THE BOUNDED PROOF PROPERTY VIA STEP ALGEBRAS AND STEP FRAMES

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Abstract. We develop a semantic criterion for a specific rule-based calculus Ax axiomatizing a given logic L to have the so-called bounded proof property. This property is a kind of an analytic subformula property limiting the proof search space. Our main tools are one-step frames and one-step algebras. These structures were used in [23], [11] to construct free algebras of modal logics via coalgebraic methods. In this paper, we use one-step algebras and one-step frames to investigating proof-theoretic aspects of modal logics.

We define conservative one-step frames and prove that every finite conservative one-step frame for Ax is a p-morphic image of a finite Kripke frame for L iff Ax has the bounded proof property and L has the finite model property. This result, combined with a 'one-step version' of the classical correspondence theory, turns out to be quite powerful in case studies. For simple logics such as **K**, **T**, **K4**, **S4**, etc, establishing basic metatheoretical properties becomes a completely automatic task (the related proof obligations can be instantaneously discharged by current first-order provers). For more complicated logics, some ingenuity is still needed, however we were able to successfully apply our uniform method to Avron's cut-free system for **GL**, to Goré's cut-free system for **S4.3**, and to Ohnishi-Matsumoto's analytic system for **S5**.

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1. Introduction

We revisit the method of describing free algebras of modal logics by approximating them with finite partial algebras. This construction is longstanding, but here we apply it in a different context. We exploit it to investigate proof-theoretic aspects of modal logics. The key points of the method are that every free algebra is approximated by partial algebras of formulas of modal complexity n, for $n \in \omega$, and that dual spaces of these approximants can be described explicitly [1][1], [26]. In a sense, the basic idea of this construction can be traced back to [24]. In [25] this method was applied to free Heyting algebras. In recent years there has been a renewed interest towards this construction e.g., [9], [11], [12], [23], [27].

In this paper[2] we study proof-theoretic consequences of this method for axiomatic systems of modal logic. In particular, we will concentrate on the *bounded proof*

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[1] The original talk was given at the BCTCS in 1988.

[2] This paper is an extended version of [10].

property. An axiomatic system Ax has the bounded proof property (the bpp, for short) if every formula ϕ of modal complexity at most n derived in Ax from some set Γ containing only formulae of modal complexity at most n, can be derived from Γ in Ax by only using formulae of modal complexity at most n. The bounded proof property is a kind of an analytic subformula property limiting the proof search space. It is an indicator of a robustness of a proof system, and hence is desirable to have. This property holds for proof systems enjoying the subformula property (the latter is a property that usually follows from cut elimination). The bounded proof property depends on an axiomatization of a logical system. That is, one axiomatization of a logic may have the bpp and the other not. The examples of such axiomatizations will be given in Section 9 of the paper.

Main tools of our method are one-step frames introduced in [23] and [11]. A one-step frame is a two-sorted structure which admits interpretations of modal formulae without nested modal operators. We show that an axiomatic system Ax axiomatizing a logic L has the bpp and the finite model property (the fmp) iff every one step-frame validating Ax is a p-morphic image of a finite Kripke frame for L. This gives a purely semantic characterization of the bpp. The main advantage of this criterion is that it is relatively easy to verify. In Section 1.1 below, we give an example explaining the details of our machinery step-by-step. Here we just list the main ingredients.

Given an axiom of a modal logic, we rewrite it into a one-step rule, that is, a rule of modal complexity 1. One-step rules can be interpreted on one-step frames. We use an analogue of the classical correspondence theory, to obtain a first-order condition (or a condition of first-order logic enriched with fixed-point operators) for a one-step frame corresponding to the one-step rule. Finally, we need to find a standard frame p-morphically mapped onto any finite one-step frame satisfying this first-order condition. This part is not automatic, but we have some standard templates. For example, we define a procedure modifying the relation of a one-step frame so that the obtained frame is standard (Kripke). In easy cases, e.g., for modal logics such as **K**, **T**, **K4**, **S4**, this frame is a frame of the logic and is p-morphically mapped onto the one-step frame. The bpp and fmp for these logics follow by our criterion. For more complicated systems such as **S4.3**, **S5** and **GL**, we show that the rules that we automatically obtain from some standard axiomatizations are not good – we prove that these axiomatizations do not have the bpp, and, by our criterion, do not admit cut elimination. However, we also show using our method, that Avron's cut-free rules for **GL** [2], Goré's cut-free rules for **S4.3** [30], and Ohnishi-Matsumoto's analytic rules for **S5** [38] provide axiomatic systems having the bpp.

In order to explain the basic idea of our technique, we proceed by giving a rather simple (but still significant) example.

1.1. **A worked out example.** Consider the modal logic obtained by adding to the basic normal modal system **K** the 'density' axiom:

$$\Box\Box x \to \Box x. \tag{1}$$

First Step: we replace (1) by equivalent derived rules having modal complexity 1. The obvious solution is to replace the modalized subformulae occurring inside the modal operator by an extra propositional variable. Thus the first candidate is the rule $y \leftrightarrow \Box x / \Box y \to \Box x$. A better solution (suggested by the proof of Proposition 3) is to take advantage of the monotonicity and to use instead the rule

$$\frac{y \to \Box x}{\Box y \to \Box x} \tag{2}$$

Often, the method suggested by the proof of Proposition 3 gives 'good' rules, but for more complicated logics one needs some ingenuity to find the right system of derived rules replacing the axioms (this is substantially the kind of ingenuity needed to find rules admitting cut-elimination).

Second Step: this step may or may not succeed, but it is entirely algorithmic. It relies on a light modification of the well-known modal correspondence machinery. We first observe that inference rules having modal complexity 1 can be interpreted in the so-called *one-step frames*. A one-step frame is a quadruple $\mathcal{S} = (W_1, W_0, f, R)$, where W_0, W_1 are sets, $f : W_1 \to W_0$ is a map and $R \subseteq W_1 \times W_0$ is a relation between W_1 and W_0. In the applications, we need two further requirements (called *conservativity* requirements) on such a one-step frame $\mathcal{S}$: for the purpose of the present discussion, we may ignore the second requirement and keep only the first one, which is just the surjectivity of f.[3] Formulae of modal complexity 1 (i.e. without nested modal operators) can be interpreted in one-step frames as follows: propositional variables are interpreted as subsets of W_0; when we apply modal operators to subsets of W_0, we produce subsets of W_1 using the modal operator $\Box_R$ canonically induced by R. In particular, for $y \subseteq W_0$ the operator $\Box_R$ is defined as $\Box_R y = \{w \in W_1 : R(w) \subseteq U\}$, where $R(w) = \{v \in W_0 : (w, v) \in R\}$, Whenever we need to compare, say y and $\Box_R x$, we apply the inverse image f (denoted by f^*) to y in order to obtain a subset of W_1. Thus, a one-step frame $\mathcal{S} = (W_1, W_0, f, R)$ validates (2) iff we have

$$\forall x, y \subseteq W_0 \ (f^*(y) \subseteq \Box_R x \Rightarrow \Box_R y \subseteq \Box_R x).$$

The standard correspondence machinery for Sahlqvist formulae shows that in the two-sorted language of one-step frames this condition has the following first-order equivalent:

$$\forall w \forall v \ (wRv \Rightarrow \exists k \ (wRf(k) \ \& \ kRv)). \tag{3}$$

In relational composition notation this becomes $R \subseteq R \circ f^o \circ R$, where f^o is the binary relation such that $wf^o v$ if $f(w) = v$. We may call (3) the *step-density* condition. In fact, notice that for standard frames, where we have $W_1 = W_0$ and $f = id$, step-density condition becomes the customary density condition, see (6) below.

Third Step: our main result states that both the finite model property and bounded proof property (for the global consequence relation) are guaranteed provided we are able to show that *any finite conservative one-step frame validating our inference rules is a p-morphic image of a standard finite frame also validating these rules*. The formal definition of a p-morphic image for one-step frames will be given in Section 6 (see Definition 10). Here we content ourselves to observing that, in our case, in order to apply the above result and obtain the finite model and bounded proof properties, we need to prove that, given a conservative finite step-dense frame $\mathcal{S} = (W_1, W_0, f, R)$, there are a finite dense frame $\mathfrak{F} = (V, S)$ and a surjective map $\mu : V \longrightarrow W_1$ such that $R \circ \mu = f \circ \mu \circ S$. In concrete examples,

[3] The second requirement is needed to state our results in full generality, but seems not be used in the applications.

the idea is to take $V := W_1$ and $\mu := id_{W_1}$. So the whole task reduces to that of finding $S \subseteq W_1 \times W_1$ such that $R = f \circ S$. That is, S should satisfy

$$\forall w \forall v\ (wRv \ \Leftrightarrow\ \exists w'\ (wSw'\ \&\ f(w') = v)). \tag{4}$$

Some ingenuity is needed in the general case to find the appropriate S (indeed our problem looks quite similar to the problem of finding appropriate filtrations case-by-case). As in the case of filtrations, there are standard templates that often work for the cases of arbitrary relations, transitive relations, etc. The basic template for the case of an arbitrary relation is that of taking S to be $f^o \circ R$, namely

$$\forall w \forall w'\ (wSw' \Leftrightarrow \exists v\ (wRv\ \&\ f(w') = v)). \tag{5}$$

Notice that what we need to prove in the end is that, assuming (3), the so-defined S satisfies (4) and (6), where

$$\forall w \forall v\ (wSv \Rightarrow \exists k\ (kSv\ \&\ wSk)). \tag{6}$$

Thus, taking into consideration that f is also surjective, i.e.,

$$\forall v\ \exists w\ f(w) = v, \tag{7}$$

(because $\mathcal{S}$ is conservative), we need the validity of the implication

$$(7)\ \&\ (3)\ \&\ (5)\ \Rightarrow\ (4)\ \&\ (6).$$

The latter is a deduction problem in first-order logic that can be solved affirmatively along the lines indicated in Section 8 below. The problem can efficiently discharged by provers like SPASS, E, Vampire.[4]

In summary, the above is a purely algorithmic procedure, that may or may not succeed (in case it does not suceed, one may try to invent better solutions for the derived rules of Step 1 and/or for the definition of the relation S in Step 3). In case the procedure suceeds, we really obtain quite a lot of information about our logic, because we get altogether: (i) completeness via finite model property; (ii) decidability; (iii) bounded proof property; (iv) first-order definability; (v) canonicity (as a consequence of (i)+(iv), via known results in modal logic). Further applications concern the step-by-step descriptions of finitely generated free algebras (following the lines of [23] and [11]). But we will not deal with free algebras in this paper.

The large amount of information that one can obtain from successful runs of the method might suggest that the event of success is quite rare. This is true in essence, but we shall see in the paper that (besides simple systems such as $\mathbf{K}, \mathbf{T}, \mathbf{K4}, \mathbf{S4}$) the procedure can be successfully applied to more interesting case studies such as the linear system $\mathbf{S4.3}$, epistemic system $\mathbf{S5}$ and the Gödel-Löb system $\mathbf{GL}$. In the case of $\mathbf{GL}$ we have definability not in first-order logic, but in first-order logic enriched with fixed-point operators. However, for finite one-step frames (as for finite standard $\mathbf{GL}$-frames), this condition boils down to a first-order condition.

[4] SPASS `http://www.spass-prover.org/` (in the default configuration) took less than half a second to solve the above problem: it derived 118 clauses, backtracked 23 clauses, performed 2 splits and kept 88 clauses; the proof produced has depth 5, length 47.

1.2. **Related Work.** Remarkable work has been conducted in the recent literature on proof theory of non classical logics. The common effort of such work is to give intrinsic algebraic and semantic characterizations of classical proof theoretic properties such as admissibility of cut-elimination and subformula properties. These properties are crucial because they usually supply decidability and complexity results as a corollary of appropriately designing the reasoning systems. The origin of our work is orthogonal to this research line, but there is certainly an overlap. As a general observation, before going into more details, we must say that our techniques have more restricted target (we say little about the concrete design of proof systems and the bpp itself is a rather weak form of analyticity, see Section 3 below for a comparison to analogous notions from the literature). However, at the same time, we point out that our techniques look robust and are widely applicable. This is because they are almost independent on syntactic presentations, in the sense that the only syntactic feature they are sensible to is the modal complexity of the inference rules. Our success in the case studies we present in Section 9 (and in many other examples the readers can easily build by themselves) should give evidence to these merits of our methodology. Here we analyse three different recent research lines concerning structural investigation on basic proof theoretic properties.

1. An algebraic approach to cut elimination via MacNeille completions for the full Lambek calculus **FL** was developed in [16–18]. The results are impressive (they relate cut-elimination with preservation under McNeille completions), but they cover just the lowest levels $\mathcal{N}_2, \mathcal{P}_3$ of the so-called substructural hierarchy. Notice also that full Lambek calculus is a non-distributive substructural logic and that the above mentioned class of axioms can include only some intermediate logics. The framework of our paper is different, as we work with modal logics which are distributive and extend classical logic. In addition, our technique is based not on completions, but on (finite) duality between distributive algebras and frames.

2. The paper [4] characterizes analyticity (and cut-elimination) from a semantic point of view and as such might be related to our research perspective. However, the semantic framework introduced in [4] is quite peculiar because it is syntax-dependent: in fact, the **G**-legal frames of [4] are defined in terms of a specific calculus **G** (not just in terms of a specific modal language). **G**-legal $\mathcal{E}$-semiframes are also defined in terms of a set of formulae $\mathcal{E}$. The semantic characterization of analyticity so-obtained seems to work well for basic standard modal systems but not, for instance, for **GL**.

3. Papers [34–36] contain interesting achievements from the syntactic side and as such might complement our work. Translations and rule saturation transformations are investigated so to produce out of Hilbert-style axiomatizations contextual sequent calculi enjoying cut and contraction admissibility properties. The method is quite successful for modal rank 1 axioms, but occasionally also for more complex logics.

1.3. **Organization of the paper.** The paper is organized as follows: In Section 2 we recall the basic definitions of logics and decision problems. In Section 3 we introduce derivable rules, reduced rules and the bounded proof property. Section 4 discusses conservative one-step modal algebras and one-step frames. In Section 5 we define diagrams of one-step modal algebras and show that diagrams encode the embeddings of one-step modal algebras. Section 6 gives a semantic criterion for a proof system to enjoy the bounded proof property and finite model property. In

Section 7 we discuss the correspondence theory for one-step frames. In Section 8 we give a few simple illustrative examples of our approach. Finally, in Section 9 we discuss in detail more sophisticated case studies concerning the modal systems **S4.3**, **GL**, **S5**.

2. Logics and Decision Problems

Modal formulae are built up from propositional variables $x, y, \dots$ by using the Booleans $(\neg, \wedge, \vee, \to, 0, 1)$ and modal operators $(\Diamond, \Box)$. For simplicity, we take $\neg$, $\wedge$, $\Diamond$ as primitive connectives, the remaining ones being defined in the customary way (in particular, $\Box$ is defined as $\neg\Diamond\neg$). We shall also use parameters $a, b, \dots$ instead of variables whenever we want to stress that uniform substitution does not apply to them. Underlined letters stand for tuples of unspecified length and formed by distinct elements, thus for instance, we may use $\underline{x}$ for a tuple such as $x_1, \dots, x_n$. When we write $\phi(\underline{x})$ we want to stress that ϕ contains at most the variables $\underline{x}$ (and no parameters) and similarly when we write $\phi(\underline{a})$ we want to stress that ϕ contains at most the parameters $\underline{a}$ (and no variables). The same convention applies to sets of formulae: if Γ is a set of formulae and we write $\Gamma(\underline{a})$, we mean that all formulae in Γ are of the kind $\phi(\underline{a})$, etc. We may occasionally replace variables with parameters in a formula: for this, we use the following self-explanatory notation. For a formula $\phi(\underline{x})$ we write $\phi(\underline{a})$ to mean that $\phi(\underline{a})$ is obtained from $\phi(\underline{x})$ by replacing $\underline{x} = x_1, \dots, x_n$ (simultaneously and respectively) by $\underline{a} = a_1, \dots, a_n$.

The modal complexity of a formula ϕ counts the maximum number of nested modal operators in ϕ (the precise definition is by an obvious induction). The polarity (positive/negative) of an occurrence of a subformula in a formula ϕ is defined inductively: ϕ is positive in ϕ, the polarity is conserved through all connectives, except $\neg$ that reverses it. When we say that a propositional variable is positive (negative) in ϕ we mean that all its occurrences are such.

A *logic* is a set of modal formulae containing tautologies, Aristotle's principle (namely $\Box(x \to y) \to (\Box x \to \Box y)$) and closed under uniform substitution, modus ponens and necessitation (namely $\phi/\Box\phi$) rules.

Logics are in bijective correspondence with varieties of modal algebras. We recall the related definitions here, see e.g., [13, Sec. 5.2] or [14, Sec. 7.6] for more details. A *modal algebra* $\mathfrak{A} = (A, \Diamond)$ is a Boolean algebra A endowed with a unary operator $\Diamond$ satisfying $\Diamond(x \vee y) = \Diamond x \vee \Diamond y, \Diamond 0 = 0$. Notice that, here and elsewhere, we use the same name for a connective and the corresponding operator in modal algebras (thus, for instance, 0 is zero, $\vee$ is join, etc.). In this way, propositional formulae can be identified with *terms* in the first order language of modal algebras. Thus, the well-known bijective correspondence between logics L and varieties V of modal algebras can be stated as follows: (i) to a logic L, one can associate the variety $V(L) = \{\mathfrak{A} \mid \text{for all } \phi \in L,\ \mathfrak{A} \models \forall \underline{x}\, \phi(\underline{x}) = 1\}$; (ii) to a variety V, we associate the logic $L(V) = \{\phi(\underline{x}) \mid \text{for all } \mathfrak{A} \in V,\ \mathfrak{A} \models \forall \underline{x}\, \phi(\underline{x}) = 1\}$.

From the semantic side, we have the notion of a frame; a *frame* $\mathfrak{F} = (W, R)$ is a set W endowed with a binary relation R. The *dual* of the frame $\mathfrak{F} = (W, R)$ is the modal algebra $\mathfrak{F}^* = (\wp(W), \Diamond_R)$, where $\wp(W)$ is the powerset Boolean algebra and $\Diamond_R$ is the semilattice morphism associated with R. The latter is defined as follows: for $A \subseteq W$, we have $\Diamond_R(A) = \{w \in W \mid R(w) \cap A \neq \emptyset\}$ (recall that $R(w)$ denotes $\{v \in W \mid (w, v) \in R\}$). Definitions regarding modal algebras can be shifted to frames by taking duals: for instance, we say that ψ is valid in $\mathfrak{F}$ iff

$\mathfrak{F}^* \models \forall \underline{x}\, \phi(\underline{x}) = 1$, that $\mathfrak{F}$ validates a logic L (or that $\mathfrak{F}$ *is a frame for* L) iff $\mathfrak{F}^*$ validates all the formulae in L, etc. It should be noticed that there is a real duality (in the categorical sense) between modal algebras and frames only if we restrict to finite modal algebras and finite frames. If we want a full duality working for arbitrary modal algebras, we must introduce some topological structures on our frames (see, e.g., [13, Sec. 5.5], [14, Sec. 7.5], [32, Ch. 4] or [41]). For the purposes of this paper, however, the duality between finite frames and finite modal algebras will suffice.

The kind of decision problems we are interested in for a logic L is the *global consequence relation decision problem.* This can be formulated as follows: given parameters $\underline{a}$, a finite set $\Gamma(\underline{a}) = \{\phi_1(\underline{a}), \dots, \phi_n(\underline{a})\}$ of propositional formulae and given a formula $\psi(\underline{a})$, decide whether we have $\Gamma \vdash_L \psi$, where the notation $\Gamma \vdash_L \psi$ means that there is a proof of ψ using tautologies, Aristotle's principle and the formulae in Γ, as well as the necessitation, modus ponens and the axioms of L (notice that uniform substitutions cannot be applied to formulae in Γ), see e.g., [32, Sec. 3.1]. In terms of the variety associated with a logic, the global consequence relation problem can be rephrased as follows.

Proposition 1. *Given a logic L, for any finite $\Gamma(\underline{a}), \psi(\underline{a})$ as above, we have that $\Gamma \vdash_L \psi$ holds iff for all $\mathfrak{A} \in V(L)$ we have*

$$\mathfrak{A} \models \forall \underline{x}\ (\bigwedge\{\phi(\underline{x}) = 1 \mid \phi \in \Gamma\} \Rightarrow \psi(\underline{x}){=}1). \tag{8}$$

Proof. We just sketch the main idea of the proof. First notice that in the quasi-equation (8), the parameters have been replaced by variables that are universally quantified. One side of the proposition is proved by induction on the length of a proof; the other side, by the standard Lindembaum algebra-like construction. For this one introduces the equivalence relation $\approx$ on formulas such that $\psi_1(\underline{a}) \approx \psi_2(\underline{a})$ holds iff $\Gamma \vdash_L \psi_1 \leftrightarrow \psi_2$. The set of formulae of the kind $\psi(\underline{a})$ are quotiented by $\approx$, and operations are defined on the representatives of equivalence classes. Finally, assuming $\Gamma \not\vdash_L \psi$, one obtains an algebra $\mathfrak{A} \in V(L)$, where the quasi-equation (8) fails. ⊣

3. Inference Rules

In proof theory, logics are specified via axiomatic systems consisting of inference rules (axioms are viewed as 0-premises rules). However, different axiomatic systems may specify the same logic, as will follow from the notion of a derivable rule introduced below. Our goal is to recognize axiomatic systems having suitable proof-theoretic properties. To this aim, we make a preliminary investigation about inference rules.

Formally, an *inference rule* is an $n+1$-tuple of formulae, written in the form

$$\frac{\phi_1(\underline{x}), \dots, \phi_n(\underline{x})}{\psi(\underline{x}).} \tag{9}$$

The ϕ's are the premises and ψ is the consequence of the rule. An *axiomatic system* Ax is a set of inference rules. We write $\vdash_{\mathrm{Ax}} \phi$ to mean that ϕ has a proof using tautologies and Aristotle's principle as well as modus ponens, necessitation and inferences from Ax. When we say that a proof uses an inference rule such as (9), we mean that the proof can introduce at any step i a formula of the kind $\psi\sigma$ provided it already introduced in the previous steps $j_1, \dots, j_n < i$ the formulae

$\phi_1\sigma, \dots, \phi_n\sigma$, respectively. Here σ is a substitution and notation $\psi\sigma$ means the application of the substitution σ to ψ.

Given parameters $\underline{a}$, a finite set $\Gamma = \{\phi_1(\underline{a}), \dots, \phi_n(\underline{a})\}$, an inference system Ax and a formula $\psi(\underline{a})$, we write $\Gamma \vdash_{\mathrm{Ax}} \psi$ to mean that ψ has a proof using tautologies, Aristotle's principle and formulae from Γ as well as modus ponens, necessitation and inferences from Ax (notice that uniform substitution cannot be applied to to members of Γ).

A modal algebra $\mathfrak{A}$ validates the rule (9) iff

$$\mathfrak{A} \models \forall \underline{x}\ (\phi_1(\underline{x})\!=\!1\ \&\cdots\&\ \phi_n(\underline{x})\!=\!1 \Rightarrow \psi(\underline{x})\!=\!1) \tag{10}$$

and it validates an axiomatic system iff it validates all inference rules in it.

We need some care when dealing with inference rules. In fact, it is clear from the above definitions that the class of algebras validating an axiom system is only a quasi-variety, whereas we are mainly interested in varieties (i.e., in logics). We need to introduce a notion recognizing equivalent axiomatic systems. This will allow us to safely limit ourselves, in case we are interested in a specific logic L, only to axiomatic systems that are equivalent to L. The notion of equivalence we require should be well behaved with respect to the decision problem (aka global consequence relation) we are interested in, in the sense that we must have $\Gamma \vdash_{\mathrm{Ax}} \psi$ iff $\Gamma \vdash_{\mathrm{Ax}'} \psi$ whenever Ax and Ax$'$ are equivalent. The key concept leading to the appropriate notion of equivalence is that of a *derivable rule*.[5] This is an inference rule satisfying one of the equivalent conditions of the next proposition.

Proposition 2. *Let* Ax *be an axiomatic system. The following conditions are equivalent for an inference rule* $\phi_1(\underline{x}), \dots, \phi_n(\underline{x})/\psi(\underline{x})$.

(i) *Every modal algebra validating* Ax *validates also* $\phi_1(\underline{x}), \dots, \phi_n(\underline{x})/\psi(\underline{x})$;
(ii) $\{\phi_1(\underline{a}), \dots, \phi_n(\underline{a})\} \vdash_{\mathrm{Ax}} \psi(\underline{a})$.

Proof. This proposition is essentially an extension of Proposition 1 from variaties to quasi-varieties and it is proved in the same way. The direction from (i) to (ii) is via a Lindembaum-like algebra construction (we take all formulae of the kind $\theta(\underline{a})$ and divide them by the equivalence relation induced by the relation $\{\phi_1(\underline{a}), \dots, \phi_n(\underline{a})\} \vdash_{\mathrm{Ax}} \theta(\underline{a}) \leftrightarrow \theta'(\underline{a})$, etc.). The direction from (ii) to (i) is a validity statement to be checked by induction on the length of a proof: what has to be checked is that, if we are given a modal algebra $\mathfrak{A}$ validating Ax and elements from its support interpreting the parameters $\underline{a}$ in such a way that $\phi_1(\underline{a}) = 1, \dots, \phi_n(\underline{a}) = 1$ hold, then whenever we have $\{\phi_1(\underline{a}), \dots, \phi_n(\underline{a})\} \vdash_{\mathrm{Ax}} \theta(\underline{a})$ we also have that $\theta(\underline{a}) = 1$ holds. Once this is shown, applying it to the special case where θ is ψ, we obtain that $\mathfrak{A}$ validates $\phi_1(\underline{x}), \dots, \phi_n(\underline{x})/\psi(\underline{x})$ (because the elements assigned to the $\underline{x}$ are arbitrary elements from the support of $\mathfrak{A}$). $\dashv$

Remark 1. In case Ax is a set of axioms (i.e. a set of 0-premises rules) one can add a further characterization to (i)-(ii) (which is in fact a deduction theorem):

(iii) *There exists* $m \geq 0$, $k_1, \dots, k_m \geq 0$ *and* $\phi_{i_1}, \dots, \phi_{i_m} \in \{\phi_1(\underline{x}), \dots, \phi_n(\underline{x})\}$ *such that* $\vdash_{\mathrm{Ax}} \Box^{k_1}\phi_{i_1} \wedge \cdots \wedge \Box^{k_m}\phi_{i_m} \to \psi$.

This characterization is useful in concrete examples to check derivability of inference rules with respect to a given logic.

[5]In contrast to derivable rules, admissible rules [40] are not appropriate, because they might have impact on global consequence relation. To see why this can be the case, recall that admissible rules are validated only by free algebras.

We say that two axiomatic systems Ax and Ax$'$ are *equivalent* iff all inference rules from Ax are derivable in Ax$'$ and vice versa. We say that Ax *axiomatizes* L iff Ax is equivalent to L (as an axiomatic system). From the previous proposition, we obtain that the above notion of equivalence is correct for our decision problems.

Corollary 1. *If* Ax *is equivalent to* Ax$'$*, then we have* $\Gamma(\underline{a}) \vdash_{\mathrm{Ax}} \psi(\underline{a})$ *iff* $\Gamma(\underline{a}) \vdash_{\mathrm{Ax}'} \psi(\underline{a})$ *for all* $\underline{a}, \Gamma(\underline{a}), \psi(\underline{a})$.

Proof. By Proposition 2, $\Gamma(\underline{a}) \vdash_{\mathrm{Ax}} \psi(\underline{a})$ is equivalent to the fact that the rule $\forall \underline{x}\ (\bigwedge \Gamma(\underline{x}) \to \phi(\underline{x}))$ is validated in all the algebras validating the rules from Ax. The same holds for Ax$'$ and then the claim is clear because Ax and Ax$'$ are equivalent. ⊣

Now we are going to discuss the assumptions we can freely make on a given axiomatic system.

Definition 1. *We say that the inference rule* (9) *is* reduced *iff (i) the formulae* $\phi_1, \ldots, \phi_n, \psi$ *have modal complexity at most 1; (ii) every propositional variable occuring in* (9) *has an occurrence within the scope of a modal operator. An axiomatic system is* reduced *iff all inference rules in it are reduced.*

Requirement (ii) above is useful in order to make more intuitive the definition of interpretation of a reduced axiomatic system into a one-step algebra. On the other hand, the proof of Proposition 3 below shows that propositional variables violating requirement (ii) can be dropped, so that requirement (ii) is also formally justified.

Proposition 3. *Every axiomatic system is equivalent to a reduced axiomatic system.*

Proof. We show how to replace every rule (9) by one or more reduced rules so as to obtain an equivalent axiomatic system. Take a formula α having modal complexity at least one and take an occurrence of it located inside a modal operator in (9). We can obtain an equivalent rule by replacing this occurrence by a new propositional variable y and by adding as a further premise $\alpha \to y$ (resp. $y \to \alpha$) if the occurrence of α is positive within ψ or negative within one of the ϕ_i's (resp. if the occurrence of α is negative within ψ or positive within one of the ϕ_i's). Continuing in this way, in the end, only formulae of modal complexity at most 1 will occur in the rule. To check the equivalence of so-obtained axiomatic systems, use Proposition 2(i).

Now suppose that a propositional variable x does not occur inside a modal operator in (9). By rewriting into the conjunctive normal form and separating conjuncts (we can split premises or the whole rule, depending on whether conjunctions are in the premises or in the conclusion), we can assume that premises and conclusions are disjunctions of propositional variables and of formulae whose main connective is a modal operator. Now, if x does not occur inside a modal operator, premises and conclusion containing it must be of the form $x \to \alpha$ or $\beta \to x$, where x does not occur in α, β. But then x can be easily eliminated: for instance, observe that $x \to \alpha, \beta \to x / x \to \gamma$ is equivalent, as a rule, to $\beta \to \alpha / \alpha \to \gamma$.[6] ⊣

Based on the above proposition, *from now on we shall consider only reduced inference rules and reduced axiomatic systems.*

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[6]There is also another (trivial and not informative) way to force requirement (ii): if x does not occur inside a modal operator in (9), just add to the rule a further premise like $\Box(x \vee \neg x)$.

Remark 2. The transformation of an axiomatic system into a reduced one can be done in different ways. The way we suggested in the proof of Proposition 3 seems to be well behaved in the examples (in the sense that it helps the application of the Ackermann rules in the correspondence phase, see Section 7 below). To obtain even better results, whenever possible, it is convenient to abstract out an occurrence of a subformula α from the conclusion only in case α has another occurrence in the conclusion of the rule (in this way, one can hope to get in the end a rule which is suitable for proving the subformula property in case cut-elimination holds). Notice that the method we suggested in the proof of Proposition 3 is *non deterministic*, so it might be applied in different ways with possibly different outcomes. We also point out that further improvements should be investigated for syntactic transformations of axiomatic systems. In particular, one could try to apply a *saturation* process to the current set of rules: the proof-theory oriented literature [16,17,34–36] developed interesting techniques for this. These techniques were originally tailored to design sequent systems where rules such as cut and contraction are admissible, but could be profitably imported in our context too.

The global consequence relation $\Gamma \vdash_{\mathrm{Ax}} \phi$ does not depend on the axiomatic system Ax chosen for a given logic L. Indeed, $\Gamma \vdash_{\mathrm{Ax}} \phi$ holds iff $\Gamma \vdash_L \phi$ holds for any axiomatic system equivalent to L, see Corollary 1. However, deciding $\Gamma \vdash_{\mathrm{Ax}} \phi$ might be easier for 'nicer' axiomatic systems. In particular, the bounded proof property below may hold only for some 'nice' axiomatic systems equivalent to a logic L.

When we write $\Gamma \vdash^n_{\mathrm{Ax}} \phi$ we mean that ϕ can be proved from Ax, Γ (in the above sense) by using a proof in which only formulae of modal complexity at most n occur.

Definition 2. *We say that* Ax *has the* bounded proof property *(bpp, for short) iff for every formula ϕ of modal complexity at most n and for every Γ containing only formulae of modal complexity at most n, we have*

$$\Gamma \vdash_{\mathrm{Ax}} \phi \quad \Rightarrow \quad \Gamma \vdash^n_{\mathrm{Ax}} \phi.$$

It should be clear that the bpp for a finite axiom system Ax equivalent to L implies the decidability of the global consequence relation for L. This is because we have a bounded search space for formulae occurring in a possible proof and for possible substitutions instantiating our rules: in fact, there are only finitely many non provably equivalent formulae containing at most a given finite set of parameters and with modal complexity bounded by a given n (notice that in a proof witnessing $\Gamma(\underline{a}) \vdash^n_{\mathrm{Ax}} \phi(\underline{a})$ we can freely suppose that only the paramters $\underline{a}$—and no variables—occur, because extra parameters or variables can be uniformly replaced by, say, 0).

Remark 3. The bpp is similar in spirit to the analyticity [4] and to the pseudo-analyticity [34] properties considered in the literature; however, it is different than those properties because the class of formulae allowed to occur in a restricted shape proof witnessing $\Gamma \vdash_{\mathrm{Ax}} \phi$ is not the same. According to the bpp, if $\Gamma(\underline{a}), \phi(\underline{a})$ have modal complexity n, *any* formula $\psi(\underline{a})$ of modal complexity at most n is allowed to occur in such a proof, whereas just subformulae of Γ, ϕ (according to the analitic restriction) or Boxed Boolean combinations of subformulae of Γ, ϕ (according to the pseudo-analytic restriction) are allowed to occur there.

The following proposition shows that we can limit our consideration to formulae of complexity 1 when checking the bpp.

Proposition 4. Ax *has the bounded proof property iff for every formula* ϕ *of modal complexity at most* 1 *and for every* Γ *containing only formulae of modal complexity at most* 1*, we have*

$$\Gamma \vdash_{\mathrm{Ax}} \phi \quad \Rightarrow \quad \Gamma \vdash^1_{\mathrm{Ax}} \phi.$$

Proof. For the purpose of this proof only, we need to apply substitutions to parameters too. To avoid confusion, we shall call such substitutions *replacements*. Formally a replacement σ is a map associating with a_i a formula $\sigma(a_i)$ (for $i = 1, \dots, m$). We denote by $\theta\sigma$ the result of the simulataneous replacement in θ of a_i by $\sigma(a_i)$. We define the modal complexity of σ to be the maximum of the complexities of the formulae $\sigma(a_i)$. Notice as a general fact that

(*) if σ has modal complexity at most k and ϕ has modal complexity at most l, then $\phi\sigma$ has modal complexity at most $k + l$.

Given our Γ, ϕ of modal complexities at most n, we define $\Gamma_i, \phi_i, \sigma_i$ $(0 \leq i \leq n-1)$ such that: (i) σ_i has modal complexity at most 1 and Γ_i, ψ_i have modal complexities at most $n - i$; (ii) $\phi_{i+1}\sigma_{i+1} = \phi_i$; (iii) $\Gamma_{i+1}\sigma_{i+1}$ is equal to the union of Γ_i with some tautologies of modal complexity at most 1; (iv) $\Gamma_{i+1} \vdash_{\mathrm{Ax}} \phi_{i+1}$ iff $\Gamma_i \vdash_{\mathrm{Ax}} \phi_i$.

We let Γ_0 be Γ, ϕ_0 be ϕ and σ_0 be the identity replacement. To define the $i + 1$-th data, consider all subformulae of the kind $\Diamond\psi$ occurring in Γ_i, ϕ_i, where ψ has complexity 0. For each such subformula, pick a fresh parameter a_ψ, replace everywhere $\Diamond\psi$ by a_ψ in Γ_i, ϕ_i. Then add $a_\psi \leftrightarrow \Diamond\psi$ to Γ_i and let σ_{i+1} be given by $\{a_\psi \mapsto \Diamond\psi\}_\psi$. Hence Properties (i)-(iv) hold.

Now suppose that $\Gamma \vdash_{\mathrm{Ax}} \phi$. Then we have $\Gamma_{n-1} \vdash_{\mathrm{Ax}} \phi_{n-1}$ by (iv) and also $\Gamma_{n-1} \vdash^1_{\mathrm{Ax}} \phi_{n-1}$ by (i) and the hypothesis of the proposition. Next, if we apply σ_{n-1} to the proof certifying $\Gamma_{n-1} \vdash^1_{\mathrm{Ax}} \phi_{n-1}$, by (ii)-(iii) and (*), we obtain $\Gamma_{n-2} \vdash^2_{\mathrm{Ax}} \phi_{n-2}$. Repeating this for $\sigma_{n-1}, \dots, \sigma_1$, we finally obtain $\Gamma \vdash^n_{\mathrm{Ax}} \phi$. $\dashv$

In the following, we shall adopt the equivalent formulation of the bpp suggested by the above proposition. We shall call finite sets $\Gamma(\underline{a})$ of formulae of modal complexity at most 1, *finite presentations*.

Remark 4. We do not consider complexity problems in this paper. However, we underline that the transformation outlined in the proof of Proposition 4 produces a global consequence relation problem whose length is *linear* in terms of the original problem. This is because the length of each Γ_i, ϕ_i increases the length of the previous Γ_{i-1}, ϕ_{i-1} by at most 4 symbols (counting $\leftrightarrow$ as a single symbol). Thus, although it may seem that a decision procedure based on the bpp requires non-elementary space search bound, this is not true: the space bound can be lowered to an elementary bound if we apply the above mentioned transformation.[7]

Remark 5. One may think that the necessitation rule does not play a prominent role in a proof witnessing $\Gamma \vdash^1_{\mathrm{Ax}} \phi$, but this is not the case at all. Suppose that Γ contains $\Box a_1 \leftrightarrow a_2, \Box a_2 \leftrightarrow a_3, \dots$. Then, if it happens that we deduce a_1, we can obtain also $\Box a_1$, then a_2, $\Box a_2$, a_3, $\Box a_3$, etc. Thus, the necessitation rule can have a prominent role even though the modal complexity of the formulae managed by the proof remains very low.

[7] Counting the number of non-equivalent formulae of modal complexity at most 1 in a given number of propositional variables [26], we obtain in this way a triple exponential bound for the search space of formulae occurring in a possible proof. This bound is still far from optimal when considering concrete systems (we believe it can be improved by further refining the complexity analysis).

4. Step algebras and step frames

The aim of this section is to supply an algebraic and a semantic framework for investigating proofs and formulae of modal complexity at most 1.

4.1. Conservative one-step algebras and one-step frames.

We first recall the definition of one-step modal algebras and one-step frames from [23] and [11], and define conservative one-step modal algebras and one-step frames.

Definition 3. *A* one-step modal algebra *is a quadruple* $\mathcal{A} = (A_0, A_1, i_0, \Diamond_0)$*, where* A_0, A_1 *are Boolean algebras,* $i_0 : A_0 \to A_1$ *is a Boolean morphism, and* $\Diamond_0 : A_0 \to A_1$ *is a semilattice morphism (i.e., it preserves only* $0, \vee$*). The algebras* A_0, A_1 *are called the* source *and the* target *Boolean algebras of the one-step modal algebra* $\mathcal{A}$*. We say that* $\mathcal{A}$ *is* conservative *iff* i_0 *is injective and the union of the images* $i_0(A_0) \cup \Diamond(A_0)$ *generates* A_1 *as a Boolean algebra.*

From the dual semantic point of view we have the following:

Definition 4. *A* one-step frame *is a quadruple* $\mathcal{S} = (W_1, W_0, f, R)$*, where* W_0, W_1 *are sets,* $f : W_1 \to W_0$ *is a map and* $R \subseteq W_1 \times W_0$ *is a relation between* W_1 *and* W_0*. We say that* $\mathcal{S}$ *is* conservative *iff* f *is surjective and the following condition is satisfied for all* $w_1, w_2 \in W_1$*:*

$$f(w_1) = f(w_2) \ \& \ R(w_1) = R(w_2) \ \Rightarrow \ w_1 = w_2. \tag{11}$$

Above, similarly to the case of Kripke frames, we used the notation $R(w_1)$ to mean the set $\{v \in W_0 \mid (w_1, v) \in R\}$ (and similarly for $R(w_2)$). The dual of a finite one-step frame $\mathcal{S} = (W_1, W_0, f, R)$ is the one-step modal algebra $\mathcal{S}^* = (\wp(W_0), \wp(W_1), f^*, \Diamond_R)$, where f^* is the inverse image operation and $\Diamond_R$ is the semilattice morphism associated with R. The latter is defined as follows: for $A \subseteq W_0$, we have $\Diamond_R(A) = \{w \in W_1 \mid R(w) \cap A \neq \emptyset\}$. Conservativity also carries over from one-step frames to one-step modal algebras.

Proposition 5. *A finite one-step frame* $\mathcal{S}$ *is conservative iff its dual one-step modal algebra* $\mathcal{S}^*$ *is conservative.*

Proof. The following is an easily established fact concerning any finite Boolean algebra $\wp(X)$: a family of subsets $G \subseteq \wp(X)$ generates $\wp(X)$ as a Boolean algebra iff the following holds for $w_1, w_2 \in X$

$$\forall g \in G \ (w_1 \in g \Leftrightarrow w_2 \in g) \Rightarrow w_1 = w_2.$$

If we now apply this to the family

$$\{f^*(v) \mid v \in Y\} \cup \{\Box_R(Y \setminus \{v\}) \mid v \in Y\}$$

we obtain precisely the statement of the lemma (here, of course, $\Box_R$ is taken to be $\neg\Diamond_R\neg$). $\dashv$

To complete our list of definitions, let us observe that a one-step modal algebra $\mathcal{A} = (A_0, A_1, i_0, \Diamond_0)$ in which we have $A_0 = A_1$ and $i_0 = id$ is nothing but a modal algebra. Similarly, a one-step frame $\mathcal{S} = (W_1, W_0, f, R)$ where we have $W_0 = W_1$ and $f = id$ is just a frame. For clarity, we shall sometimes call modal algebras and frames *standard* or *plain* modal algebras and frames, respectively.

4.2. **Inference Validation in Step Algebras.** We spell out what it means for a one-step modal algebra and a one-step frame to validate an axiomatic system Ax and a finite presentation Γ. Notice that only formulae of modal complexity at most 1 are involved, according to our conventions from Section 3.

Let us fix a finite set of variables $\underline{x} = x_1, \ldots, x_n$ and a finite set of parameters $\underline{a} = a_1, \ldots, a_m$ (either $\underline{x}$ or $\underline{a}$ can be empty). An $\underline{a}$-*augmented* one-step modal algebra $\mathcal{A} = (A_0, A_1, i_0, \Diamond_0, \underline{\mathtt{a}})$ is a one-step modal algebra together with displayed elements $\underline{\mathtt{a}} = \mathtt{a_1}, \ldots, \mathtt{a_m} \in A_0$ (these elements will interpret parameters).

Given an $\underline{a}$-augmented one-step modal algebra as above, an $\mathcal{A}$-*valuation* is a map associating with each variable x an element $\mathtt{v}(x) \in A_0$. For every formula ϕ of complexity 0, we define $\phi^{\mathtt{v}0} \in A_0$ as follows:

$$x^{\mathtt{v}0} = \mathtt{v}(x) \text{ (for every variable } x); \qquad a_i^{\mathtt{v}0} = \mathtt{a_i} \ (a_i \in \underline{a});$$
$$(\phi * \psi)^{\mathtt{v}0} = \phi^{\mathtt{v}0} * \psi^{\mathtt{v}0} \ (* = \wedge, \vee); \qquad (\neg\phi)^{\mathtt{v}0} = \neg(\phi^{\mathtt{v}0}).$$

For every formula ϕ of complexity 0, we define $\phi^{\mathtt{v}1} \in A_1$ as $i_0(\phi^{\mathtt{v}0})$. For ϕ of complexity 1, $\phi^{\mathtt{v}1} \in A_1$ is defined as follows:

$$(\Diamond\phi)^{\mathtt{v}1} = \Diamond(\phi^{\mathtt{v}0}); \qquad (\phi * \psi)^{\mathtt{v}1} = \phi^{\mathtt{v}1} * \psi^{\mathtt{v}1} \ (* = \wedge, \vee); \qquad (\neg\phi)^{\mathtt{v}1} = \neg(\phi^{\mathtt{v}1}).$$

Definition 5. *We say that* $\mathcal{A}$ validates the inference rule (9) *iff for every* $\mathcal{A}$-*valuation* $\mathtt{v}$,[8] *we have that if* ($\phi_1^{\mathtt{v}1} = 1$ *and* $\cdots$ *and* $\phi_n^{\mathtt{v}1} = 1$)*, then* $\psi^{\mathtt{v}1} = 1$*. We say that* $\mathcal{A}$ validates an axiomatic system Ax *(written* $\mathcal{A} \models \mathrm{Ax}$*) iff* $\mathcal{A}$ *validates all inferences from* Ax*.*

Notice that it might well be that $\mathrm{Ax}_1, \mathrm{Ax}_2$ both axiomatize the same logic L, but that only one of them is validated by a given $\mathcal{A}$. This phenomenon, however, cannot happen in case $\mathcal{A}$ is standard (i.e., it is a modal algebra).

For formulae $\phi(\underline{a})$ where the variables $\underline{x}$ do not occur, the valuation $\mathtt{v}$ is not relevant. Thus, in such cases, we may write $\phi^{\underline{\mathtt{a}}0}, \phi^{\underline{\mathtt{a}}1}$ instead of $\phi^{\mathtt{v}0}, \phi^{\mathtt{v}1}$, respectively, to stress the fact that the augmentation $\underline{\mathtt{a}}$ is the essential part of the definition. We write $\mathcal{A} \models \phi(\underline{a})$ for $\phi^{\underline{\mathtt{a}}1} = 1$. We say that $\mathcal{A}$ *validates the presentation* Γ (in symbols, $\mathcal{A} \models \Gamma(\underline{a})$) iff we have that $\mathcal{A} \models \phi(\underline{a})$ for all $\phi(\underline{a}) \in \Gamma$.

The notion of an $\mathcal{S}$-*valuation* for a one-step frame $\mathcal{S}$ is the expected one, namely $\mathtt{v}$ is an $\mathcal{S}$-*valuation* iff it is an $\mathcal{S}^*$-*valuation*. In the same way the other notions introduced above (augmentation, $\phi^{\mathtt{v}0}, \phi^{\mathtt{v}1}$, validation of a presentation, of an inference, of an axiomatic system) can be extended by duality to one-step frames. We shall turn on valuations in one-step frames in Section 7.

We can specialize the above notions to standard modal algebras and frames. An $\underline{a}$-augmentation in a modal algebra $\mathfrak{A} = (A, \Diamond)$ is a tuple $\underline{\mathtt{a}}$ of elements from the support of A, matching the length of $\underline{a}$. For frames $\mathfrak{F} = (W, R)$, we dually take a tuple from $\wp(W)$, i.e., a tuple of subsets. Given such $\underline{a}$-augmentation, we can define $\mathfrak{A} \models \Gamma(\underline{a})$ and $\mathfrak{F} \models \Gamma(\underline{a})$ for a presentation $\Gamma(\underline{a})$, just specializing the above definitions (standard modal algebras and frames are special one-step modal algebras and frames). Notice that $\mathfrak{F} \models \Gamma(\underline{a})$ is *global* validity in terms of the Kripke forcing from the modal logic literature, see e.g., [32, Sec. 3.1].

Lemma 1. *Let* $\mathcal{A} = (A, B, i, \Diamond, \underline{\mathtt{a}})$ *be an augmented conservative one-step modal algebra that validates the axioms* Ax *and the presentation* $\Gamma(\underline{a})$*. Then* $\Gamma \vdash^1_{\mathrm{Ax}} \psi(\underline{a})$ *implies* $\mathcal{A} \models \psi(\underline{a})$*.*

[8]Recall that our inference rules are all reduced: in view of this fact, we used A_0 as the codomain of our valuations $\mathtt{v}$.

Proof. Let $\phi_1, \dots, \phi_n$ be the proof of ψ witnessing $\Gamma \vdash^1_{\mathrm{Ax}} \psi$. Notice that all formulae in such a proof must have modal complexity at most one. In addition, we can freely suppose that variables do not occur in the proof. If they are there, they can be replaced by a tautology, still obtaining a proof witnessing $\Gamma \vdash^1_{\mathrm{Ax}} \psi$, because variables do not occur in Γ, ψ. Then each ϕ_i belongs to Γ or is obtained from the previous ϕ_i's by applying the rules of Ax, modus ponens and necessitation.

The cases of modus ponens is easy. Now assume that ϕ_j is obtained from ϕ_i by applying the rule of necessitation. Then ϕ_i is of complexity 0 (otherwise the complexity of ϕ_j will be greater than one) and ϕ_j is $\Box\phi_i$. The induction hypothesis yields $\mathcal{A} \models \phi_i$, that is $\phi_i^{\mathtt{a1}} = 1$, and $i(\phi_i^{\mathtt{a0}}) = 1$. Since $\mathcal{A}$ is conservative, i is injective, which yields $\phi_i^{\mathtt{a0}} = 1$, thus $(\Box\phi_i)^{\mathtt{a1}} = \Box\phi_i^{\mathtt{a0}} = 1$ and finally $\mathcal{A} \models \phi_j$.

For the case of inference rules, we use the fact that rules are reduced. In fact, suppose that ϕ_i is obtained from $\phi_{i_1}, \dots, \phi_{i_m}$ by applying the reduced rule $\psi_1, \dots, \psi_m/\psi \in \mathrm{Ax}$. This means that for some substitution σ, we have that $\phi_{i_1}, \dots, \phi_{i_m}$ coincide with $\psi_1\sigma, \dots, \psi_m\sigma$ and ϕ_i coincides with $\psi\sigma$. Since every propositional variable occurring in the rule must have an occurrence inside a $\Diamond$ and the formulae occurring in the proof have modal complexity at most 1, the substitution σ must map variables to formulae of complexity 0. In other words, if $x_1, \dots, x_l$ are the variables occurring in $\psi_1, \dots, \psi_m, \psi$, we have that $\sigma(x_i) = \theta_i(\underline{a})$ where the θ_i have modal complexity 0 ($1 \le i \le l$). If we take a valuation $\mathtt{w}$ such that $\mathtt{w}(x_i) = \theta_i^{\mathtt{a0}}$ ($1 \le i \le l$), we can easily check by induction that, for every θ of complexity less or equal to 1, we have that $\theta^{\mathtt{w1}}$ is equal to $(\theta\sigma)^{\mathtt{a1}}$. Thus, from the induction hypothesis we have $(\psi_1\sigma)^{\mathtt{a1}} = 1, \dots, (\psi_m\sigma)^{\mathtt{a1}} = 1$, that is $\psi_1^{\mathtt{w1}} = 1, \dots, \psi_m^{\mathtt{w1}} = 1$. Since $\mathcal{A}$ validates the rules, we must have that $1 = \psi^{\mathtt{w1}} = (\psi\sigma)^{\mathtt{a1}} = \phi_i^{\mathtt{a1}}$, namely $\mathcal{A} \models \phi_i$. ⊣

5. Embeddings and Extensions

In this section we introduce the morphisms of one-step modal algebras and one-step frames. We also define the notion of a diagram of a finite one-step algebra and prove that the diagram encodes the embedability of this one-step modal algebra into other one-step modal algebras.

Definition 6. *An* embedding *between one-step modal algebras* $\mathcal{A} = (A_0, A_1, i_0, \Diamond_0)$ *and* $\mathcal{A}' = (A'_0, A'_1, i'_0, \Diamond'_0)$ *is a pair of injective Boolean morphisms* $h : A_0 \longrightarrow A'_0$, $k : A_1 \longrightarrow A'_1$ *such that*

$$k \circ i_0 = i'_0 \circ h \text{ and } k \circ \Diamond_0 = \Diamond'_0 \circ h \ . \tag{12}$$

$$\begin{tikzcd}
A_0 \arrow[r, "h"] \arrow[d, "i_0"'] & A'_0 \arrow[d, "i'_0"] \\
A_1 \arrow[r, "k"] & A'_1
\end{tikzcd}$$

$$\begin{tikzcd} A_0 \arrow[r, "h"] \arrow[d, "\Diamond_0"'] & A'_0 \arrow[d, "\Diamond'_0"] \\ A_1 \arrow[r, "k"] & A'_1 \end{tikzcd}$$

Notice that, when $\mathcal{A}'$ is standard (i.e. $A'_1 = A'_0 =$ and $i'_0 = id$), h must be $k \circ i_0$ and (12) reduces to

$$k \circ \Diamond_0 = \Diamond'_0 \circ k \circ i_0. \tag{13}$$

The following lemma is immediate.

Lemma 2. *Let (h, k) be an embedding between one-step modal algebras $\mathcal{A} = (A_0, A_1, i_0, \Diamond_0)$ and $\mathcal{A}' = (A'_0, A'_1, i'_0, \Diamond'_0)$. Suppose they are both augmented and that for the respective interpretations $\underline{\mathsf{a}}, \underline{\mathsf{a}}'$ of the parameters $\underline{a} = a_1, \ldots, a_n$, we have $h(\underline{\mathsf{a}}) = \underline{\mathsf{a}}'$, that is, $h(\mathsf{a}_i) = \mathsf{a}'_i$ for all $i = 1, \ldots, n$. Then for every formula $\phi(\underline{a})$, we have $\mathcal{A} \models \phi(\underline{a})$ iff $\mathcal{A}' \models \phi(\underline{a})$.*

Corollary 2. *Suppose that there is an embedding between the one-step modal algebras $\mathcal{A}$ and $\mathcal{A}'$;. Then, if $\mathcal{A}'$ validates an axiomatix system* Ax*, so does $\mathcal{A}$.*

For frames we have the dual definition. In the definition below, we use $\circ$ to denote relational composition: for $R_1 \subseteq X \times Y$ and $R_2 \subseteq Y \times Z$, we have $R_2 \circ R_1 := \{(x, z) \in X \times Z \mid \exists y \in Y \ ((x, y) \in R_1 \ \& \ (y, z) \in R_2)\}$. Notice that the relational composition applies also when one or both of R_1, R_2 is a function.

Definition 7. *A* p-morphism *between step frames $\mathcal{F}' = (W'_1, W'_0, f', R')$ and $\mathcal{F} = (W_1, W_0, f, R)$ is a pair of surjective maps $\mu : W'_1 \longrightarrow W_1, \quad \nu : W'_0 \longrightarrow W_0$ such that*

$$f \circ \mu = \nu \circ f' \text{ and } R \circ \mu = \nu \circ R'. \tag{14}$$

$$\begin{tikzcd} W'_1 \arrow[r, "\mu"] \arrow[d, "f'"'] & W_1 \arrow[d, "f"] \\ W'_0 \arrow[r, "\nu"] & W_0 \end{tikzcd}$$

$$\begin{tikzcd} W'_1 \arrow[r, "\mu"] \arrow[d, "R'"'] & W_1 \arrow[d, "R"] \\ W'_0 \arrow[r, "\nu"] & W_0 \end{tikzcd}$$

Notice that, when $\mathcal{F}'$ is standard (i.e., $W'_1 = W'_0$ and $f' = id$), ν must be $f \circ \mu$ and (14) reduces to

$$R \circ \mu = f \circ \mu \circ R'. \tag{15}$$

The dual of Lemma 2 holds for step frames too.

We now introduce an important ingredient of our proofs, namely *diagrams*. These are adaptations to our step contexts of classical methods in mathematical logic, due to A. Robinson in the model-theoretic environment [15, Ch. 2.1] and due to Jankov and Fine in the modal logic environment (see, e.g., [32, Sec. 7.3], [14, Sec. 9.4]).

Let $\mathcal{A} = (A, B, i, \Diamond)$ be a finite conservative one-step algebra. For each $\mathtt{a} \in A$ we introduce a parameter $p_{\mathtt{a}}$ (below we call $\underline{a}$ the tuple of such parameters). Let

$$\begin{array}{rcl}\Gamma^0_{\mathcal{A}}(\underline{a}) & := & \{p_{\mathtt{a}\vee\mathtt{b}} \leftrightarrow p_{\mathtt{a}} \vee p_{\mathtt{b}} : \mathtt{a}, \mathtt{b} \in A\} \cup \\ & & \{p_{\mathtt{a}\wedge\mathtt{b}} \leftrightarrow p_{\mathtt{a}} \wedge p_{\mathtt{b}} : \mathtt{a}, \mathtt{b} \in A\} \cup \\ & & \{p_{\neg\mathtt{a}} \leftrightarrow \neg p_{\mathtt{a}} : \mathtt{a} \in A\}.\end{array}$$

We augment $\mathcal{A}$ by interpreting every parameter $p_{\mathtt{a}}$ as $\mathtt{a}$. By the conservativity of $\mathcal{A}$, for every $\mathtt{b} \in B$, there is $\theta_{\mathtt{b}}$ such that $\mathtt{b}$ is equal to $\theta_{\mathtt{b}}^{\underline{\mathtt{a}}1}$. Notice that from our definitions, it follows in particular that for $\mathtt{a} \in A$, we have $\theta_{i(\mathtt{a})} = p_{\mathtt{a}}$.

Now let

$$\begin{array}{rcl}\Gamma^1_{\mathcal{A}}(\underline{a}) & := & \{\theta_{\mathtt{a}\vee\mathtt{b}} \leftrightarrow \theta_{\mathtt{a}} \vee \theta_{\mathtt{b}} : \mathtt{a}, \mathtt{b} \in B\} \cup \\ & & \{\theta_{\mathtt{a}\wedge\mathtt{b}} \leftrightarrow \theta_{\mathtt{a}} \wedge \theta_{\mathtt{b}} : \mathtt{a}, \mathtt{b} \in B\} \cup \\ & & \{\theta_{\neg\mathtt{a}} \leftrightarrow \neg\theta_{\mathtt{a}} : \mathtt{a} \in B\} \cup \\ & & \{\theta_{\Diamond\mathtt{a}} \leftrightarrow \Diamond p_{\mathtt{a}} : \mathtt{a} \in A\}.\end{array}$$

The *positive diagram* of $\mathcal{A}$ is the set of formulae $\Gamma_{\mathcal{A}}(\underline{a}) := \Gamma^0_{\mathcal{A}}(\underline{a}) \cup \Gamma^1_{\mathcal{A}}(\underline{a})$ and the *negative diagram* of $\mathcal{A}$ is the set of formulae

$$\Delta_{\mathcal{A}}(\underline{a}) \quad := \quad \{\theta_{\mathtt{a}} \leftrightarrow \theta_{\mathtt{b}} : \mathtt{a} \neq \mathtt{b}, \text{ for } \mathtt{a}, \mathtt{b} \in B\}.$$

We say that an augmented modal algebra $\mathcal{C}$ refutes $\Gamma_{\mathcal{A}} \vdash \Delta_{\mathcal{A}}$ iff we have $\mathcal{C} \models \phi(\underline{a})$ forall $\phi(\underline{a}) \in \Gamma_{\mathcal{A}}$ and $\mathcal{C} \not\models \psi(\underline{a})$ forall $\psi(\underline{a}) \in \Delta_{\mathcal{A}}$.

Lemma 3. *Let $\mathcal{A}$ be a conservative finite one-step algebra with the natural augmentation $\underline{\mathtt{a}}$ (interpreting every parameter $p_{\mathtt{a}}$ to $\mathtt{a}$). Then*

(1) *$\mathcal{A}$ refutes $\Gamma_{\mathcal{A}} \vdash \Delta_{\mathcal{A}}$.*
(2) *For each conservative one-step algebra $(C_0, C_1, j, \Diamond)$, there is an augmentation $\underline{\mathtt{c}}$ such that $\mathcal{C} = (C_0, C_1, j, \Diamond, \underline{\mathtt{c}})$ refutes $\Gamma_{\mathcal{A}} \vdash \Delta_{\mathcal{A}}$ iff $\mathcal{A}$ is embeddable into $\mathcal{C}$.*

Proof. (1) That $\mathcal{A}$ augmented with $\underline{\mathtt{a}}$ refutes $\Gamma_{\mathcal{A}} \vdash \Delta_{\mathcal{A}}$ is easy. (2) Now suppose $\mathcal{C} = (C_0, C_1, j, \Diamond, \underline{\mathtt{c}})$ refutes $\Gamma_{\mathcal{A}} \vdash \Delta_{\mathcal{A}}$. We define $h : A \to C_0$ so that we have $h(\underline{\mathtt{a}}) = \underline{\mathtt{c}}$. The map $k : B \to C_1$ is taken to be $k(\mathtt{b}) := \theta_{\mathtt{b}}^{\underline{\mathtt{c}}1}$. Since $\mathcal{C} \models \Gamma^0_{\mathcal{A}}(\underline{a})$ and j is injective, h is a Boolean morphism and since $\mathcal{C} \models \Gamma^1_{\mathcal{A}}(\underline{a})$, k is a Boolean morphism too. The fact that $k \circ i = j \circ h$ holds by construction (recall that $\theta_{i(\mathtt{a})}$ is $p_{\mathtt{a}}$ for all $\mathtt{a} \in A_0$). The preservation of $\Diamond$ follows from the validitation of the sentences $\theta_{\Diamond\mathtt{a}} \leftrightarrow \Diamond p_{\mathtt{a}}$. Since formulae from $\Delta_{\mathcal{A}}(\underline{a})$ are not validated, k is injective. Then h must also be injective because the injective morphism $k \circ i$ factors through it.

Now assume that $\mathcal{A}$ is embedded into $(C_0, C_1, i, \Diamond)$ via (h, k). Notice that k is uniquely determined by h because $\mathcal{A}$ is conservative. Augment $(C_0, C_1, i, \Diamond)$ by using $h(\underline{\mathtt{a}})$ to interpret the parameters. Now the claim follows by Lemma 2 and the fact that $\mathcal{A}$ (augmented with $\underline{\mathtt{a}}$) refutes $\Gamma_{\mathcal{A}} \vdash \Delta_{\mathcal{A}}$. $\dashv$

The aim of the next section is to formulate semantically, in terms of one-step frames, the bounded proof property. For this we need to introduce extensions [11]:

Definition 8. *Let $\mathcal{A}_0 = (A_0, A_1, i_0, \Diamond_0)$ be a one-step modal algebra. A* one-step extension *of $\mathcal{A}_0$ is a one-step modal algebra $\mathcal{A}_1 = (A_1, A_2, i_1, \Diamond_1)$ satisfying $i_1 \circ \Diamond_0 =$*

$\Diamond_1 \circ i_0$. *Dually, a* one-step extension *of the one-step frame* $\mathcal{S}_0 = (W_1, W_0, f_0, R_0)$ *is a one-step frame* $\mathcal{S}_1 = (W_2, W_1, f_1, R_1)$ *satisfying* $R_0 \circ f_1 = f_0 \circ R_1$.

The following lemma relates embeddings into standard algebras with extensions.

Lemma 4. *Let* $\mathcal{A} = (A_0, A_1, i_0, \Diamond_0)$ *be a one-step modal algebra and let* $(k \circ i_0, k)$ *be an embedding of it into a standard modal algebra* $\mathfrak{A} = (A, \Diamond)$. *Then the one-step modal algebra* $\mathcal{A}' = (A_1, A, k, \Diamond \circ k)$ *is an extension of* $\mathcal{A}$. *Moreover, if* $\mathfrak{A}$ *validates an axiom system* Ax, *then so does* $\mathcal{A}'$.

Proof. The first claim is by (13); the last claim is due to the fact that $\mathcal{A}'$ also embeds into $\mathfrak{A}$ (via the pair of Boolean morphisms (k, id)), so that Corollary 2 applies. $\dashv$

6. Semantic Characterizations of the bpp and fmp

In this section we prove our main result, namely a semantic characterization of the fact that an axiomatic system has the bounded proof property and the finite model property. The following definition introduces the semantic notion leading to this characterization.

Definition 9. *A class of one-step modal algebras has the* extension property *iff every conservative one-step modal algebra* $\mathcal{A}_0 = (A_0, A_1, i_0, \Diamond_0)$ *in the class has an extension* $\mathcal{A}_1 = (A_1, A_2, i_1, \Diamond_1)$ *such that* i_1 *is injective and* $\mathcal{A}_1$ *is also in the class.*

A class of one-step modal frames has the extension property *iff every conservative one-step frame* $\mathcal{S}_0 = (W_1, W_0, f_0, R_0)$ *in the class has an extension* $\mathcal{S}_1 = (W_2, W_1, f_1, R_1)$ *such that* f_1 *is surjective and* $\mathcal{S}_1$ *is also in the class.*

Theorem 1. *An axiomatic system* Ax *has the bpp iff the class of finite one-step modal algebras (equivalenly, the class of finite one-step frames) validating* Ax *has the extension property.*

Proof. Suppose that the class of one-step modal algebras validating Ax has the extension property and let $\Gamma(\underline{a})$ be a finite presentation (i.e., a finite set of formulae of modal complexity at most 1) such that $\Gamma \not\vdash^1_{\mathrm{Ax}} \phi$ for a formula $\phi(\underline{a})$ of modal complexity at most 1. We use a Lindembaum-like construction to build a one-step modal algebra $\mathcal{A}_0 = (A_0, A_1, i_0, \Diamond_0)$ as follows. For formulae $\psi_1(\underline{a}), \psi_2(\underline{a})$ of modal complexity at most 1, let us put $\psi_1 \approx \psi_2$ iff $\Gamma \vdash^1_{\mathrm{Ax}} \psi_1 \leftrightarrow \psi_2$. This is clearly an equivalence relation and we can build the Boolean algebras A_0, A_1 by considering the equivalence classes of the formulae of complexity 0 and (0 or 1), respectively. The Boolean morphism i_0 associates with the equivalence class of ψ the equivalence class of ψ inside the set of formulae of complexity at most 1. This is clearly injective. We also define $\Diamond_0$ to be the map associating with the equivalence class of ψ the equivalence class of $\Diamond\psi$. The one-step modal algebra we introduced is obviously conservative: this is because i_0 is injective and because the formulae of modal complexity 1 can all be obtained as Boolean combinations of formulae of modal complexity 0 and of formulae of the kind $\Diamond\psi$, where ψ has complexity 0. $\mathcal{A}_0$ also validates Ax by construction. We can define an augmentation by taking $\mathtt{a}$ to be the tuple of the equivalence classes in A_0 of the parameters $\underline{a}$. In this way, we have $(\bigwedge\Gamma)^{\mathtt{a}1} = 1$ and $\phi^{\mathtt{a}1} \neq 1$.

By the extension property, there is an extension $\mathcal{A}_1 = (A_1, A_2, i_1, \Diamond_1)$, with injective i_1, also validating Ax. We can freely assume that $\mathcal{A}_1$ is conservative;

otherwise, we replace A_2 by the subalgebra generated by the images of i_1 and $\Diamond_1$, which as a subalgebra also trivially validates Ax. If we continue in this way, we generate a chain

$$A_0 \xrightarrow{i_0} A_1 \to \cdots \to A_k \xrightarrow{i_k} A_{k+1} \to \cdots \tag{16}$$

of Boolean algebras equipped with semilattice morphisms

$$A_0 \xrightarrow{\Diamond_0} A_1 \to \cdots \to A_k \xrightarrow{\Diamond_k} A_{k+1} \to \cdots \tag{17}$$

satisfying the conditions $\Diamond_{k+1} \circ i_k = i_{k+1} \circ \Diamond_k$ and such that for every $k \geq 0$, the one-step modal algebra $(A_k, A_{k+1}, i_k, \Diamond_k)$ validates Ax. Thus, the Boolean algebra A obtained by taking the colimit of (16) can be endowed with a semilattice morphism $\Diamond : A \longrightarrow A$ in such a way that $\mathfrak{A} := (A, \Diamond)$ is a standard modal algebra validating Ax by construction. In this algebra, under the obvious augmentation obtained by composing the previous augmentation a with the inclusion of A_0 into the colimit, since the i_k are all injective, we have $\mathfrak{A} \models \Gamma(\underline{a})$ and $\mathfrak{A} \not\models \phi(\underline{a})$. Thus, we found an augmented (standard) modal algebra validating Ax, Γ but not ϕ: this implies that $\Gamma \not\vdash_{\mathrm{Ax}} \phi$.

Conversely, suppose that the bpp holds and take a conservative finite one-step modal algebra $\mathcal{A} = (A_0, A_1, i_0, \Diamond)$. Let $\underline{a}$ be a list of parameters naming the elements of A_0. Since $\mathcal{A}$ (with the natural augmentation) refutes $\Gamma_{\mathcal{A}} \vdash \Delta_{\mathcal{A}}$, by Lemma 1, we have that $\Gamma_{\mathcal{A}} \not\vdash^1_{\mathrm{Ax}} \delta$ for every $\delta \in \Delta_{\mathcal{A}}$. By the bpp, we have

$$\Gamma_{\mathcal{A}} \not\vdash_{\mathrm{Ax}} \delta \tag{18}$$

for every $\delta \in \Delta_{\mathcal{A}}$.

Now consider the sets S_0, S_1, S_2 of formulae $\theta(\underline{a})$ of complexity at most 0, 1, 2, respectively. Introduce in these sets the Lindembaum equivalence relations $\psi_1 \approx \psi_2$ iff $\Gamma_{\mathcal{A}} \vdash_{\mathrm{Ax}} \psi_1 \leftrightarrow \psi_2$ and let B_0, B_1, B_2 be the Boolean algebras of the equivalence classes. We have Boolean and semilattice morphisms:

$$B_0 \xrightarrow{i_0} B_1 \xrightarrow{i_1} B_2, \qquad B_0 \xrightarrow{\Diamond_0} B_1 \xrightarrow{\Diamond_1} B_2,$$

making $\mathcal{B}_0 = (B_0, B_1, i_0, \Diamond_0)$ and $\mathcal{B}_1 = (B_1, B_2, i_1, \Diamond_1)$ one-step conservative modal algebras validating Ax; $\mathcal{B}_1$ is an extension of $\mathcal{B}_0$ with injective i_1. We can augment $\mathcal{B}_0$ by interpreting the parameters $\underline{a}$ as their own equivalence classes. By (18) and Lemma 3, $\mathcal{A}$ embeds into $\mathcal{B}_0$. The embedding from the proof of Lemma 3 is an isomorphism by the construction of $\mathcal{B}_0$: in fact, the equivalence classes of the parameters $\underline{a}$ are in the image of the embedding, they generate B_0 as a Boolean algebra, whereas the Boolean algebra B_1 is generated by their images along i_0 and $\Diamond_0$ (in fact, B_1 consists of the equivalence classes of the formulae $\psi(\underline{a})$ having modal complexity at most 1). Since $\mathcal{B}_0 \simeq \mathcal{A}$ and $\mathcal{B}_0$ has an extension $\mathcal{B}_1$ validating Ax with injective i_1, the result follows. $\dashv$

The characterization of bpp from Theorem 1 may not be easy to handle, because in practical cases one would like to avoid managing one-step extensions and would prefer to work with standard frames instead. This is possible, if we combine the bpp with the finite model property.

Definition 10. *An axiomatic system* Ax *has the (global)* finite model property*, the* fmp *for short, if for every tuple $\underline{a}$ of parameters, for every finite set of formulae $\Gamma(\underline{a})$ and for every formula $\phi(\underline{a})$ we have $\Gamma \not\vdash_{\mathrm{Ax}} \phi$ iff there exists a finite $\underline{a}$-augmented modal algebra $\mathfrak{A}$ such that $\mathfrak{A} \models \mathrm{Ax}$, $\mathfrak{A} \models \Gamma(\underline{a})$ and $\mathfrak{A} \not\models \phi(\underline{a})$ (equivalently, iff* *there exists a finite $\underline{a}$-augmented Kripke frame $\mathfrak{F}$ such that $\mathfrak{F} \models \mathrm{Ax}$, $\mathfrak{F} \models \Gamma(\underline{a})$ and $\mathfrak{F} \not\models \phi(\underline{a})$).*

Notice that a finite $\underline{a}$-augmented Kripke frame $\mathfrak{F}$ is nothing but a Kripke model based on $\mathfrak{F}$ (here we consider Kripke valuations restricted to the parameters only).

We are ready for our main result:

Theorem 2. *An axiomatic system* Ax *has both the bpp and the fmp iff every finite conservative one-step frame validating* Ax *is a p-morphic image of some finite frame validating* Ax *(equivalently, iff every finite conservative one-step modal algebra validating* Ax *has an embedding into some finite modal algebra validating* Ax*).*

Proof. First assume that every finite conservative one-step modal algebra validating Ax has an embedding into some standard finite modal algebra validating Ax. Since this implies that the class of finite one-step modal algebras validating Ax has the extension property (by Lemma 4), Ax has the bpp by Theorem 1. Now, to show that fmp holds, suppose that $\Gamma \not\vdash_{\mathrm{Ax}} \varphi$, for a finite set $\Gamma(\underline{a})$ and for a formula $\phi(\underline{a})$. By applying the same method as in the proof of Proposition 4, we can freely assume that Γ, ϕ have complexity 1. Let us build the Lindembaum algebra for Γ, Ax. This is the algebra defined in the following way: for formulae $\psi_1(\underline{a}), \psi_2(\underline{a})$, let us put $\psi_1 \approx \psi_2$ iff $\Gamma \vdash_{\mathrm{Ax}} \psi_1 \leftrightarrow \psi_2$. This is clearly an equivalence relation and we can build an augmented modal algebra out of it by defining all operations on equivalence classes. In particular, the selected tuple $\underline{\mathsf{a}}$ will be the tuple of the equivalence classes of the $\underline{a}$. We obtain Boolean subalgebras $A_0, A_1, \dots$ by considering the equivalence classes of the formulae of modal complexity at most 0, 1, ... Then ϕ is refuted in the augmented one-step modal algebra $\mathcal{A} = (A_0, A_1, i_0, \Diamond_0, \underline{\mathsf{a}})$ which is such that $\mathcal{A} \models \Gamma(\underline{a})$ and $\mathcal{A} \models \mathrm{Ax}$. Here i_0 is inclusion and $\Diamond_0$ is the restriction of $\Diamond$ to A_0 in the domain and A_1 in the codomain. By our assumption, $\mathcal{A}$ embeds (via some k satisfying (13)) into a finite modal algebra $\mathfrak{A} = (A, \Diamond)$ validating Ax. We can augment $\mathfrak{A}$ by taking the $k(i_0(\underline{\mathsf{a}}))$ as the selected tuple interpreting the parameters $\underline{a}$. As embeddings preserve the interpretation of formulas (see Lemma 2), we have that $(A, \Diamond, k(i_0(\underline{\mathsf{a}})))$ also refutes $\phi(\underline{a})$, validates $\Gamma(\underline{a})$. Hence, $(A, \Diamond, k(i_0(\underline{\mathsf{a}})))$ is a countermodel to $\Gamma \vdash_{\mathrm{Ax}} \phi$, and thus Ax has the finite model property.

Now suppose Ax has both the bpp and the fmp and let $\mathcal{A}$ be a finite conservative one-step modal algebra that validates Ax. Since $\mathcal{A}$ (with the natural augmentation) refutes $\Gamma_{\mathcal{A}} \vdash \Delta_{\mathcal{A}}$, by Lemma 1, we have that $\Gamma_{\mathcal{A}} \not\vdash^1_{\mathrm{Ax}} \delta$ for every $\delta \in \Delta_{\mathcal{A}}$. By the bpp, we have

$$\Gamma_{\mathcal{A}} \not\vdash_{\mathrm{Ax}} \delta \tag{19}$$

for every $\delta \in \Delta_{\mathcal{A}}$ and by the fmp there exists an augmented finite modal algebra $\mathfrak{C}_\delta$ witnessing this. Taking the (finite) product of the $\mathfrak{C}_\delta$, we obtain a finite augmented modal algebra $\mathfrak{C}$ refuting $\Gamma_{\mathcal{A}} \vdash \Delta_{\mathcal{A}}$ and validating Ax. By Lemma 3, this implies that $\mathcal{A}$ is embedded into $\mathfrak{C}$. ⊣

7. One-Step Correspondence

In this section we develop the correspondence theory for one-step frames based on the classical correspondence theory for standard frames.

Let us turn to Definition 5. We make a little reformulation of it in the following. Notice that a one-step modal algebra $=(A_0, A_1, i, \Diamond)$ is in fact a two-sorted structure for the first-order language $\mathcal{L}_a$ having two sorts, Boolean operations for each of them, and two sorted unary function symbols. We may express inference validation using Tarski first order semantics, provided we turn modal formulae ϕ of complexity at most 1 into terms for such a language. This is easily done as follows: just replace every occurrence of a variable x *which is not located inside a modal connective* in ϕ by $i(x)$. Let us call $\tilde{\phi}$ the result of such a replacement. The following fact is then clear.

Proposition 6. *We have that $\mathcal{A}$ validates the reduced inference rule*

$$\frac{\phi_1(\underline{x}), \dots, \phi_n(\underline{x})}{\psi(\underline{x})} \tag{20}$$

iff as a two-sorted $\mathcal{L}_a$-structure, it satisfies

$$\mathcal{A} \models \forall \underline{x}\, (\tilde{\phi}_1 = 1 \;\&\cdots\&\; \tilde{\phi}_n = 1 \to \tilde{\psi} = 1). \tag{21}$$

If $\mathcal{A}$ is of the kind $\mathcal{F}^*$ for a one-step frame $\mathcal{F}$, we can turn (21) into a formula in the language $\mathcal{L}_f$ of one-step frames. Such a language is also two-sorted, but it is much simpler because it has just a unary function and a binary relation symbol. The conversion is obvious and can be adapted from standard translations in modal logic [5]. The problem however is that the conversion introduces second order quantifiers (the $\underline{x}$ are subsets when dealing with frames) and so may lead to a condition that is difficult to handle in the applications. The idea is to make same symbolic manipulations on it and try to convert it into a first-order $\mathcal{L}_f$-condition. This is not always possible, but may succeed in many practical cases. Following the extensive literature on this topic [5–8, 19–22, 28, 29, 42] (see also [13, Sec. 3.6], [14, Sec. 10.3], [32, Ch. 5]), we introduce the symbolic procedure at the algebraic language by first enriching $\mathcal{L}_a$.

The enrichment comes from the following observations. Suppose that $\mathcal{F} = (W_0, W_1, f, R)$ is a one-step frame. First of all, the Boolean algebras $\wp(W_0), \wp(W_1)$ are atomic and moreover the morphism $i := f^* : \wp(W_0) \to \wp(W_1)$ has a left i^* and a right adjoint $i_!$. In fact i^* is direct image along f and $i_!$ is $\neg i^* \neg$. The operator $\Diamond : \wp(W_0) \to \wp(W_1)$ (we skip the index R) also has a right adjoint, which is the Box operator $\blacksquare$ induced by the converse relation R^o of R. We shall make use also of the related Diamond $\blacklozenge$ defined as $\neg\blacksquare\neg$. Thus we enrich $\mathcal{L}_a$ with extra unary function symbols $i^*, i_!, \blacksquare, \blacklozenge$ of appropriate sorts. In addition, we shall make use of the letters w_i^0, w_i^1 to denote *nominals*, namely quantified variables ranging over atoms (i.e., singleton subsets) of $\wp(W_0), \wp(W_1)$, respectively. For simplicity and for readability, we shall avoid the superscript $(-)^1, (-)^0$ indicating the sort of nominals. However, we shall adopt the convention of using preferably the variables $w, w_0, w', \dots$ for nominals of sort 1, the variables $v, v_0, v', \dots$ for nominals of sort 0 and the letters $u, u_0, u', \dots$ for variables of unspecified sort (i.e., for variables that might be of both sorts, which are useful in preventing, e.g., rule duplications). We call $\mathcal{L}_a^+$ the enriched language.

The idea is the following. We want to analyze validity of the inference rule (20) in a one-step frame $\mathcal{F}$. We initialize our procedure to:

$$\forall \underline{x}\, (1 \le \tilde{\phi}_1 \;\&\cdots\&\; 1 \le \tilde{\phi}_n \to 1 \le \tilde{\psi}) \tag{22}$$

Here and below, we use abbreviations such as $\alpha \le \beta$ to mean $\alpha \to \beta = 1$. Usually, we omit external quantifiers $\forall \underline{x}$ and use sequent notation, so that (22) is written as

$$1 \le \tilde{\phi}_1,\ \dots,\ 1 \le \tilde{\phi}_n \Rightarrow 1 \le \tilde{\psi}\ . \tag{23}$$

$\dfrac{\tilde\phi \leq \tilde\psi}{\forall u\ (u\leq\tilde\phi\ \to\ u\leq\tilde\psi)}$	$\dfrac{u \leq \tilde\psi_1\wedge\tilde\psi_2}{u\leq\tilde\psi_1\ \&\ u\leq\tilde\psi_2}$	$\dfrac{u \leq \tilde\psi_1\vee\tilde\psi_2}{u\leq\tilde\psi_1\ \text{or}\ u\leq\tilde\psi_2}$
$\dfrac{u \leq \neg\tilde\psi}{u\not\leq\tilde\psi}\quad\dfrac{u\not\leq\tilde\psi}{\tilde\psi\leq\neg u}$	$\dfrac{w \leq \Diamond\tilde\psi}{\exists v\ (w\leq\Diamond v\ \&\ v\leq\tilde\psi)}$	$\dfrac{v \leq \blacklozenge\tilde\psi}{\exists w\ (v\leq\blacklozenge w\ \&\ w\leq\tilde\psi)}$
$\dfrac{u \leq 1}{\top}$	$\dfrac{u \leq 0}{\bot}$	$\dfrac{v \leq i^*(\tilde\psi)}{\exists w\ (v\leq i^*(w)\ \&\ w\leq\tilde\psi)}$

TABLE 1. Nominals Rules

$\dfrac{\tilde\phi \leq \Box\tilde\psi}{\blacklozenge\tilde\phi \leq \tilde\psi}$	$\dfrac{\tilde\phi \leq \blacksquare\tilde\psi}{\Diamond\tilde\phi \leq \tilde\psi}$
$\dfrac{\tilde\phi \leq i(\tilde\psi)}{i^*(\tilde\phi) \leq \tilde\psi}$	$\dfrac{\tilde\phi \leq i_!(\tilde\psi)}{i(\tilde\phi) \leq \tilde\psi}$

TABLE 2. Adjunction Rules

We then try to find a sequence of applications of the rules below ending with a formula where *only quantifiers for nominals* occur (that is, the variables $\underline{x}$ have been eliminated). If we succeed, the standard translation can easily and automatically *convert the final formula into a first-order formula in the language* $\mathcal{L}_f$. It is possible to characterize syntactic classes (e.g., Sahlqvist-like classes and beyond) where the procedure succeeds, but for the purposes of this paper we are not interested in the details of such characterizations. They can be obtained in a straightforward way by extending well-known characterizations, see e.g. [5, 20, 22]).

The rules we use are divided into three groups:

(a): Any set of invertible rules in classical first-order sequent calculus; we refer the reader to proof-theory textbooks such as [37] for more details on this;
(b): Rules for managing nominal quantifiers (see Table 1);
(c): Adjunction rules (see Table 2);
(d): Ackermann rules (see Table 3).

Rules (a)-(b)-(c) are local, in the sense that they can be applied simply by replacing the upper formula by the lower formula (or vice versa); rules (d) on the contrary require checking global monotonicity conditions at the whole sequent level. Ackermann rules eliminate the quantified variables $\underline{x}$ one by one in successful runs.

When we start from a logic L, we first need to convert the axioms into reduced inference rules. The method indicated in the proof of Proposition 3 has the big advantage of *introducing new quantified variables that can be easily eliminated via the adjunction and Ackermann rules*, as is shown in the example below.

$\dfrac{\Gamma, x \leq \tilde{\phi} \Rightarrow \Delta}{\Gamma(\tilde{\phi}/x) \Rightarrow \Delta(\tilde{\phi}/x)}$	(x is not in ϕ, is positive in all Γ, negative in all Δ)
$\dfrac{\Gamma, \tilde{\phi} \leq x \Rightarrow \Delta}{\Gamma(\tilde{\phi}/x) \Rightarrow \Delta(\tilde{\phi}/x)}$	(x is not in ϕ, is negative in all Γ, positive in all Δ)

TABLE 3. Ackermann Rules

Example 1. Let us consider the system **K4** that is axiomatized by the single axiom $\Box x \to \Box\Box x$. Since this axiom does not have modal complexity 1, we turn it into the inference rule

$$\frac{\Box x \leq y}{\Box x \leq \Box y} \tag{24}$$

following the proof of Proposition 3. We initialize our procedure to

$$\Box x \leq i(y) \Rightarrow \Box x \leq \Box y.$$

by adjunction rules, we obtain

$$i^*(\Box x) \leq y \Rightarrow \Box x \leq \Box y.$$

We can immediately eliminate y via the Ackermann rules

$$\Box x \leq \Box i^*(\Box x).$$

We now use atom rules together with rules (a) (i.e., invertible rules in classical sequent calculus) and arrive at:

$$w \leq \Box x \Rightarrow w \leq \Box i^*(\Box x).$$

Notice that we have also the nominal variable w implicitly universally quantified here. By adjointness we obtain a sequent

$$\blacklozenge w \leq x \Rightarrow w \leq \Box i^*(\Box x)$$

to which the Ackermann rules apply yieldying

$$w \leq \Box i^*(\Box \blacklozenge w).$$

This is a condition involving only (one) quantified variable for nominals. Thus in the language $\mathcal{L}_f$ for one-step frames it is first-order definable. To do the unfolding, it is sufficient to notice that the nominal w stands in fact for the set $\{w' \in W_1 \mid w' = w\}$. Therefore, by turning $\leq$ into a set-theoretic inclusion and letting the modal operators have their standard relational meaning, we obtain

$$\forall w \forall w' \ (w = w' \to \forall v \ (R(w', v) \to \exists w_1 \ (f(w_1) = v \ \& \\ \& \ \forall v_1 (R(w_1, v_1) \to \exists w_2 \ (R(w_2, v_1) \ \& \ w_2 = w)))))$$

that simplifies to

$$\forall w \forall v \ (R(w, v) \to \exists w_1 \ (f(w_1) = v \ \& \ R(w_1) \subseteq R(w))) \ . \tag{25}$$

Using relational composition notation, condition above can be rewritten as

$$R \subseteq f \circ \geq_R, \tag{26}$$

where $w_1 \geq_R w_2$ is defined to be $R(w_1) \supseteq R(w_2)$.

8. Examples

In this section we show how to apply Theorem 2 in basic cases (more elaborated examples will be analyzed in the next section). The methodology is the following. We have three steps, as pointed out in Subsection 1.1:

- starting from a logic L, we produce an equivalent axiomatic system Ax_L with reduced rules (for the lack of better ideas, there is a default procedure for that, see the proof of Proposition 3);
- we apply the correspondence machinery of Section 7 and try to obtain a first-order formula α_L in the two-sorted language of one-step frames characterizing the one-step frames validating Ax_L;
- we apply Thorem 2 and try to prove that conservative finite one-step frames satisfying α_L are p-morphic images of standard frames for L.

If we succeed, we obtain both the fmp and bpp for L. In the examples, given a finite conservative one-step frame $\mathcal{F} = (X, Y, f, R)$ satisfying α_L, the finite frame required by Theorem 2 is often based on X and the p-morphism is the identity. Thus one must simply define a relation S on X in such a way that (15) holds (with $R' = S$). Condition (15), taking into consideration that μ is the identity, reduces to

$$R = f \circ S. \tag{27}$$

There are standard templates for S (e.g., for the basic case, reflexive case, transitive case, etc.). Once the right template is chosen, checking (27) becomes a first-order deduction problem. As such, the problem can also be solved by automated reasoning tools (or also manually).

We give some examples below (we shall use a relational formalism, which is more succint than full first-order logic).

- $\boxed{L = \mathbf{K}}$: this is the smallest normal modal logic. Take $S := f^o \circ R$.[9] Then we have $f \circ S = f \circ f^o \circ R = R$, showing (27) (we used that $f \circ f^o = id$, which holds by the surjectivity of f).
- $\boxed{L = \mathbf{T}}$: this is the logic axiomatized by $\Box x \to x$. The one-step correspondence gives $f \subseteq R$ as the semantic condition equivalent to being a one-step frame for L. We still take $S := f^o \circ R$ and get (27) as above. In addition, we must show that S is reflexive, namely that $id \subseteq S$: this is obtained from $id \subseteq f^o \circ f \subseteq f^o \circ R = S$.
- $\boxed{L = \mathbf{K4}}$: this is the logic axiomatized by $\Box x \to \Box\Box x$. As we know, this axiom can be turned into the equivalent rule (24) and the one-step correspondence gives (26) (namely $R \subseteq f \circ \geq_R$) as the semantic condition equivalent to being a one-step frame for $\mathbf{K4}$ (recall that $w \geq_R w'$ is defined as $R(w) \supseteq R(w')$). We take S to be $(f^o \circ R) \cap \geq_R$.[10] To check (27), notice that
$$f \circ S \ = \ f \circ ((f^o \circ R) \cap \geq_R) \ = \ R \cap (f \circ \geq_R) \ = \ R$$
where we used (26) together with the relational identity $R \cap (f \circ H) = f \circ ((f^o \circ R) \cap H)$, holding for all R, H. To check that S is transitive,

[9]This is the same as saying that wSz iff $f(z) \in R(w)$.

[10]This is the same as saying that wSw' iff $R(w) \supseteq \{f(w')\} \cup R(w')$.

observe that

$$\begin{aligned} S \circ S &= ((f^o \circ R) \cap \geq_R) \circ ((f^o \circ R) \cap \geq_R) \subseteq \\ &\subseteq (\geq_R \circ \geq_R) \cap (f^o \circ R \circ \geq_R) \subseteq \geq_R \cap (f^o \circ R) = S \end{aligned}$$

because $R \circ \geq_R \subseteq R$.

- $\boxed{L = \mathbf{S4}}$ Here one can combine the previous two cases. However, the definition of S as $(f^o \circ R) \cap \geq_R$ simplifies to $\geq_R$. In fact, we have $w((f^o \circ R) \cap \geq_R)w'$ iff $wRf(w')$ & $R(w) \supseteq R(w')$ iff $R(w) \supseteq R(w')$ (given that $f(w') \in R(w')$ holds by reflexivity).

Some of the above computations might look involved, but all of them can be automatized: one can use (an adaptation of) the Sahlqvist correspondence algorithm SQEMA together with a first-order prover to discharge them.[11] Only the definition of S needs to be supplied, but standard solutions (such as $S := f^o \circ R$ or $S := (f^o \circ R) \cap \geq_R$ or $S := \geq_R$ used above) may work well in practice.

Remark 6. Notice that the definition of a conservative finite one-step frame (Definition 4) has two conditions. However, only the first one (namely surjectivity of f) is used in the computations above. In fact, it is not clear whether Theorem 2 holds if we drop the second condition (11) in the definition of a one-step conservative frame. All what we can say at the moment is that in the applications we use only the right-to-left side of the theorem and that the second condition of the definition of conservativety does not seem to play any role. Contrary to this, although the requirement of the finiteness of the one-step frame which is required to be a p-morphic image of a standard frame has not been used so far, it will be essentially applied in more complicated cases in the next section.

9. Case Studies

In this Section we show that our methodology can be fruitfully applied to significant logics taken from the proof-theoretic literature.

9.1. **A case study: S4.3.** As a first example we take **S4.3**, which is **S4** plus the axiom

$$\Box(\Box x \to y) \vee \Box(\Box y \to x).$$

Applying the procedure of Proposition 3, we obtain the equivalent rule

$$\frac{x' \leq \Box x, \ y' \leq \Box y}{\Box(x' \to x) \vee \Box(y' \to y).} \tag{28}$$

Correspondence applied to this rule gives the following condition

$$\forall w \forall v_1 \forall v_2 (wRv_1 \ \&\ wRv_2 \to \exists w_1(f(w_1) = v_1 \ \&\ w_1Rv_2) \vee \exists w_2(f(w_2) = v_2 \ \&\ w_2Rv_1)). \tag{29}$$

This condition is 'bad'. Indeed the above axiomatization for **S4.3** lacks the bpp as follows from the counterexample described below.

Example 2. Since it is well-known that the fmp holds for **S4.3**, it is sufficient, in view of Theorem 2, to exhibit a finite one-step conservative frame satisfying (29) which is not a p-morphic image of a standard frame for **S4.3**. We sketch the counterexample. The key observation is that a finite one-step frame can be extended to

[11] The SPASS prover solves each of the above problems in less than half a second on a common laptop.

a finite frame for **S4.3** iff it satisfies condition (31) below (one side of the equivalence will be proved below, the other side is trivial, because a standard frame for **S4.3** satisfies (31) and so does any step-frame which is a p-morphic image of it). Hence it is sufficient to exhibit a reflexive, transitive one-step finite frame satisfying (29) but not (31) (reflexivity and transitivity are intended in the sense of step correspondence, they are characterized in Section 8). In order to obtain such a counterexample consider the following one-step frame (it is sufficient to take any $n \geq 3$):

$$X := \{0\} \cup \{1, \ldots, n\}^2, \quad Y := \{0\} \cup \{1, \ldots, n\}$$

with

$$R(0) := Y, \; R(\langle i, j \rangle) = \{i, j\}, \quad f(0) := 0, \; f(\langle i, j \rangle) = i.$$

Reflexivity, transitivity and (29) are trivial (for transitivity, use the pairs $\langle j, j \rangle$). On the other hand, (31) fails with $w := 0$ and $S := \{1, \ldots, n\}$.

Instead of the rule obtained by the procedure of Proposition 3, we axiomatize **S4.3** by using the reflexivity axiom for **T** and the following infinitely many rules proposed by R. Goré [30]:

$$\frac{\cdots \Box y \to x_j \vee \bigvee_{j \neq i} \Box x_i \cdots}{\Box y \to \bigvee_{i=1}^{n} \Box x_i} \tag{30}$$

The rules are indexed by n and the n-th rule has n premises, according to the values $j = 1, \ldots, n$.

We do the correspondence manually on the n-th rule. We directly use a set-theoretic language for one-step frames instead of the rules from the tables of Section 7. The reason is that first-order language is not sufficient for our procedures.

The n-th rule is not valid in a one-step frame (X, Y, f, R) iff there exist $w \in X, Q \subseteq Y, P_1 \ldots, P_n \subseteq Y$ such that

$$R(w) \subseteq Q \;\&\; \exists v_1, \ldots, v_n \bigwedge_{i=1}^{n} (wRv_i \;\&\; v_i \notin P_i) \;\&\; \bigwedge_{j=1}^{n} (\Box_R Q \subseteq f^*(P_j) \cup \bigcup_{j \neq i} \Box_R P_i).$$

We can now apply Ackermann rule with $Q := R(w)$, $P_i := Y \setminus \{v_i\}$ and obtain

$$\exists w \; \exists v_1, \ldots, v_n \bigwedge_{i=1}^{n} wRv_i \;\&\; \bigwedge_{j=1}^{n} (\Box_R R(w) \subseteq f^*(\{v_j\}^C) \cup \bigcup_{j \neq i} \Box_R \{v_i\}^C)$$

(where $(-)^C$ is the set-theoretic complement to Y). This is a first-order condition. If we negate it, we obtain the following condition for the validity of the n-th rule:

$$\forall w \; \forall v_1, \ldots, v_n (\{v_1, \ldots, v_n\} \subseteq R(w) \to \bigvee_{j=1}^{n} \Box_R R(w) \not\subseteq f^*(\{v_j\}^C) \cup \bigcup_{j \neq i} \Box_R \{v_i\}^C) \; .$$

This can be rewritten into

$$\begin{aligned} \forall w \; \forall v_1, \ldots, v_n \qquad & (\{v_1, \ldots, v_n\} \subseteq R(w) \to \\ & \to \textstyle\bigvee_{j=1}^{n} \exists w_j (R(w_j) \subseteq R(w) \;\&\; f(w_j) = v_j \;\&\; \bigwedge_{j \neq i} w_j R v_i)) \; . \end{aligned}$$

Now, we are interested only in *finite* step frames, where the simultaneous validity of all the rules can be restated as follows:

$$\forall w \; \forall S \subseteq R(w) \; \exists v \in S \; \exists w' \; (f(w') = v \;\&\; S \setminus \{v\} \subseteq R(w') \subseteq R(w)),$$

where S is assumed to range over non empty sets. Since we also have the reflexivity condition $f \subseteq R$ coming from the validity of the axiom of **T**, we finally obtain

$$\forall w\, \forall S \subseteq R(w)\, \exists v \in S\, \exists w'\, (f(w') = v \;\&\; S \subseteq R(w') \subseteq R(w)). \tag{31}$$

Thus, we arrived at the following

Lemma 5. *Every one-step frame of* **S4.3** *axiomatized by Goré's rules* (30) *satisfies* (31).

In order to apply Theorem 2 we must prove that a finite one-step (X, Y, f, R) frame satisfying (31) can be extended to a finite frame (X', R') which is quasi-linear (i.e., it is a frame for **S4.3**)

$$\forall w\, \forall w_1\, \forall w_2\, (wRw_1 \;\&\; wRw_2 \to w_1Rw_2 \text{ or } w_2Rw_1). \tag{32}$$

Note that (31) is equivalent to (32) if interpreted in standard frames. For proving the next theorem contrary to what we did in Section 7, we shall not take X' to be X, but we shall define X' via an *ad hoc* definition.

Theorem 3. **S4.3** *axiomatized by Goré's rules* (30) *has the bpp and fmp.*

Proof. Let (X, Y, f, R) be a conservative finite one-step validating (30), and hence satisfying (31). Notice that we can define a preorder $\geq_R$ on X by putting $w_1 \geq_R w_2$ iff $R(w_1) \supseteq R(w_2)$. An $\geq_R$-*chain* is a nonempty subset C of X such that we have $w_1 \geq_R w_2$ or $w_2 \geq_R w_1$ for all $w_1, w_2 \in C$. For such a chain C, we write $C := C_w$ to emphasize that w is a $\geq_R$-biggest element of C (notice that this $\geq_R$-biggest element may not be unique). We take X' to be the disjoint union of the $\geq_R$-chains C of X satisfying the following condition:

$$w \in C \;\&\; wRv \to \exists w' \in C\; (w \geq_R w' \;\&\; f(w') = v). \tag{33}$$

Let $\mu : X' \longrightarrow X$ be the disjoint union of the inclusions. X' can be turned into a quasi-linear preorder by taking the disjoint union of the relations $\geq_R$. Since (15) holds by construction and by reflexivity, we only need to check the surjectivity of μ, namely that each $w \in X$ belongs to a chain C_w satisfying condition (33). We do that by induction on the cardinality of $R(w)$. Suppose that this is true for all w' such that $R(w')$ has cardinality smaller than $R(w)$. Let U be the subset of X consisting of $z \in X$ such that $R(z) = R(w)$. This is a single cluster in X. In case it satisfies condition (33), the result follows. If it does not, let S be the subset of those elements $v \in R(w)$ for which there is no $w' \in U$ such that $f(w') = v$. According to our assumption S is not empty and hence by condition (31) there are $v \in S$ and z such that $f(z) = v$ and $S \subseteq R(z) \subseteq R(w)$. Since $f(z) = v$ and $v \in S$, we must have $z \notin U$, that is $R(z) \neq R(w)$. However, $R(z) \subseteq R(w)$, thus we have the strict inclusion $R(z) \subset R(w)$ and, as a consequence, our induction hypothesis applies to z. This means that z belongs to a chain C'_z satisfying (33) and we can put $C_w = U \cup C'_z$ to find the desired chain containing w. To see this, notice that for any $v \in R(w)$ either there is a $w' \in U$ such that $f(w') = v$ or v is in $S \subseteq R(z) \subseteq R(w)$. Hence there is some $w' \in C'_z$ such that $f(w') = v$. The same applies to the case $v \in R(\tilde{w})$ for $\tilde{w} \in U$, because $R(\tilde{w}) = R(w)$ by the definition of U. Thus, we have found a finite standard frame for **S4.3** (namely X' endowed with the disjoint unions of the $\geq_R$) which the given conservative finite one-step frame (X, Y, f, R) validating Goré's rules (30) is a p-morphic image of. The result follows by Theorem 2. ⊣

9.2. **A case study: GL.** The Gödel-Löb modal logic **GL** can be axiomatized by the single axiom $\Box(\Box x \to x) \to \Box x$. This system is known to have the fmp and to be complete with respect to the class of finite irreflexive transitive frames. From the proof-theoretic side, the following rule

$$\frac{x \wedge \Box x \wedge \Box y \to y}{\Box x \to \Box y} \tag{34}$$

has been proposed in [2], and shown to admit cut-elimination.[12] We are going to analyze **GL** *when equivalently axiomatized by rule* (34) and show by using Theorem 2, that it has both the fmp and bpp.

We manually analyze the validity of rule (34) in a one-step frame. It would also be possible (by introducing specific Ackermann rules for fixpoint first-order logic) to apply the machinery of Section 7 for the first part of our argument below, see the techniques of [8]. The rule (34) is not valid on a one step frame (W_1, W_0, R, f) iff

$$\exists P, \exists Q \subseteq W_0 \ (f^*(P) \cap \Box_R P \cap \Box_R Q \subseteq f^*(Q) \ \& \ \Box_R P \not\subseteq \Box_R Q)$$

i.e. iff

$$\exists w \in W_1,\ \exists P, \exists Q \subseteq W_0 \ \ (\exists_f [f^*(P) \cap \Box_R P \cap \Box_R Q] \subseteq Q \ \& \ R(w) \subseteq P \ \& \ R(w) \not\subseteq Q)$$

(here $\exists_f$ is direct image along f). We can eliminate now P by the Ackermann rule and obtain

$$\exists w \in W_1,\ \exists Q \subseteq W_0 \ (\exists_f [f^*(R(w)) \cap \Box_R R(w) \cap \Box_R Q] \subseteq Q \ \& \ R(w) \not\subseteq Q).$$

We let $Q := \mu(Y, w)\ \exists_f (f^*(R(w)) \cap \Box_R R(w) \cap \Box Y)$ be the minimal assignment for Q. Then we have $\exists w \in W_1 (R(w) \not\subseteq \mu(Y, w) \exists_f (f^*(R(w)) \cap \Box_R R(w) \cap \Box_R Y)$. This means that our LFP(FO)-correspondent is the formula

$$\forall w \ R(w) \subseteq \mu(Y, w)\ \exists_f (f^*(R(w)) \cap \Box_R R(w) \cap \Box_R Y). \tag{35}$$

We want to have a better version of (35) in finite step frames. For this we need the following lemma.

Lemma 6. *Let (W_1, W_0, R, f) be a one-step frame. Then for every $w \in W_1$, we have*

$$\mu(Y, w)\ \exists_f (f^*(R(w)) \cap \Box_R R(w) \cap \Box_R Y) \subseteq \{f(w') \mid R(w') \subset R(w)\} \tag{36}$$

(*here* $\subset$ *stands for strict inclusion*).

Proof. Since $\mu(Y, w)$ is the minimum prefix point, it suffices to show that $\{f(w') \mid R(w') \subset R(w)\}$ is also a prefix point, namely that we have

$$\exists_f (f^*(R(w)) \cap \Box_R R(w) \cap \Box_R \{f(w') \mid R(w') \subset R(w)\}) \subseteq \{f(w') \mid R(w') \subset R(w)\}$$

i.e. by adjunction between direct and inverse image

$$f^*(R(w)) \cap \Box_R R(w) \cap \Box_R \{f(w') \mid R(w') \subset R(w)\} \subseteq f^*(\{f(w') \mid R(w') \subset R(w)\}) \tag{37}$$

To show (37), pick z such that (i) $f(z) \in R(w)$; (ii) $R(z) \subseteq R(w)$; (iii) $R(z) \subseteq \{f(w') \mid R(w') \subset R(w)\}$. We need to find w' such that $f(z) = f(w')$ and $R(w') \subset R(w)$. Now, if $f(z) \in R(z)$ the result follows by (iii). Otherwise, we can take $w' := z$ and obtain $R(z) \subseteq R(w)$ by (ii). Since $f(z) \in R(w) \setminus R(z)$ by (i), the inclusion is indeed proper. The result follows. ⊣

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[12] To be precise, [2] has a sequent notation for the rule (he has commas where we have $\wedge$ and a finite list Γ where we have a single variable x), but this difference is immaterial for our purposes.

We have the following corollary of Lemma 6 and (35).

Corollary 3. *A finite one-step frame validates Avron's rule* (34) *iff we have*

$$\forall w\ (R(w) \subseteq \{f(w') \mid R(w') \subset R(w)\}). \tag{38}$$

Proof. By the above, we only need to show that (38) implies (35) in finite frames. Assume (38) and pick $v \in R(w)$. By (38) there is w' such that $f(w') = v$ and $R(w') \subset R(w)$. Let k be the cardinality of $R(w')$. We prove by induction on k that $v \in N_{k+1}$, where N_{k+1} is the $k+1$-th member of the ascending approximation chain of the fixpoint (i.e. we have $N_0 := \emptyset$ and $N_{k+1} := \exists_f(f^*(R(w)) \cap \Box_R R(w) \cap \Box_R N_k)$). For $k = 0$, the claim is easy; for $k > 0$, we show that $w' \in f^*(R(w)) \cap \Box_R R(w) \cap \Box_R N_k$. That $w' \in f^*(R(w)) \cap \Box_R R(w)$ is clear by the choice of w'. To prove that $R(w') \subseteq N_k$, pick $\tilde{v} \in R(w')$. By (38) there is $\tilde{w}'$ such that $f(\tilde{w}') = \tilde{v}$ and $R(\tilde{w}') \subset R(w') \subset R(w)$. Thus the induction hypothesis applies and we obtain $\tilde{v} \in N_k$. ⊣

Notice that condition (38) implies the one-step transitivity condition (25). Corollary 3 allows us to prove that

Theorem 4. GL *axiomatized by Avron's rule* (34) *has the bpp and fmp.*

Proof. In order to apply Theorem 2, let us fix a finite conservative one-step frame (W_1, W_0, f, R) satisfying (38). We need to build a standard finite frame (W', S) (with irrefelexive and transitive S) and a surjective map $\mu : W' \longrightarrow W_1$ satisfying (15). The idea is to build for every $w \in W_1$ an irreflexive transitive finite tree (T_w, S_w) and a function $\mu : T_x \longrightarrow W_1$ mapping the root of T_w to w and satisfying (15). Once this is done, we can simply take the disjoint union of the (T_w, S_w) to get the desired (W', S). The constructions of μ_x and of (T_w, S_w) are indeed easily obtained by induction on the cardinality of $R(w)$ (this induction works because our step frame is finite and (38) holds). In the induction step, if $R(w) = \{v_1, \dots, v_n\}$, we take $w_1, \dots, w_n \in W_1$ such that for all $i = 1, \dots, n$ we have $f(w_i) = v_i$ and $R(w_i) \subset R(w)$; then we add a root to the disjoint union of the already built trees $(T_{w_1}, S_{w_1}), \dots, (T_{w_n}, S_{w_n})$. Thus, we have found a finite standard frame for **GL** (namely (W', S) above) which the given conservative finite one-step frame (W_1, W_0, f, R) validating Avron's rule (34) is a p-morphic image of. The result follows by Theorem 2. ⊣

We point out that the standard axiomatization of **GL** is bad (i.e. it does not satisfies the bpp). The standard axiomatization for **GL** consists of the transitivity rule (24) for **K4** plus Löb's rule

$$\frac{\Box x \to x}{x} \tag{39}$$

If we apply correspondence machinery to this rule, we obtain the following LFP(FO)-condition for one-step conservative frames (W_1, W_0, f, R):

$$\forall v \in W_0\ v \in \mu(Y, v)\,(\exists_f \Box_R Y). \tag{40}$$

Now, it follows from the above, that conservative finite one-step frames which are p-morphic images of standard finite frames for **GL** are precisely those that satisfy (38). This, together with the fact that **GL** has the fmp, implies that *either* (40) is equivalent to (38) for transitive conservative finite one-step frames, *or* that the axiomatization of **GL** given by the transitivity rule plus Löb's rule does not satify the bpp.

Example 3. The following (W_1, W_0, f, R) is an example of a finite conservative transitive one-step frame satisfying (40) but not (38). Take $W_0 := \{v_1, v_2\}$, $W_1 := \{w_1, w_2, w_3\}$, $R := \{(w_1, v_1), (w_2, v_2)\}$, $f(w_1) := f(w_2) := v_1$, $f(w_3) = v_2$. If we compute $\mu(Y, w)\,(\exists_f \Box_R Y)$, we converge in three steps with $\emptyset \subseteq \{v_2\} \subseteq \{v_1, v_2\}$, thus (40) is true. Conservativity (Definition 4) and transitivity (25) are easily seen to be true. However, (38) fails because we have that w_1 is the only element which is such that $f(w_1) \in R(w_1) = \{v_1\}$ and the inclusion $R(w_1) \subseteq R(w_1)$ is obviously not strict.

Finally, one may wonder what happens if we apply the procedure of Proposition 3 to the **GL** axiom $\Box(\Box x \to x) \to \Box x$. We obtain the rule

$$\frac{x \to (\Box y \to y)}{\Box x \to \Box y} \tag{41}$$

This may be seen as just a reformulation of Avron's rule. In fact, the two rules are inter-derivable with derivations of modal complexity 1. Firstly, the derivability of (41) from (34) is clear (because $(x \to (\Box y \to y)) \to (x \wedge \Box x \wedge \Box y \to y)$ is a tautology). For the other direction, use the fact that

$$(x \wedge \Box x \wedge \Box y \to y) \to (x \to (\Box(x \wedge y) \to (x \wedge y)))$$

is a theorem in any normal modal logic and apply rule (41) with y replaced by $x \wedge y$. Thus, we have an example where the general procedure of Proposition 3 suggests a 'good' inference rule.

9.3. **A case study: S5.** The modal logic **S5** is obtained from **T** by adding it the axiom $\Diamond x \to \Box\Diamond x$; semantically, the system **S5** is complete with respect to (finite) Kripke frames whose accessibility relation is an equivalence relation. Despite its semantic simplicity, **S5** is challenging for proof-theoretic design. In a usual formulation [38] (see also the discussion in [3, 39]), **S5** is axiomatized by adding to a cut-eliminating sequent calculus for **T** the following rule:

$$\frac{\Box\Gamma \Rightarrow y, \Box\Delta}{\Box\Gamma \Rightarrow \Box y, \Box\Delta} \tag{42}$$

In the resulting system, cuts cannot be completely eliminated, but can be limited to subformulae of the sequent to be proved. This 'analytic' cut-elimination property is sufficient to imply the bpp, and thus we should be able to obtain the bpp directly by our methods. We show that it is indeed so.

A preliminary remark is in order. In sequent calculi, one adopts the meta-notation $\Box\Gamma$ for the finite set $\{\Box x \mid x \in \Gamma\}$. Notice however that the comma represents conjunction on the left of the sequent implication $\Rightarrow$ and disjunction on the right of it. Since we do not care about equivalence modulo complexity 1 proofs, we are legitimate to replace a formula such as $\bigwedge_{x\in\Gamma} \Box x$ by $\Box \bigwedge_{x\in\Gamma} x$, hence we can shrink a finite set $\Box\Gamma$ on the left of the sequent implication to a single formula $\Box x$ without compromizing our analysis. We cannot however do the same on the right of the sequent implication because $\Box$ does not distribute over disjunctions. This is why (42) *is in fact an infinite sequence of rules for our purposes*; this infinite sequence can be more conveniently written as

$$\frac{\Box x \to y \vee \Box z_1 \vee \cdots \vee \Box z_n}{\Box x \to \Box y \vee \Box z_1 \vee \cdots \vee \Box z_n} \tag{43}$$

In particular, the correspondence algorithm must try to find a condition which is equivalent to the whole infinite sequence of rules (43) (similarly to the **S4.3** case above).

To facilitate our task let us rewrite (43) as

$$\frac{\Box x \wedge \Diamond z_1 \wedge \cdots \wedge \Diamond z_n \to y}{\Box x \wedge \Diamond z_1 \wedge \cdots \wedge \Diamond z_n \to \Box y} \tag{44}$$

(this is a modal complexity 1 transformation). The n-th rule (44) is not valid in a finite one-step frame (X, Y, f, R) iff there exists $Q, R, P_1 \ldots, P_n \subseteq Y$ such that

$$\Box Q \cap \Diamond P_1 \cap \cdots \cap \Diamond P_n \subseteq f^*(R) \ \&\ \Box Q \cap \Diamond P_1 \cap \cdots \cap \Diamond P_n \not\subseteq \Box R$$

Using adjunction between direct image $\exists_f$ and inverse image f^* and Ackermann rule, we equivalently get

$$\Box Q \cap \Diamond P_1 \cap \cdots \cap \Diamond P_n \not\subseteq \Box \exists_f (\Box Q \cap \Diamond P_1 \cap \cdots \cap \Diamond P_n).$$

Thus the rule is not valid iff there is $w \in X$ such that

$$R(w) \subseteq Q \ \&\ w \in \Diamond P_1 \cap \cdots \cap \Diamond P_n \ \&\ R(w) \not\subseteq \exists_f (\Box Q \cap \Diamond P_1 \cap \cdots \cap \Diamond P_n).$$

After eliminating Q by Ackermann lemma and unravelling the definition of $\Diamond$, we obtain

$$\exists v_1, \ldots, v_n (\bigwedge_{i=1}^{n} wRv_i \ \&\ v_i \in P_i) \ \&\ R(w) \not\subseteq \exists_f (\Box R(w) \cap \Diamond P_1 \cap \cdots \cap \Diamond P_n).$$

At this point we can also eliminate the P_i and arrive at

$$\exists v_1, \ldots, v_n \ (\bigwedge_{i=1}^{n} wRv_i \ \&\ R(w) \not\subseteq \exists_f (\Box R(w) \cap \bigcap_{i=1}^{n} \Diamond \{v_i\})).$$

If we look at all these conditions (varying n) and keep in mind that our step frames are finite, we realize that it is sufficient to take $\{v_1, \ldots, v_n\} := R(w)$ to get all of them simultaneously. Thus we obtain a single condition, namely

$$R(w) \not\subseteq \exists_f (\Box R(w) \cap \bigcap_{v \in R(w)} \Diamond \{v\})).$$

The negation of this sentence is

$$\forall w \in X, \forall v \in Y \ (wRv \to \exists \tilde{w} \in X \ (f(\tilde{w}) = v \ \&\ R(w) = R(\tilde{w})). \tag{45}$$

Theorem 5. **S5**, *as axiomatized by adding to* **T** *the rules* (43), *has the bpp and the fmp.*

Proof. We know that a finite conservatiove one-step frame (X, Y, f, R) for this system satisfies the step-reflexivity condition $\forall w \ (f(w) \in R(w))$ together with (45). To build a standard frame (W', S) for **S5** and a surjective map $\mu : W' \longrightarrow W_1$ satisfying (15) it is sufficient to take $W' := W, \mu := id$ and S to be the relation defined by $wS\tilde{w}$ iff $R(w) = R(\tilde{w})$. ⊣

Again, notice that if we apply the procedure of Proposition 3 to the axiom $\Diamond x \to \Box \Diamond x$, we obtain the rule

$$\frac{\Diamond x \leq y}{\Diamond x \to \Box y.} \tag{46}$$

Correspondence applied to this rule gives the following condition

$$\forall w \forall v_1 \forall v_2 (wRv_1 \ \&\ wRv_2 \to \exists w_1 (f(w_1) = v_1 \ \&\ w_1 Rv_2). \tag{47}$$

This condition is 'bad' because it does not guarantee the extension property: the latter is guaranteed iff (45) holds and it is easy to see (by using for instance the step frame of Example 2 as a counterexample) that (47) is weaker than (45). Thus, if we axiomatize **S5** by adding to **T** the rule (46), we do not get the bpp.

10. Conclusions

We have developed a uniform method for obtaining information on a logic and its axiomatizations. The method relies on embedding properties of finite one-step modal algebras. By a step-version of the classical correspondence theory, it is possible to dualize the procedure to one-step frames and to make the application of our methodology completely algorithmic in the most simple cases. This makes concrete the possibility of mechanizing the metatheory of propositional modal logic.

We also analyzed our approach in three nontrivial cases, namely for the cut-free axiomatizations of **S4.3**, **GL** and **S5** known from the literature and we succeeded in all three cases in proving the fmp and bpp by our methods. The proof is not entirely mechanical in these cases, but it is still based on a common feature: induction on the cardinality of accessible worlds in finite one-step frames.

We still have to face a large amount of questions, that are hard to answer at the moment because our methodology is quite novel. In particular, it would be interesting to see whether the method can fruitfully apply to complicated logics arising in computer science applications (such as dynamic logic, linear or branching time temporal logics, the modal μ-calculus, etc.).

Another potentially interesting question concerns finite axiomatizability: one in fact can always trivially force the bpp by taking one rule for each derivable formula, the point is to show that finitely many rules suffice to achieve the bpp (this is indeed what brings decidability from the bpp). The notion of finite axiomatizability itself requires a careful formulation: a rule like 42 should be seen as a single rule, despite the fact that we cannot shrink the formulae Δ occurring on the right to a single formula. What we need here is a notion of a rule with context, like in [4, 35, 36].

A final important series of questions concerns the clarification of the relationship between our techniques and standard techniques employed in filtrations and analytic tableaux. The connections with hypersequents approaches [3, 17] need special investigation; in fact, hypersequents have shown to be a powerful technique capable of dealing with a large class of modal semantic conditions [33] they also seem to provide nice and relatively simple axiomatizations (compare for instance the finite axiomatization for **S4.3** given in [31] with Goré rules from Subsection 9.1 above). Notice that since an hypersequent H is interpreted (e.g. in **S4**-systems) by taking the necessitation of the Boxes of the interpretations of the sequents occurring in H, the use of an hypersequent of modal complexity 1 yields always a modal complexity 2 proofs, so our results are not applicable to hypersequent calculi. We are nevertheless confident that our machinery can be modified so to encompass also applications to hypersequent calculi.

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