics coincide. Consequently, KMT is axiomatized over K by Δ. $\square$

We next turn our attention to examples of $\mathsf{K4}$-stable logics. The examples will illustrate that $\mathsf{K4}$-stability is in a way "more frequent" than stability. Roughly speaking, the reason is that some first-order properties become positively definable modulo transitivity and rootedness.

We start by showing that $\mathsf{D4} := \mathsf{K4} \vee \mathsf{D}$, $\mathsf{S4} := \mathsf{K4} \vee \mathsf{T}$, and $\mathsf{K4B} := \mathsf{K4} + p \to \Box\Diamond p$ are $\mathsf{K4}$-stable logics. That $\mathsf{D4}$ and $\mathsf{S4}$ are $\mathsf{K4}$-stable can, for example, be inferred from the stability of D and T and Proposition 3.10. It is well known that $\mathsf{K4B}$ is the logic of symmetric $\mathsf{K4}$-frames. It is easy to see that this class is not preserved under stable images and hence is not definable by positive formulas. Nevertheless, $\mathsf{K4B}$ is characterized by the stable class of rooted frames satisfying $\forall xy\,(xRy) \vee \forall xy\,(x = y)$, and so $\mathsf{K4B}$ is a $\mathsf{K4}$-stable logic. Note that the additional condition of transitivity is not needed since the latter clause implies transitivity.

Theorem 6.4. *The following are axiomatizations of the* $\mathsf{K4}$*-stable logics* $\mathsf{D4}$*,* $\mathsf{S4}$*, and* $\mathsf{K4B}$ *in terms of stable formulas:*

(1) $\mathsf{D4} = \mathsf{K4} + \gamma(\bullet)$*;*
(2) $\mathsf{S4} = \mathsf{K4} + \gamma(\bullet) + \gamma(\substack{\circ\\ \uparrow\\ \bullet})$*;*
(3) $\mathsf{K4B} = \mathsf{K4} + \gamma(\substack{\circ\\ \uparrow\\ \circ})$*;*

Proof. (1). Let $\mathfrak{X}$ be a $\mathsf{K4}$-space. It is sufficient to show that $\mathfrak{X} \models \Box p \to \Diamond p$ iff $\mathfrak{X} \models \gamma(\bullet)$. If $\mathfrak{X} \not\models \Box p \to \Diamond p$, then there is $x \in X$ such that $x\not R y$ for all $y \in X$. Therefore, $\{x\}$ is a closed generated subframe of X, and $\mathfrak{Y} = (\{x\}, \varnothing)$ is a finite rooted $\mathsf{K4}$-frame. The unique map from $\mathfrak{Y}$ onto $\bullet$ is stable, and so we conclude that $\mathfrak{X} \not\models \gamma(\bullet)$. Conversely, suppose that $\mathfrak{X} \not\models \gamma(\bullet)$. Then there is a stable map from a topo-rooted closed generated subframe $\mathfrak{Y}$ of $\mathfrak{X}$ onto $\bullet$. This implies that $\mathfrak{Y}$ is a singleton with no R-successors, and hence $\mathfrak{X}$ contains a point with no R-successors. Thus, $\mathfrak{X} \not\models \Box p \to \Diamond p$.

(2). Let $\mathfrak{X}$ be a $\mathsf{K4}$-space. It is sufficient to show that $\mathfrak{X} \models p \to \Diamond p$ iff $\mathfrak{X} \models \gamma(\bullet), \gamma(\overset{\circ}{\underset{\bullet}{\uparrow}})$. Suppose $\mathfrak{X} \not\models \gamma(\bullet)$ or $\mathfrak{X} \not\models \gamma(\overset{\circ}{\underset{\bullet}{\uparrow}})$. Then there is a topo-rooted closed generated subframe $\mathfrak{Y}$ of $\mathfrak{X}$ and a stable map from $\mathfrak{Y}$ onto $\bullet$ or $\overset{\circ}{\underset{\bullet}{\uparrow}}$. Observe that under a stable map a preimage of an irreflexive point has to be irreflexive. Now both of the latter frames contain an irreflexive point, so in either case $\mathfrak{Y}$ contains an irreflexive point. Therefore, so does $\mathfrak{X}$. Thus, $\mathfrak{X}$ is not reflexive, and so $\mathfrak{X} \not\models p \to \Diamond p$. For the converse, suppose that x is an irreflexive point of $\mathfrak{X}$. Consider the closed generated subframe $Y := R^+[x]$ of $\mathfrak{X}$, and let $\mathfrak{Y}$ be the corresponding $\mathsf{K4}$-space. Clearly x is a unique root of $\mathfrak{Y}$. Since $x \notin R[x]$, there is a clopen subset of $\mathfrak{X}$ separating x from $R[x]$. Therefore, x is an isolated point of Y. Thus, $\mathfrak{Y}$ is topo-rooted. If $Y = \{x\}$, then the unique map from $\mathfrak{Y}$ onto $\bullet$ is stable, and so $\mathfrak{X} \not\models \gamma(\bullet)$. Otherwise mapping x to the root of $\overset{\circ}{\underset{\bullet}{\uparrow}}$ and the rest of Y to the top point of $\overset{\circ}{\underset{\bullet}{\uparrow}}$ gives rise to a stable map, and hence $\mathfrak{X} \not\models \gamma(\overset{\circ}{\underset{\bullet}{\uparrow}})$.

(3). Since $\mathsf{K4B}$ is a $\mathsf{K4}$-stable logic, it has the fmp. Also, since $\mathsf{K4} + \gamma(\overset{\circ}{\underset{\circ}{\uparrow}})$ is axiomatized over $\mathsf{K4}$ by the stable formula of a finite rooted $\mathsf{K4}$-frame with a reflexive root, it has the fmp by Proposition 4.13. Therefore, it is sufficient to show that for any finite rooted $\mathsf{K4}$-frame $\mathfrak{F} = (X, R)$, we have $\mathfrak{F} \models p \to \Box\Diamond p$ iff $\mathfrak{F} \models \gamma(\overset{\circ}{\underset{\circ}{\uparrow}})$. Suppose $\mathfrak{F} \not\models p \to \Box\Diamond p$. Then $\mathfrak{F}$ is not symmetric, and so there are $x, y \in X$ such that xRy but $y \not R x$. Define $f : \mathfrak{F} \to \overset{\circ}{\underset{\circ}{\uparrow}}$ by mapping $R^+[y]$ to the top node of $\overset{\circ}{\underset{\circ}{\uparrow}}$ and the rest to the root of $\overset{\circ}{\underset{\circ}{\uparrow}}$. It is easy to see that f is an onto stable map. Therefore, $\mathfrak{F} \not\models \overset{\circ}{\underset{\circ}{\uparrow}}$. Conversely, if $\mathfrak{F} \not\models \gamma(\overset{\circ}{\underset{\circ}{\uparrow}})$, then since $\mathfrak{F}$ is rooted, by Lemma 4.3, there is a stable map from $\mathfrak{F}$ onto $\overset{\circ}{\underset{\circ}{\uparrow}}$. Let x be a root of $\mathfrak{F}$ and let $y \in X$ be such that $f(y)$ is the top point of $\overset{\circ}{\underset{\circ}{\uparrow}}$. Since f is stable, xRy but $y \not R x$. Thus, $\mathfrak{F}$ is not symmetric. This yields that $\mathfrak{F} \not\models p \to \Box\Diamond p$. $\Box$

We next provide axiomatizations of some $\mathsf{S4}$-stable logics. Recall that $\mathsf{S4Alt}_n := \mathsf{S4} + \mathsf{alt}_n$, where

$$\mathsf{alt}_n := \Box p_1 \vee \Box(p_1 \to p_2) \vee \cdots \vee \Box(p_1 \wedge \cdots \wedge p_n \to p_{n+1}).$$

The $\mathsf{S4Alt}_n$-frames are the $\mathsf{S4}$-frames such that each point has $\leq n$ alternatives; that is,

$$\forall x x_1 \dots x_{n+1} \left(\bigwedge_{1 \leq i \leq n+1} x R x_i \to \bigvee_{1 \leq i < j \leq n+1} x_i = x_j \right).$$

Clearly this formula is not positive. It is not hard to see that this property is not preserved by stable maps, and hence is not definable by positive formulas. But the rooted $\mathsf{S4Alt_n}$-frames are characterized by the positive formula

$$\exists r \forall x\, (rRx) \wedge \forall x_1 \dots x_{n+1} \left(\bigvee_{1 \leq i < j \leq n+1} x_i = x_j \right),$$

implying that $\mathsf{S4Alt_n}$ is an $\mathsf{S4}$-stable logic.

Proposition 6.5. *The logics* $\mathsf{S5}$ *and* $\mathsf{S4Alt}_n$ *are* $\mathsf{S4}$*-stable. They are axiomatized over* $\mathsf{S4}$ *by the following stable formulas:*

(1) $\mathsf{S5} = \mathsf{S4} + \gamma(\overset{\circ}{\underset{\circ}{\uparrow}})$.
(2) $\mathsf{S4Alt}_n = \mathsf{S4} + \gamma(\boxed{\circ \cdots \circ})$.

Proof. (1). Since $\mathsf{S5} = \mathsf{S4} \vee \mathsf{K4B}$, this follows from Lemma 5.5 connecting $\mathsf{S4}$-stability and $\mathsf{K4}$-stability.

(2). Observe that there is a stable map from a finite rooted $\mathsf{S4}$-frame $\mathfrak{F}$ onto the $(n+1)$-cluster $\boxed{\circ \cdots \circ}$ iff the cardinality of $\mathfrak{F}$ is greater than n. The result follows since both $\mathsf{S4Alt}_n$ and $\mathsf{S4} + \gamma(\boxed{\circ \cdots \circ})$ have the fmp. $\Box$

We next consider the following normal extensions of $\mathsf{S4}$:

$\mathsf{S4.2} = \mathsf{S4} + \Diamond\Box p \to \Box\Diamond p$, the logic of directed $\mathsf{S4}$-frames;
$\mathsf{S4.3} = \mathsf{S4} + \Box(\Box p \to q) \vee \Box(\Box q \to p)$, the logic of upward connected $\mathsf{S4}$-frames;;
$\mathsf{S4BW}_n = \mathsf{S4} + \mathsf{bw}_n$, the logic of $\mathsf{S4}$-frames of width $\leq n$, where

$$\mathsf{bw}_n := \bigwedge_{i=0}^{n} \Diamond p_i \to \bigvee_{0\leq i\neq j\leq n} \Diamond(p_i \wedge \Diamond p_j);$$

$\mathsf{S4BTW}_n$, the logic of $\mathsf{S4}$-frames of top width $\leq n$.

The definitions of the first-order properties of $\mathsf{S4}$-frames mentioned above are:

(strong) directedness: $\forall xuv\,((xRu \wedge xRv) \to \exists y\,(uRy \wedge vRy))$.
(strong) upward connectedness: $\forall xuv\,((xRu \wedge xRv) \to (uRv \vee vRu))$.
bounded width: $\forall y \forall x_1 \dots x_{n+1} \left(\bigwedge_{1\leq i\leq n+1} yRx_i \to \bigvee_{1\leq i\neq j\leq n+1} x_iRx_j\right)$.
bounded top width:

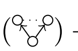
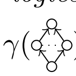
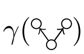
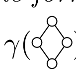
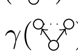
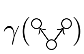
$$\forall y\forall x_1 \dots x_{n+1}\left(\bigwedge_{1\leq i\leq n+1} yRx_i \wedge \bigwedge_{1\leq i\leq n+1} \forall z(x_iRz \to zRx_i) \to \bigvee_{1\leq i\neq j\leq n+1} x_iRx_j\right).$$

Clearly none of these formulas is positive. It is not hard to see that none of the properties is preserved by stable maps, and hence is not definable by positive formulas. Nevertheless, these logics are $\mathsf{S4}$-stable. One way to see this is to look at their intuitionistic fragments.

$\mathsf{LC} = \mathsf{IPC} + (p \to q) \vee (q \to p)$, the Gödel-Dummett logic;
$\mathsf{KC} = \mathsf{IPC} + \neg p \vee \neg\neg p$, the logic of weak excluded middle;
$\mathsf{BW}_n = \mathsf{IPC} + \bigvee_{i=0}^{n}(p_i \to \bigvee_{j\neq i} p_j)$;
$\mathsf{BTW}_n = \mathsf{IPC} + \bigwedge_{0\leq i\leq j\leq n} \neg(\neg p_i \wedge \neg p_j) \to \bigvee_{i=0}^{n}(\neg p_i \to \bigvee_{j\neq i} \neg p_j)$.

We have that $\mathsf{S4.2} = \tau(\mathsf{LC})$, $\mathsf{S4.3} = \tau(\mathsf{KC})$, and more generally, $\mathsf{S4BW}_n = \tau(\mathsf{BW}_n)$ and $\mathsf{S4BTW}_n = \tau(\mathsf{BTW}_n)$ for every n. Lemma 5.1 together with the axiomatizations provided in [2, Thm. 7.5] then yields:

Proposition 6.6. *The logics* $\mathsf{S4.2}$ *and* $\mathsf{S4.3}$ *are* $\mathsf{S4}$*-stable. More generally,* $\mathsf{S4BW}_n$ *and* $\mathsf{S4BTW}_n$ *are* $\mathsf{S4}$*-stable for every* n*. These logics are axiomatized by the following stable formulas:*

(1) $\mathsf{S4BW}_n = \mathsf{S4} + \gamma($$) + \gamma($$)$. *In particular,* $\mathsf{S4.3} = \mathsf{S4} + \gamma($$) + \gamma($$)$.
(2) $\mathsf{S4BTW}_n = \mathsf{S4} + \gamma($$)$. *In particular,* $\mathsf{S4.2} = \mathsf{S4} + \gamma($$)$.

We define $\mathsf{K4.2} := \mathsf{S4.2}^+$, $\mathsf{K4.3} := \mathsf{S4.3}^+$, $\mathsf{K4BW}_n := (\mathsf{S4BW}_n)^+$, $\mathsf{K4BTW}_n := (\mathsf{S4BTW}_n)^+$, and $\mathsf{K4Alt}_n := (\mathsf{S4Alt}_n)^+$. Since for a $\mathsf{K4}$-frame $\mathfrak{F}$, we have $\mathfrak{F} \models \mathsf{L}$ iff $\mathfrak{F}^+ \models \mathsf{L}^+$, from the first-order characterizations of the corresponding logics above $\mathsf{S4}$, we obtain:

- $\mathfrak{F}$ is a $\mathsf{K4.2}$-frame iff $\forall xuv\,((xRu \wedge xRv \wedge u \neq v) \to (\exists y\,(uRy \wedge vRy) \vee uRv \vee vRv))$.
- $\mathfrak{F}$ is a $\mathsf{K4.3}$-frame iff $\forall xuv\,((xRu \wedge xRv \wedge u \neq v) \to (uRv \vee vRu))$.
- $\mathfrak{F}$ is a $\mathsf{K4BW}_n$-frame iff the width of $\mathfrak{F}$ is $\leq n$.
- $\mathfrak{F}$ is a $\mathsf{K4BTW}_n$-frame iff the top width of $\mathfrak{F}$ is $\leq n$.
- $\mathfrak{F}$ is a $\mathsf{K4Alt}_n$-frame iff

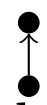
$$\forall xx_1 \dots x_{n+1}\left(\bigwedge_{1\leq i\leq n+1} xRx_i \to \left(\bigvee_{1\leq i<j\leq n+1} x_i = x_j \vee \bigvee_{1\leq i\leq n+1} x_i = x\right)\right).$$

Remark 6.7. In [13], the definitions of $\mathsf{K4.2}$ and $\mathsf{K4Alt}_n$ are slightly different. Namely, $\mathsf{K4.2}$ is defined as $\mathsf{K4} + \mathsf{dir}$ where $\mathsf{dir} = \Diamond(\Box p \wedge q) \to \Box(\Diamond p \vee q)$, and $\mathsf{K4Alt}_n$ is defined as $\mathsf{K4} + \mathsf{alt}_n$ for $n \geq 1$. The first-order condition corresponding to dir is

$$\forall xuv\,((xRu \wedge xRv \wedge u \neq v) \to \exists y\,(uRy \wedge vRy))$$

and the first-order condition corresponding to alt_n is

$$\forall x x_1 \ldots x_{n+1} \, (\bigwedge_{1 \le i \le n+1} xRx_i \to \bigvee_{1 \le i < j \le n+1} x_i = x_j).$$

If we define K4.2 and $\mathsf{K4Alt}_n$ as in [13], then it is no longer the case that $\mathsf{K4.2} = \mathsf{S4.2}^+$ and $\mathsf{K4Alt}_n = (\mathsf{S4Alt}_n)^+$. Moreover, these logics are not K4-stable. To see that $\mathsf{K4} + \mathsf{dir}$ is not K4-stable, observe that the K4-frame  validates all these logics but its stable image  refutes dir, yielding that $\mathsf{K4} + \mathsf{dir}$ is not K4-stable. The same example shows that $\mathsf{K4} + \mathsf{alt}_1$ is not K4-stable, and that $\mathsf{K4} + \mathsf{alt}_n$ is not K4-stable can be shown similarly. These facts and Proposition 6.8 below justify our usage of the names K4.2 and $\mathsf{K4Alt}_n$.

Clearly none of these formulas is positive. It is not hard to see that none of the properties is preserved by stable maps, and hence is not definable by positive formulas. In fact, the classes of transitive frames of the logics just described are not stable. Nevertheless, all these logics are K4-stable. One way to see this is that in all these cases the classes of their *transitive rooted frames* are definable by positive formulas:

- K4.2 is characterized by transitive frames satisfying

$$\exists r \forall x \, (r = x \vee rRx) \wedge \forall uv (\exists z (uRz \wedge vRz) \vee u = v \vee uRv \vee vRu).$$

- K4.3 is characterized by transitive frames satisfying

$$\exists r \forall x \, (r = x \vee rRx) \wedge \forall xy \, (x = y \vee xRy \vee yRx).$$

- $\mathsf{K4BW}_n$ is characterized by transitive frames satisfying

$$\exists r \forall x \, (r = x \vee rRx) \wedge \forall y \forall x_1 \ldots x_{n+1} \left(\bigvee_{1 \le i \ne j \le n+1} x_i R x_j \vee \bigvee_{1 \le i \le n+1} x_i = r \right).$$

- $\mathsf{K4BTW}_n$ is characterized by transitive frames satisfying

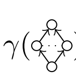
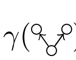
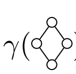
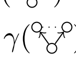
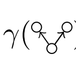
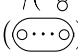
$$\exists r \forall x \, (r = x \vee rRx) \wedge \exists m_1, \ldots m_n \left(\forall y \bigvee_{1 \le i \le n} (yRm_i \vee y = m_i) \right).$$

- $\mathsf{K4Alt}_n$ is characterized by transitive frames satisfying

$$\exists r \forall x \, (r = x \vee rRx) \wedge \forall x_1 \ldots x_{n+1} \left(\bigvee_{1 \le i < j \le n+1} x_i = x_j \vee x_i = r \right).$$

Since $\mathsf{K4.2} = \mathsf{S4.2}^+$, $\mathsf{K4.3} = \mathsf{S4.3}^+$, $\mathsf{K4BW}_n = (\mathsf{S4BW}_n)^+$, $\mathsf{K4BTW}_n = (\mathsf{S4BTW}_n)^+$, and $\mathsf{K4Alt}_n = (\mathsf{S4Alt}_n)^+$, from Proposition 6.6 and Lemma 5.4 we conclude:

Proposition 6.8.

(1) $\mathsf{K4BW}_n = \mathsf{K4} + \gamma($$) + \gamma($$)$. *In particular,* $\mathsf{K4.3} = \mathsf{K4} + \gamma($$) + \gamma($$)$.
(2) $\mathsf{K4BTW}_n = \mathsf{K4} + \gamma($$)$. *In particular,* $\mathsf{K4.2} = \mathsf{K4} + \gamma($$)$.
(3) $\mathsf{K4Alt}_n = \mathsf{K4} + \gamma($$)$.

In the following table we summarize the axiomatizations of K4-stable and S4-stable logics obtained above.

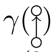
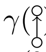
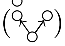
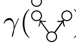
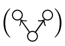
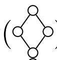
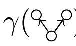
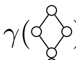
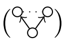
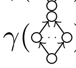
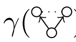
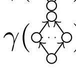
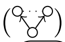
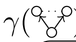
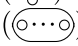
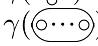
D4	=	$\mathsf{K4} + \gamma(\bullet)$	S4	=	$\mathsf{K4} + \gamma(\bullet) + \gamma($$)$
K4B	=	$\mathsf{K4} + \gamma($$)$	S5	=	$\mathsf{S4} + \gamma($$)$
K4.2	=	$\mathsf{K4} + \gamma($$)$	S4.2	=	$\mathsf{S4} + \gamma($$)$
K4.3	=	$\mathsf{K4} + \gamma($$) + \gamma($$)$	S4.3	=	$\mathsf{S4} + \gamma($$) + \gamma($$)$
$\mathsf{K4BW}_n$	=	$\mathsf{K4} + \gamma($$) + \gamma($$)$	$\mathsf{S4BW}_n$	=	$\mathsf{S4} + \gamma($$) + \gamma($$)$
$\mathsf{K4BTW}_n$	=	$\mathsf{K4} + \gamma($$)$	$\mathsf{S4BTW}_n$	=	$\mathsf{S4} + \gamma($$)$
$\mathsf{K4Alt}_n$	=	$\mathsf{K4} + \gamma($$)$	$\mathsf{S4Alt}_n$	=	$\mathsf{S4} + \gamma($$)$

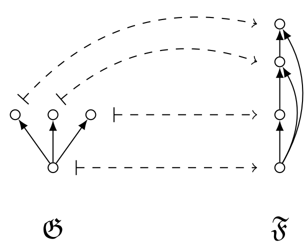
TABLE 2. Axiomatizations of some K4-stable and S4-stable logics

Next, as promised, we show that several well-known logics are not stable. We point out that to prove that a given logic L is not stable it is not sufficient to show that the class of all finite L-frames is not stable. The difficulty is in proving that L is not characterized by *any* stable class of finite L-frames. Consider the following well-known logics (see, e.g., [13, p. 116]):

- $\mathsf{KB} = \mathsf{K} + p \to \Box\Diamond p$, the logic of symmetric frames;
- $\mathsf{K5} = \mathsf{K} + \Diamond\Box p \to \Box p$, the logic of Euclidean frames;
- $\mathsf{GL} = \mathsf{K4} + \Box(\Box p \to p) \to \Box p$, the logic of dually well-founded K4-frames;
- $\mathsf{S4.Grz} = \mathsf{S4} + \Box(\Box(p \to \Box p) \to p) \to p$, the logic of Noetherian S4-frames;
- $\mathsf{K4.1} = \mathsf{K4.1} + \Box\Diamond p \to \Diamond\Box p$, the logic of K4-frames with degenerate final clusters;
- $\mathsf{S4.1} = \mathsf{S4} \vee \mathsf{K4.1}$, the logic of S4-frames with degenerate final clusters.

Theorem 6.9. *None of the logics* K4, S4, KB, *and* K5 *is stable. Neither are the logics* GL, S4.Grz, K4.1, *and* S4.1. *In fact,* GL *and* K4.1 *are not* K4*-stable and* S4.Grz *and* S4.1 *are neither* K4*-stable nor* S4*-stable.*

Proof. We start by showing that K4 is not stable. If K4 were stable, then by Theorem 3.8, there would exist a stable class $\mathcal{K}$ of finite rooted K4-frames whose logic is K4. Consider the finite rooted frames $\mathfrak{F}, \mathfrak{G}$ and an onto stable map $\mathfrak{G} \twoheadrightarrow \mathfrak{F}$ shown below.

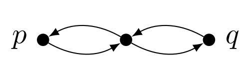


Note that $\mathfrak{G}$ is transitive, but $\mathfrak{F}$ is not. Since $\mathfrak{G}$ is a K4-frame and $\mathfrak{G} \not\models \gamma(\mathfrak{G})$, we see that $\mathsf{K4} \not\models \gamma(\mathfrak{G})$. Therefore, there is $\mathfrak{H} \in \mathcal{K}$ such that $\mathfrak{H} \not\models \gamma(\mathfrak{G})$. As $\mathfrak{G}$ has a reflexive root, by Proposition 4.13(1), $\mathfrak{G}$ is a stable image of $\mathfrak{H}$. Thus, since $\mathcal{K}$ is stable, $\mathfrak{G} \in \mathcal{K}$. The same reasoning yields $\mathfrak{F} \in \mathcal{K}$. But this is a contradiction as $\mathfrak{F}$ is not transitive. Consequently, K4 is not a stable logic.

A similar reasoning gives that S4 is not a stable logic. We next show that KB is not a stable logic. If it were, then by Theorem 3.8, there would exist a stable class $\mathcal{K}$ of finite rooted KB-frames whose logic is KB.

Claim 6.10. *There is $\mathfrak{F} \in \mathcal{K}$ containing distinct x, y that are not R-related to each other.*

Proof. Clearly the KB-model



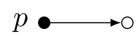
refutes $\mathsf{bw}_1 = \Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond^+ q) \vee \Diamond(q \wedge \Diamond^+ p)$. Therefore, $\mathsf{KB} \nvdash \mathsf{bw}_1$. Thus, there is $\mathfrak{F} \in \mathcal{K}$ such that $\mathfrak{F} \not\models \mathsf{bw}_1$. It is easy to see that $\mathfrak{F}$ has the desired property. □

For such an $\mathfrak{F} = (X, R)$ define $\mathfrak{F}' = (X, R')$, where $R' = R \cup \{(x, y)\}$. Then the identity map is a stable map from $\mathfrak{F}$ onto $\mathfrak{F}'$. Since $\mathcal{K}$ is stable, $\mathfrak{F}' \in \mathcal{K}$. But this is a contradiction as $\mathfrak{F}'$ is not symmetric. Thus, KB is not a stable logic.

Next we show that $\mathsf{K5}$ is not a stable logic. If $\mathsf{K5}$ were stable, then there would be a stable class $\mathcal{K}$ of finite rooted $\mathsf{K5}$-frames whose logic is $\mathsf{K5}$.

Claim 6.11. *There is $\mathfrak{F} \in \mathcal{K}$ containing x, y such that xRy and $x \not R x$.*

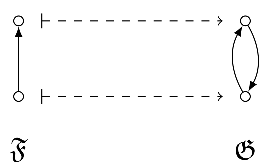
Proof. Clearly the $\mathsf{K5}$-model



refutes the formula $\varphi := p \to \Diamond p \vee \Box\bot$. Therefore, $\mathsf{K5} \nvdash \varphi$. Thus, there is $\mathfrak{F} \in \mathcal{K}$ such that $\mathfrak{F} \not\models \varphi$. It is easy to see that $\mathfrak{F}$ has the desired property. □

For such an $\mathfrak{F} = (X, R)$ define $\mathfrak{F}' = (X, R')$, where $R' = R \cup \{(y, x)\}$. Then the identity map is a stable map from $\mathfrak{F}$ onto $\mathfrak{F}'$. Since $\mathcal{K}$ is stable, $\mathfrak{F}' \in \mathcal{K}$. But this is a contradiction as $\mathfrak{F}'$ is not Euclidean because in an Euclidean frame every successor is reflexive. Thus, $\mathsf{K5}$ is not a stable logic.

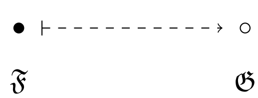
Next we show that $\mathsf{S4.Grz}$ is not a stable logic. By Proposition 3.10(1), it is sufficient to show that $\mathsf{S4.Grz}$ is not $\mathsf{S4}$-stable. It is easy to see that the map $\mathfrak{F} \twoheadrightarrow \mathfrak{G}$ between finite rooted $\mathsf{S4}$-frames depicted below is stable.



Note that $\mathfrak{F}$ is a $\mathsf{S4.Grz}$-frame, while $\mathfrak{G}$ is not. Therefore, by Corollary 4.15(6), $\mathsf{S4.Grz}$ is not $\mathsf{S4}$-stable. Thus, by Lemma 5.5(1), $\mathsf{S4.Grz}$ is not $\mathsf{K4}$-stable.

The same argument yields that $\mathsf{S4.1}$ is not $\mathsf{S4}$-stable. Therefore, by Lemma 5.5(1), $\mathsf{S4.1}$ is not $\mathsf{K4}$-stable. Since $\mathsf{S4.1} = \mathsf{S4} \vee \mathsf{K4.1}$, Lemma 5.5(2) yields that $\mathsf{K4.1}$ is not $\mathsf{K4}$-stable. Thus, neither $\mathsf{S4.1}$ nor $\mathsf{K4.1}$ is stable by Proposition 3.10(1).

Finally, we show that GL is not stable. For this it is sufficient to show that GL is not $\mathsf{K4}$-stable. It is easy to see that the map depicted below is a stable map from a finite rooted GL-frame $\mathfrak{F}$ onto a finite rooted $\mathsf{K4}$-frame $\mathfrak{G}$, which is not a GL-frame.



The rest of the argument is the same as in the case of $\mathsf{S4.Grz}$. □

We conclude the paper by providing examples that show that the classes of $\mathsf{K4}$-stable logics, transitive subframe, cofinal subframe, and union-splitting logics (these classes of logics are discussed in detail in [13, Sec. 10.5 and 11.3]) are all different.

	transitive subframe	**transitive cofinal subframe**	K4-**stable**	S4-**stable**	**union** K4-**splitting**	**union** S4-**splitting**
S4.2	-	✓	✓	✓	✓	✓
S4.Grz	✓	✓	-	-	✓	✓
GL	✓	✓	-	×	-	×
τL	-	-	✓	✓	✓	✓
$\mathsf{K4BTW}_3$	-	✓	✓	×	-	×
$\mathsf{S4BTW}_3$	-	✓	✓	✓	-	-

"✓" means the logic belongs to the class; "-" means the logic does not belong to the class; "×" means not applicable.

TABLE 3

- By Proposition 6.6, S4.2 is S4-stable. Therefore, by Lemma 5.5(1), S4.2 is K4-stable. It is well known that S4.2 is S4-splitting (see, e.g., [29]). Since S4 is a union K4-splitting, it follows that S4.2 is a union K4-splitting. Finally, it is well known that S4.2 is a cofinal subframe logic (see, e.g., [13, Sec. 9.4]), and it is easy to see that S4.2 is not a subframe logic.
- By Theorem 6.9, S4.Grz is neither S4-stable nor K4-stable. On the other hand, it is well known that S4.Grz is a subframe logic (see, e.g., [13, Sec. 9.4]). Therefore, S4.Grz is a cofinal subframe logic. Finally, it is well known that S4.Grz is a union S4-splitting (see, e.g., [12, Exm. 1.11]). Thus, S4.Grz is a union K4-splitting.
- By Theorem 6.9, GL is not K4-stable, and it is well known that GL is not a union K4-splitting (see, e.g., [13, Exe. 9.13]). On the other hand, it is well known that GL is a subframe logic (see, e.g., [13, Sec. 9.4]). Thus, GL is a cofinal subframe logic.
- It was shown in [4] that there is a stable si-logic L which is not a cofinal subframe logic. Therefore, neither is τL. Thus, τL is not a subframe logic. By Lemma 5.1, τL is S4-stable. Since L is a tabular logic, it is a union splitting si-logic (see, e.g., [6, Thm. 3.4.27]). By [13, Cor. 9.64], τL is a union S4-splitting logic, hence a union K4-splitting logic.
- It is easy to see that neither $\mathsf{S4BTW}_3$ nor $\mathsf{K4BTW}_3$ is a subframe logic. It follows from [13, Sec. 9.4 and Cor. 9.64] that $\mathsf{S4BTW}_3$ is a cofinal subframe logic. Since $\mathsf{K4BTW}_3 = \mathsf{S4BTW}_3{}^+$, it follows that $\mathsf{K4BTW}_3$ is a cofinal subframe logic. An adaptation of the proof of [13, Prop. 9.50] shows that $\mathsf{K4BTW}_3$ is not a union K4-splitting logic and $\mathsf{S4BTW}_3$ is not a union S4-splitting logic. On the other hand, by Proposition 6.6, $\mathsf{S4BTW}_3$ is S4-stable, and by Proposition 6.8, $\mathsf{K4BTW}_3$ is K4-stable.

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