COALGEBRAIC GEOMETRIC LOGIC: BASIC THEORY

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Abstract. Using the theory of coalgebra, we introduce a uniform framework for adding modalities to the language of propositional geometric logic. Models for this logic are based on coalgebras for an endofunctor on some full subcategory of the category of topological spaces and continuous functions. We investigate derivation systems, soundness and completeness for such geometric modal logics, and we we specify a method of lifting an endofunctor on **Set**, accompanied by a collection of predicate liftings, to an endofunctor on the category of topological spaces, again accompanied by a collection of (open) predicate liftings. Furthermore, we compare the notions of modal equivalence, behavioural equivalence and bisimulation on the resulting class of models, and we provide a final object for the corresponding category.

1. Introduction

Propositional geometric logic arose at the interface of (pointfree) topology, logic and theoretical computer science as the logic of *finite observations* [1, 39]. Its language is constructed from a set of proposition letters by applying finite conjunctions and arbitrary disjunctions, these being the propositional operations preserving the property of finite observability. Through an interesting topological connection, formulas of geometric logic can be interpreted in the frame of open sets of a topological space. Central to this connection is the well-known dual adjunction between the category **Frm** of frames and frame morphisms and the category **Top** of topological spaces and continuous maps, which restricts to several interesting Stone-type dualities [20].

Coalgebraic logic is a framework in which generalised versions of modal logics are developed parametric in the signature of the language and a functor $\mathsf{T} : \mathbf{C} \to \mathbf{C}$ on some base category $\mathbf{C}$. With classical propositional logic as base logic, two natural choices for the base category are **Set**, the category of sets and functions, and **Stone**, the category of Stone spaces and continuous functions, i.e. the topological dual to the algebraic category of

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Boolean algebras. Coalgebraic logic for endofunctors on **Set** has been well investigated and still is an active area of research, see e.g. [12, 28]. In this setting, modal operators can be defined using the notion of relation lifting [31] or predicate lifting [32]. Coalgebraic logic in the category of Stone coalgebras has been studied in [27, 18, 14, 6], and there is a fairly extensive literature on the design of a coalgebraic modal logic based on a general Stone-type duality (or adjunction), see for instance [11, 24] and references therein.

In this paper we investigate some links between coalgebraic logic and geometric logic. That is, we use methods from coalgebraic logic to introduce modal operators to the language of geometric logic, with the intention of studying interpretations of these logics in certain topological coalgebras. Note that extensions of geometric logic with the basic modalities $\Box$ and $\Diamond$, which are closely related to the topological Vietoris construction, have received much attention in the literature, see [39] for some early history. A first step towards developing coalgebraic geometric logic was taken in [37], where a method is explored to lift a functor on **Set** to a functor on the category **KHaus** of compact Hausdorff spaces, and the connection is investigated between the lifted functor and a relation-lifting based "cover" modality.

Our aim here is to develop a framework for the coalgebraic geometric logics that arise if we extend geometric logic with modalities that are induced by appropriate *predicate liftings*. Guided by the connection between geometric logic and topological spaces, we choose the base category of our framework to be **Top** itself, or one of its full subcategories such as **Sob** (sober spaces), **KSob** (compact sober spaces) or **KHaus** (compact Hausdorff spaces). On this base category $\mathbf{C}$ we then consider an arbitrary endofunctor T which serves as the type of our topological coalgebras. Furthermore, we shall see that if we want our formulas to be interpreted as *open sets* of the coalgebra carrier, we need the predicate liftings that interpret the modalities of the language to satisfy some natural *openness* condition. Summarizing, we shall study the coalgebraic geometric logic induced by (1) a functor $\mathsf{T} : \mathbf{C} \to \mathbf{C}$, where $\mathbf{C}$ is a full subcategory of **Top**, and (2) a set Λ of open predicate liftings for T. As running examples we take the combination of the basic modalities for the Vietoris functor, and that of the monotone box and diamond modalities for various topological manifestations of the monotone neighborhood functor on **Set**. The structures providing the semantics for our coalgebraic geometric logics are the T-*models* consisting of a T-coalgebra together with a valuation mapping proposition letters to open sets in the coalgebra carrier.

The main results that we report on here are the following:

- Section 4 contains a detailed description of the *monotone neighbourhood functor* on **KHaus**, which naturally extends the monotone functor on **Stone** [18] that corresponds to monotone modal logic.
- In Section 5 we discuss derivation systems for coalgebraic geometric logic, based on *consequence pairs*, and derive a general completeness result.
- After that, in Section 6 we adapt the method of [25] in order to lift a **Set**-functor together with a collection of predicate liftings to an endofunctor on **Top**. We obtain the Vietoris functor and monotone functor on **KHaus** as restrictions of such lifted functors.
- In Section 7, we construct a final object in the category of T-models, where T is an endofunctor on **Top** which preserves sobriety and admits a Scott-continuous, characteristic geometric modal signature.

- Finally, in Section 8 we transfer the notion of Λ-bisimilarity from [15, 3] to our setting, and we compare this to geometric modal equivalence, behavioural equivalence and Aczel-Mendler bisimilarity. Our main finding is that on the categories **Top**, **Sob** and **KSob**, the first three notions coincide, provided Λ and T meet some reasonable conditions.

We finish the paper with listing some questions for further research.

2. Preliminaries

We briefly fix notation and review some preliminaries.

2.1. **Categories and functors.** We use a **bold font** for categories. We assume familiarity with the following categories and functors:

- **Set** is the category of sets and functions;
- **Top** is the category of topological spaces and continuous functions;
- **KHaus** and **Stone** are the full subcategories of **Top** whose objects are compact Hausdorff spaces and Stone spaces respectively;
- **BA** is the category of Boolean algebras and Boolean algebra morphisms.

Categories can be connected by functors. We use a sans serif font for functors. In particular, the following functors are regularly used in this paper:

- $\mathsf{U} : \mathbf{Top} \to \mathbf{Set}$ is the forgetful functor sending a topological space to its underlying set. The functor U restricts to every subcategory of **Top**, in which case we shall abuse notation and also call it U;
- $\mathsf{P} : \mathbf{Set} \to \mathbf{Set}$ and $\breve{\mathsf{P}} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ are the covariant and contravariant powerset functor respectively;
- $\mathsf{Q} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{BA}$ sends a set to its powerset Boolean algebra and a function to the inverse image map viewed as morphism in **BA**;
- $\Omega : \mathbf{Top} \to \mathbf{Set}$ sends a topological space to the set of opens.

Note that $\breve{\mathsf{P}} = \mathsf{U} \circ \mathsf{Q}$. More categories and functors will be defined along the way. We use the symbol $\equiv$ for categorical equivalence.

2.2. **Coalgebra.** Let $\mathbf{C}$ be a category and T an endofunctor on $\mathbf{C}$. A T-*coalgebra* is a pair (X, γ) where X is an object in $\mathbf{C}$ and $\gamma : X \to \mathsf{T}X$ is a morphism in $\mathbf{C}$. A T-*coalgebra morphism* between two T-coalgebras (X, γ) and (X', γ') is a morphism $f : X \to X'$ in $\mathbf{C}$ satisfying $\gamma' \circ f = \mathsf{T}f \circ \gamma$. The collection of T-coalgebras and T-coalgebra morphisms forms a category, which we shall denote by $\mathbf{Coalg}(\mathsf{T})$. The category $\mathbf{C}$ is called the *base category* of $\mathbf{Coalg}(\mathsf{T})$.

The notion of an *algebra* for T is defined dually, and gives rise to the category $\mathbf{Alg}(\mathsf{T})$.

Example 2.1 (Kripke frames)**.** Kripke frames correspond 1-1 with P-coalgebras. For a Kripke frame (X, R) define $\gamma_R : X \to \mathsf{P}X : x \mapsto \{y \mid xRy\}$. Then (X, γ_R) is a P-coalgebra. Conversely, for a P-coalgebra (X, γ) define R_γ by $xR_\gamma y$ iff $y \in \gamma(x)$. Then (X, R_γ) is a Kripke frame. It is not hard to see that $R_{\gamma_R} = R$ and $\gamma_{R_\gamma} = \gamma$, so we obtain a bijection between Kripke frames and P-coalgebras. Moreover, bounded morphisms between Kripke frames are precisely P-coalgebra morphisms. Thus, we have

$$\mathbf{Krip} \cong \mathbf{Coalg}(\mathsf{P}),$$

where **Krip** is the category of Kripke frames and bounded morphisms.

Example 2.2 (Monotone neighbourhood frames)**.** Let $\mathsf{D} : \mathbf{Set} \to \mathbf{Set}$ be the functor given on objects by

$$\mathsf{D}X = \{W \subseteq \mathsf{P}X \mid \text{ if } a \in W \text{ and } a \subseteq b \text{ then } b \in W\},$$

where X is a set. For a morphism $f : X \to X'$ define

$$\mathsf{D}f : \mathsf{D}X \to \mathsf{D}X' : W \mapsto \{a' \in \mathsf{P}X' \mid f^{-1}(a') \in W\}.$$

Then the category of monotone frames a bounded morphisms is isomorphic to $\mathbf{Coalg}(\mathsf{D})$ [10, 17, 18].

2.3. **Coalgebraic logic for Set-coalgebras.** Let T be a $\mathbf{Set}$-functor and Φ a set of proposition letters. A T-*model* is a triple (X, γ, V) where (X, γ) is a T-coalgebra and $V : \Phi \to \mathsf{P}X$ is a valuation of the proposition letters. An n*-ary predicate lifting for* T is a natural transformation

$$\lambda : \breve{\mathsf{P}}^n \to \breve{\mathsf{P}} \circ \mathsf{T},$$

where $\breve{\mathsf{P}}^n$ denotes the n-fold product of the contravariant powerset functor. A predicate lifting is called *monotone* if for all sets X and subsets $a_1, \ldots, a_n, b \subseteq X$ we have $\lambda_X(a_1, \ldots, a_i, \ldots, a_n) \subseteq \lambda_X(a_1, \ldots, a_i \cup b, \ldots, a_n)$.

For a set Λ of predicate liftings for T, define the *language* $\mathrm{ML}(\Lambda)$ by

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n),$$

where $p \in \Phi$ and $\lambda \in \Lambda$ is n-ary. The *semantics* of $\varphi \in \mathrm{ML}(\Lambda)$ on a T-model $\mathfrak{X} = (X, \gamma, V)$ is given recursively by

$$\llbracket p \rrbracket^{\mathfrak{X}} = V(p), \quad \llbracket \varphi_1 \wedge \varphi_2 \rrbracket^{\mathfrak{X}} = \llbracket \varphi_1 \rrbracket^{\mathfrak{X}} \cap \llbracket \varphi_2 \rrbracket^{\mathfrak{X}}, \quad \llbracket \neg\varphi \rrbracket^{\mathfrak{X}} = X \setminus \llbracket \varphi \rrbracket^{\mathfrak{X}},$$
$$\llbracket \heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n) \rrbracket^{\mathfrak{X}} = \gamma^{-1}(\lambda(\llbracket \varphi_1 \rrbracket^{\mathfrak{X}}, \ldots, \llbracket \varphi_n \rrbracket^{\mathfrak{X}})),$$

where $p \in \Phi$ and λ ranges over Λ.

Example 2.3 (Kripke models)**.** Consider for P-models the predicate liftings $\lambda^{\Box}, \lambda^{\Diamond} : \breve{\mathsf{P}} \to \breve{\mathsf{P}} \circ \mathsf{P}$ given by

$$\lambda^{\Box}_X(a) = \{b \in \mathsf{P}X \mid b \subseteq a\}, \qquad \lambda^{\Diamond}_X(a) = \{b \in \mathsf{P}X \mid b \cap a \neq \emptyset\}.$$

Then $\lambda^{\Box}$ and $\lambda^{\Diamond}$ yield the usual Kripke semantics of $\Box$ and $\Diamond$.

Example 2.4 (Monotone neighbourhood frames)**.** Monotone neighbourhood models are precisely D-models, where D is the functor defined in Example 2.2. The usual semantics for the box and diamond in this setting can be obtained from the predicate liftings given by

$$\lambda^{\Box}_X(a) = \{W \in \mathsf{D}X \mid a \in W\}, \qquad \lambda^{\Diamond}_X(a) = \{W \in \mathsf{D}X \mid X \setminus a \notin W\}. \tag{2.1}$$

We refer to [28] for many more examples of coalgebraic logic for $\mathbf{Set}$-functors.

2.4. **Geometric logic.** Let Φ be a set of proposition letters. The language $\mathrm{GL}(\Phi)$ of geometric formulas is given by

$$\varphi ::= \top \mid \bot \mid p \mid \varphi \wedge \varphi \mid \bigvee_{i \in I} \varphi_i$$

where $p \in \Phi$ and I is some index set. *Coherent* formulas are defined in the same way, but without infinitary disjuctions. These are also known as *positive* logic.

We describe the logical system as a collection of binary consequence pairs, written as $\varphi \lhd \psi$. A *geometric logic* is a collection consequence pairs closed under the following rules: *Identity*

$$\varphi \lhd \varphi,$$

cut

$$\frac{\varphi \lhd \psi \quad \psi \lhd \chi}{\varphi \lhd \chi},$$

the *conjunction rules*

$$\varphi \lhd \top, \qquad \varphi \wedge \psi \lhd \varphi, \qquad \varphi \wedge \psi \lhd \psi, \qquad \frac{\varphi \lhd \psi \quad \varphi \lhd \chi}{\varphi \lhd \psi \wedge \chi},$$

the *disjunction rules*

$$\varphi \lhd \bigvee S \quad (\varphi \in S), \qquad \frac{\varphi \lhd \psi \quad (\text{all } \varphi \in S)}{\bigvee S \lhd \psi}$$

and *frame distributivity*

$$\varphi \wedge \bigvee S \lhd \bigvee \{\varphi \wedge \psi \mid \psi \in S\}.$$

Note that these are in fact all schemata. We write $\mathcal{GL}$ for the minimal geometric logic, i.e. the smallest collection of consequence pairs closed under the axioms and rules given above. We write $\varphi \vdash_{\mathcal{GL}} \psi$ if the consequence pair $\varphi \lhd \psi$ is in $\mathcal{GL}$.

Note that frame distributivity allows us to reduce every formula to a disjunction of finite conjunctions of proposition letters. Therefore, modulo equivalence, the formulas form a set. A collection S of geometric formulas is called *directed* if for every pair $\varphi, \psi \in S$ there exists $\chi \in S$ such that $\varphi \vdash \chi$ and $\psi \vdash \chi$.

The topological semantics and algebraic semantics of geometric logic are given by topological spaces and frames.

2.5. **Frames and spaces.** A *frame* is a complete lattice F in which for all $a \in F$ and $S \subseteq F$ the infinite distributive law holds:

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}.$$

A *frame homomorphism* is a function between frames that preserves finite meets and arbitrary joins.

For $a, b \in F$ we say that a is *well inside* b, notation: $a \lessdot b$, if there is a $c \in F$ such that $c \wedge a = \bot$ and $c \vee b = \top$. An element $a \in F$ is called *regular* if $a = \bigvee \{b \in F \mid b \lessdot a\}$ and a frame is called *regular* if all of its elements are regular. The *negation* of $a \in F$ is defined as ${\sim}a = \bigvee \{b \in F \mid a \wedge b = \bot\}$. A frame is said to be *compact* if $\bigvee S = \top$ implies that there is a finite subset $S' \subseteq S$ such that $\bigvee S' = \top$.

Lemma 2.5. *For all elements a, b in a frame F we have $a \lessdot b$ iff ${\sim}a \vee b = \top$.*

Proof. See [20, III1.1]. $\square$

Lemma 2.6. *Finite meets and arbitrary joins of regular elements are regular.*

Proof. It is known that $d \leq c \lessdot a \leq b$ implies $d \lessdot b$. We first show that $c \lessdot a$ and $d \lessdot b$ implies $c \wedge d \lessdot a \wedge b$. It is clear that $c \wedge d \lessdot a$ and $c \wedge d \lessdot b$. Since ${\sim}(c \wedge d) \vee (a \wedge b) = ({\sim}(c \wedge d) \vee a) \wedge ({\sim}(c \wedge d) \vee b) = \top \wedge \top = \top$ we know $c \wedge d \lessdot a \wedge b$.

Now suppose a and b are regular elements, then

$$a \wedge b = \bigvee \{c \mid c \lessdot a\} \wedge \bigvee \{d \mid d \lessdot b\} = \bigvee \{c \wedge d \mid c \lessdot a, d \lessdot b\} \leq \bigvee \{c \mid c \lessdot a \wedge b\} \leq a \wedge b,$$

so $a \wedge b$ is regular. If a_i is regular for all i in some index set I, then

$$\bigvee_{i \in I} a_i = \bigvee_{i \in I} \Big(\bigvee \{c \mid c \lessdot a_i\} \Big) \leq \bigvee \Big\{ c \mid c \lessdot \bigvee_{i \in I} a_i \Big\} \leq \bigvee_{i \in I} a_i,$$

so an arbitrary join of regular elements is regular. □

Frames can be presented by generators and relations.

Definition 2.7. A *presentation* is a pair $\langle G, R \rangle$ where G is a set of generators and R is a collection of relations between expressions constructed from the generators using arbitrary joins and finite meets.

Let F be a frame and $\mathsf{Z}F$ its underlying set. We say that $\langle G, R \rangle$ *presents* F if there is an assignment $f : G \to \mathsf{Z}F$ of the generators such that (i), (ii) and (iii) hold:

(i) The set $\{f(g) \mid g \in G\}$ generates F.

The assignment f can be extended to an assignement $\widetilde{f}$ for any expression x build from the generators in G using $\wedge$ and $\bigvee$. We require:

(ii) If $x = x'$ is a relation in R, then $\widetilde{f}(x) = \widetilde{f}(x')$ in F.

(iii) For any F' and assignment $f' : G \to \mathsf{Z}F'$ satisfying property (ii) there exists a frame homomorphism $h : F \to F'$ such that the diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr, "f'"'] & \mathsf{Z}F \arrow[d, "\mathsf{Z}h"] \\ & \mathsf{Z}F' \end{tikzcd}$$

commutes.

The frame homomorphism from (iii) is necessarily unique, because the image of the generating set $\{f(g) \mid g \in G\}$ under h is determined by the diagram. A detailed account of frame presentations may be found in chapter 4 of [39].

Remark 2.8. We will regularly define a frame homomorphism $F \to F'$ from a frame F presented by $\langle G, R \rangle$ to some frame F'. By Definition 2.7 it suffices to give an assignment $f' : G \to F'$ such that (ii) holds, because this yields a unique frame homomorphism $F \to F'$. By abuse of notation, we will denote the unique frame homomorphism $F \to F'$ such that the diagram in (iii) commutes with f' as well.

The next fact allows us to define a frame by specifying generators and relations. A proof can be found in [20, Proposition II2.11].

Fact 2.9. Any presentation by generators and relations presents a unique frame.

The collection of open sets of a topological space $\mathbb{X}$ forms a frame, denoted $\mathsf{opn}\mathbb{X}$. A continuous map $f : \mathbb{X} \to \mathbb{X}'$ induces $\mathsf{opn}f = f^{-1} : \mathsf{opn}\mathbb{X}' \to \mathsf{opn}\mathbb{X}$ and with this definition opn is a contravariant functor $\mathbf{Top} \to \mathbf{Frm}$. A frame is called *spatial* if it isomorphic to $\mathsf{opn}\mathbb{X}$ for some topological space $\mathbb{X}$.

A *point* of a frame F is a frame homomorphism $p : F \to 2$, with $2 = \{\top, \bot\}$ the two-element frame. Let $\mathsf{pt}F$ be the collection of points of F endowed with the topology $\{\widetilde{a} \mid a \in F\}$, where $\widetilde{a} = \{p \in \mathsf{pt}F \mid p(a) = \top\}$. For a frame homomorphism $f : F \to F'$ define $\mathsf{pt}f : \mathsf{pt}F' \to \mathsf{pt}F$ by $p \mapsto p \circ f$. The assignment pt defines a functor $\mathbf{Frm} \to \mathbf{Top}$. A topological space that arises as the space of points of a lattice is called *sober*. The *sobrification* of a topological space $\mathbb{X}$ is $\mathsf{pt}(\mathsf{opn}\mathbb{X})$.

We denote by $\mathbf{Sob}$ and $\mathbf{KSob}$ the full subcategories of $\mathbf{Top}$ whose objects are sober spaces and compact sober spaces, respectively. Where $\mathbf{Frm}$ is the category of frames and frame homomorphisms, $\mathbf{SFrm}$, $\mathbf{KSFrm}$ and $\mathbf{KRFrm}$ are the full subcategories of $\mathbf{Frm}$ whose objects are spatial frames, compact spatial frames and compact regular frames, respectively. The functor $\mathsf{Z} : \mathbf{Frm} \to \mathbf{Set}$ is the forgetful functor sending a frame to the underlying set, and restricts to every subcategory of $\mathbf{Frm}$. Note that $\Omega = \mathsf{Z} \circ \mathsf{opn}$.

Fact 2.10. The functor pt is a right adjoint to opn. This adjunction restricts to a duality between the category of spatial frames and the category of sober spaces,

$$\mathbf{SFrm} \equiv \mathbf{Sob}^{\mathrm{op}}.$$

This duality restricts to the dualities

$$\mathbf{KSFrm} \equiv \mathbf{KSob}^{\mathrm{op}}$$

and

$$\mathbf{KRFrm} \equiv \mathbf{KHaus}^{\mathrm{op}}.$$

For a more thorough exposition of frames and spaces, and a proof of the statements in Fact 2.10 we refer to section C1.2 of [22]. We explicitly mention one isomorphism which is part of this duality, for we will encounter it later on.

Remark 2.11. Let $\mathbb{X}$ be a sober space. Then Fact 2.10 entails that there is an isomorphism $\mathbb{X} \to \mathsf{pt}(\mathsf{opn}\mathbb{X})$. This isomorphism is given by $x \mapsto p_x$, where p_x is the point given by

$$p_x : \mathsf{opn}\mathbb{X} \to 2 : \begin{cases} a \mapsto \top & \text{if } x \in a \\ a \mapsto \bot & \text{if } x \notin a \end{cases}$$

for all $x \in \mathbb{X}$ and $a \in \Omega\mathbb{X}$.

3. Logic for topological coalgebras

Although not all of our results can be proved for every full subcategory of $\mathbf{Top}$, we will give the basic definitions in full generality. To this end, we let $\mathbf{C}$ be some full subcategory of $\mathbf{Top}$ and define coalgebraic logic over base category $\mathbf{C}$. In particular $\mathbf{C} = \mathbf{KHaus}$ and $\mathbf{C} = \mathbf{Sob}$ will be of interest. Throughout this section T is an arbitrary endofunctor on $\mathbf{C}$ and we view Ω as a functor $\mathbf{C} \to \mathbf{Frm}$. Recall that Φ is an arbitrary but fixed set of proposition letters. We begin with defining the topological version of a predicate lifting, called an *open* predicate lifting.

3.1. Open predicate liftings.

Definition 3.1. An *open predicate lifting* for T is a natural transformation

$$\lambda : \Omega^n \to \Omega \circ \mathsf{T}.$$

An open predicate lifting is called *monotone in its i-th argument* if for every $\mathbb{X} \in \mathbf{C}$ and all $a_1, \ldots, a_n, b \in \Omega\mathbb{X}$ we have $\lambda_{\mathbb{X}}(a_1, \ldots, a_i, \ldots, a_n) \subseteq \lambda_{\mathbb{X}}(a_1, \ldots, a_i \cup b, \ldots, a_n)$, and *monotone* if it is monotone in every argument. It is called *Scott-continuous* in its i-th argument if for every $\mathbb{X} \in \mathbf{C}$ and every directed set $A \subseteq \Omega\mathbb{X}$ we have

$$\lambda_{\mathbb{X}}(a_1, \ldots, \bigcup A, \ldots, a_n) = \bigcup_{b \in A} \lambda_{\mathbb{X}}(a_1, \ldots, b, \ldots, a_n)$$

and *Scott-continuous* if it is Scott-continuous in every argument.

A collection of open predicate liftings for T is called a *geometric modal signature* for T. A geometric modal signature for a functor T is called *monotone* if every open predicate lifting in it is monotone, *Scott-continuous* if every predicate lifting in it is Scott-continuous, and *characteristic* if for every topological space $\mathbb{X}$ in $\mathbf{C}$ the collection

$$\{\lambda_{\mathbb{X}}(a_1, \ldots, a_n) \mid \lambda \in \Lambda \ n\text{-ary}, a_i \in \Omega\mathbb{X}\}$$

is a sub-base for the topology on $\mathsf{T}\mathbb{X}$.

Remark 3.2. Using the fact that for any two (open) sets a, b the set $\{a, a \cup b\}$ is directed, it is easy to see that Scott-continuity implies monotonicity.

Scott-continuity will play a rôle in Section 7, where it is used to show that the collection of formulas modulo (semantic) equivalence is a set, rather than a proper class.

Let $\mathbb{S}$ be the Sierpinski space, i.e. the two element set $2 = \{0, 1\}$ topologised by $\{\emptyset, \{1\}, 2\}$. For a topological space $\mathbb{X}$ and $a \subseteq \mathsf{U}\mathbb{X}$ let $\chi_a : \mathbb{X} \to \mathbb{S}$ be the characteristic map (i.e. $\chi_a(x) = 1$ iff $x \in a$). Note that χ_a is continuous if and only if $a \in \Omega\mathbb{X}$. Analogously to predicate liftings for **Set**-functors [35, Proposition 43], one can classify n-ary predicate liftings as open subsets of $\mathsf{T}\mathbb{S}^n$. This elucidates the analogy with predicate liftings for **Set**-functors.

Proposition 3.3. *Suppose $\mathbb{S} \in \mathbf{C}$, then there is a bijective correspondence between n-ary open predicate liftings and elements of $\Omega\mathsf{T}\mathbb{S}^n$. This correspondence is given as follows: To an open predicate lifting λ assign the set $\lambda_{\mathbb{S}^n}(\pi_1^{-1}(\{1\}), \ldots, \pi_n^{-1}(\{1\})) \in \Omega\mathsf{T}\mathbb{S}^n$, where $\pi_i : \mathbb{S}^n \to \mathbb{S}$ be the i-th projection, and conversely, for $c \in \Omega\mathsf{T}\mathbb{S}^n$ define $\lambda^c : \Omega^n \to \Omega\mathsf{T}$ by $\lambda^c_{\mathbb{X}}(a_1, \ldots, a_n) = (\mathsf{T}\langle\chi_{a_1}, \ldots, \chi_{a_n}\rangle)^{-1}(c)$.*

Definition 3.4. The *language* induced by a geometric modal signature Λ is the collection $\mathrm{GML}(\Phi, \Lambda)$ of formulas defined by the grammar

$$\varphi ::= \top \mid p \mid \varphi_1 \wedge \varphi_2 \mid \bigvee_{i \in I} \varphi_i \mid \heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n),$$

where p ranges over the set Φ of proposition letters, I is some index set, and $\lambda \in \Lambda$ is n-ary. Abbreviate $\bot := \bigvee \emptyset$. We call a formula in $\mathrm{GML}(\Phi, \Lambda)$ *coherent* if it does not involve any infinite disjunctions.

3.2. **Interpretation and examples.** The language $\mathrm{GML}(\Phi, \Lambda)$ is interpreted in so-called geometric T-models.

Definition 3.5. A *geometric* T*-model* is a triple $\mathfrak{X} = (\mathbb{X}, \gamma, V)$ where $(\mathbb{X}, \gamma)$ is a T-coalgebra and $V : \Phi \to \Omega\mathbb{X}$ is a valuation of the proposition letters. A map $f : \mathbb{X} \to \mathbb{X}'$ is a *geometric* T*-model morphism* from $(\mathbb{X}, \gamma, V)$ to $(\mathbb{X}', \gamma', V')$ if f is a coalgebra morphism between the underlying coalgebras and $f^{-1} \circ V' = V$. The collection of geometric T-models and geometric T-model morphisms forms a category, which we denote by $\mathbf{Mod}(\mathsf{T})$.

Definition 3.6. The *semantics* of $\varphi \in \mathrm{GML}(\Phi, \Lambda)$ on a geometric T-model $\mathfrak{X} = (\mathbb{X}, \gamma, V)$ is given recursively by

$$\llbracket \top \rrbracket^{\mathfrak{X}} = X, \quad \llbracket p \rrbracket^{\mathfrak{X}} = V(p), \quad \llbracket \varphi \wedge \psi \rrbracket^{\mathfrak{X}} = \llbracket \varphi \rrbracket^{\mathfrak{X}} \cap \llbracket \psi \rrbracket^{\mathfrak{X}}, \quad \llbracket \bigvee_{i \in I} \varphi_i \rrbracket^{\mathfrak{X}} = \bigcup_{i \in I} \llbracket \varphi_i \rrbracket^{\mathfrak{X}},$$

$$\llbracket \heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n) \rrbracket^{\mathfrak{X}} = \gamma^{-1}(\lambda_{\mathbb{X}}(\llbracket \varphi_1 \rrbracket^{\mathfrak{X}}, \ldots, \llbracket \varphi_n \rrbracket^{\mathfrak{X}})).$$

We write $\mathfrak{X}, x \Vdash \varphi$ iff $x \in \llbracket \varphi \rrbracket^{\mathfrak{X}}$. Two states x and x' are called *modally equivalent* if they satisfy the same formulas, notation: $x \equiv_{\Lambda} x'$. We say that φ is a *semantic consequence* of ψ in $\mathbf{Mod}(\mathsf{T})$, notation: $\varphi \Vdash_{\mathsf{T}} \psi$, if $\llbracket \varphi \rrbracket^{\mathfrak{X}} \subseteq \llbracket \psi \rrbracket^{\mathfrak{X}}$ for all $\mathfrak{X} \in \mathbf{Mod}(\mathsf{T})$.

The following proposition shows that morphisms preserve truth. Its proof is similar to the proof of theorem 6.17 in [36].

Proposition 3.7. *Let Λ be a geometric modal signature for T. Let $\mathfrak{X} = (\mathbb{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathbb{X}', \gamma', V')$ be geometric T-models and let $f : \mathfrak{X} \to \mathfrak{X}'$ be a geometric T-model morphism. Then for all $\varphi \in \mathrm{GML}(\Phi, \Lambda)$ and $x \in \mathbb{X}$ we have*

$$\mathfrak{X}, x \Vdash \varphi \quad \textit{iff} \quad \mathfrak{X}', f(x) \Vdash \varphi.$$

We state the notion of behavioural equivalence for future reference.

Definition 3.8. Let $\mathfrak{X} = (\mathbb{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathbb{X}', \gamma', V')$ be two geometric T-models and $x \in \mathbb{X}$, $x' \in \mathbb{X}'$ two states. We say that x and x' are *behaviourally equivalent* in $\mathbf{Mod}(\mathsf{T})$, notation: $x \simeq_{\mathbf{Mod}(\mathsf{T})} x'$, if there exists a geometric T-model $\mathfrak{Y}$ and T-model morphisms

$$\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xleftarrow{f'} \mathfrak{X}'$$

such that $f(x) = f'(x')$.

As an immediate consequence of Proposition 3.7 we find that behavioural equivalence implies modal equivalence. We will see in Section 7 that, under mild conditions, the converse is true as well.

Let us give some concrete examples of functors.

Example 3.9 (Trivial functor)**.** Let $\mathbb{2} = \{0, 1\}$ be topologised by $\{\emptyset, \{0, 1\}\}$ (the trivial topology). Define the functor $\mathsf{F} : \mathbf{Top} \to \mathbf{Top}$ by $\mathsf{F}\mathbb{X} = \mathbb{2}$ for every $\mathbb{X} \in \mathbf{Top}$ and $\mathsf{F}f = \mathrm{id}_{\mathbb{2}}$, the identity map on $\mathbb{2}$, for every continuous function f. This is clearly a functor. Consider the open predicate lifting $\lambda : \Omega \to \Omega \circ \mathsf{F}$ given by $\lambda_{\mathbb{X}}(a) = \mathsf{U}\mathbb{2}$ for all $a \in \Omega\mathbb{X}$. For a F-model $\mathfrak{X} = (\mathbb{X}, \gamma, V)$ we then have $\mathfrak{X}, x \Vdash \heartsuit^{\lambda}\varphi$ iff $\gamma(x) \in \lambda(\llbracket \varphi \rrbracket^{\mathfrak{X}})$ iff $\llbracket \varphi \rrbracket^{\mathfrak{X}} \in \Omega\mathbb{X}$. So $\heartsuit^{\lambda} = \top$.

Next we have a look at the Vietoris functor on $\mathbf{KHaus}$. Coalgebras for this functor have also been studied in [5], where they are used to interpret the positive modal logic from [13, 9]. In Section 4 we study the example of the *monotone functor*, which gives rise to monotone modal geometric logic.

Example 3.10 (Vietoris functor)**.** For a compact Hausdorff space $\mathbb{X}$, let $\mathsf{V}_{\mathbf{kh}}\mathbb{X}$ be the collection of closed subsets of $\mathbb{X}$ topologized by the subbase

$$\boxdot a := \{b \in \mathsf{V}_{\mathbf{kh}}\mathbb{X} \mid b \subseteq a\}, \quad \diamonddot a := \{b \in \mathsf{V}_{\mathbf{kh}}\mathbb{X} \mid a \cap b \neq \emptyset\},$$

where a ranges over $\Omega\mathbb{X}$. For a continuous map $f : \mathbb{X} \to \mathbb{X}'$ define $\mathsf{V}_{\mathbf{kh}}f : \mathsf{V}_{\mathbf{kh}}\mathbb{X} \to \mathsf{V}_{\mathbf{kh}}\mathbb{X}'$ by $\mathsf{V}_{\mathbf{kh}}f(a) = f[a]$. If $\mathbb{X}$ is compact Hausdorff, then so is $\mathsf{V}_{\mathbf{kh}}\mathbb{X}$ [30, Theorem 4.9], and if $f : \mathbb{X} \to \mathbb{X}'$ is a continuous map between compact Hausdorff spaces, then $\mathsf{V}_{\mathbf{kh}}f$ is well defined and continuous [27, Lemma 3.8], so $\mathsf{V}_{\mathbf{kh}}$ defines an endofunctor on **KHaus**.

Let $\mathfrak{X} = (\mathbb{X}, \gamma, V)$ be a $\mathsf{V}_{\mathbf{kh}}$-model. If we set

$$\lambda^{\Box}_{\mathbb{X}} : \Omega\mathbb{X} \to \Omega(\mathsf{V}_{\mathbf{kh}}\mathbb{X}) : a \mapsto \{b \in \mathsf{V}_{\mathbf{kh}}\mathbb{X} \mid b \subseteq a\},$$

where $\mathbb{X} \in \mathbf{Top}$, then we have $\mathfrak{X}, x \Vdash \Box\varphi$ iff $\gamma(x) \in \lambda^{\Box}_{\mathbb{X}}([\![\varphi]\!]^{\mathfrak{X}})$ iff $\gamma(x) \subseteq [\![\varphi]\!]^{\mathfrak{X}}$ iff every successor of x satisfies φ. Similarly $\lambda^{\Diamond}_{\mathbb{X}} : \Omega\mathbb{X} \to \Omega \circ \mathsf{V}_{\mathbf{kh}}\mathbb{X}$, given by $\lambda^{\Diamond}_{\mathbb{X}}(a) = \diamonddot a$ yields the usual semantics of the diamond modality.

3.3. **Strong predicate liftings.** In Section 8 it turns out to be useful to have a slightly stronger notion of open predicate liftings, called strong open predicate liftings, as this allows us to prove that behavioural equivalence implies so-called Λ-bisimilarity. Whereas the action of open predicate liftings is defined only on open subsets, a strong open predicate lifting acts on *every* subset of elements of a topological space. Recall that $\mathsf{U} : \mathbf{Top} \to \mathbf{Set}$ is the forgetful functor.

Definition 3.11. A *strong open predicate lifting* for $\mathsf{T} : \mathbf{C} \to \mathbf{C}$ is a natural transformation

$$\mu : (\breve{\mathsf{P}} \circ \mathsf{U})^n \to \breve{\mathsf{P}} \circ \mathsf{U} \circ \mathsf{T}$$

such that for all $\mathbb{X} \in \mathbf{C}$ and $a_1, \ldots, a_n \in \Omega\mathbb{X}$ the set $\lambda_{\mathbb{X}}(a_1, \ldots, a_n)$ is open in $\mathsf{T}\mathbb{X}$. Monotonicity and Scott-continuity of strong open predicate liftings are defined in the standard way.

We call an open predicate lifting (from Definition 3.1) *strong* if it is the restriction of some strong open predicate lifting and *strongly monotone* if it is the restriction of a monotone strong open predicate lifting.

Evidently, every strong open predicate lifting restricts to an open predicate lifting, and it is only this weaker notion of open predicate lifting that has an effect on the semantics. Our notion of strong open predicate lifting is similar to the notion of a *topological predicate lifting* for endofunctors on **Stone**, which were introduced in [14].

Example 3.12. The predicate lifting corresponding to the box modality from Example 3.10 is strong, for it is the restriction of $\mu : \mathsf{U} \to \mathsf{U} \circ \mathsf{V}_{\mathbf{kh}}$ given by $\mu_{\mathbb{X}}(u) = \{b \in \mathsf{V}_{\mathbf{kh}}\mathbb{X} \mid b \subseteq u\}$. Likewise, all other predicate liftings from Examples 3.9, 3.10 and the monotone functor from Section 4 are strong as well.

We devote the remainder of this section to investigating strong open predicate liftings. Recall from Example 3.9 that $\mathbb{2}$ denotes the two-element set with the trivial topology. We claim that natural transformations $\mu : (\breve{\mathsf{P}} \circ \mathsf{U})^n \to \breve{\mathsf{P}} \circ \mathsf{U} \circ \mathsf{T}$ correspond one-to-one with elements of $\breve{\mathsf{P}}\mathsf{U}\mathsf{T}\mathbb{2}$, provided $\mathbb{2} \in \mathbf{C}$: To a natural transformation μ associate the set $\mu_{\mathbb{2}}(p_1^{-1}(\{1\}), \ldots, p_n^{-1}(\{1\}))$, where $p_i : \mathbb{2}^n \to \mathbb{2}$ denotes the i-th projection. Conversely, for $c \in \breve{\mathsf{P}}\mathsf{U}\mathsf{T}\mathbb{2}$ define μ^c by $\mu^c_{\mathbb{X}}(a_1, \ldots, a_n) = (\mathsf{T}\langle \chi'_{a_1}, \ldots, \chi'_{a_n} \rangle)^{-1}(c)$, where $\mathbb{X}$ is a topological space, $a \subseteq \mathsf{U}\mathbb{X}$ and $\chi'_a : \mathbb{X} \to \mathbb{2}$ is the characteristic map. Note that χ'_a is continuous regardless of whether a is open or not, hence T acts on all χ'_a. Details of the bijection are left to the reader.

Proposition 3.13. *Let T be an endofunctor on $\mathbf{C}$ and suppose that $\mathbf{C}$ contains the spaces $\mathbb{2}$ and $\mathbb{S}$. Let $s : \mathbb{S} \to \mathbb{2}$ be the identity map and let $c \in \breve{\mathsf{P}}\mathsf{U}\mathsf{T}\mathbb{2}^n$. The natural transformation μ^c is a strong open predicate lifting if and only if $(\mathsf{T}s^n)^{-1}(c) \subseteq \mathsf{T}\mathbb{S}^n$ is open.*

Proof. We give the proof for the case $n = 1$, the general case being similar. Left to right follows from the fact that $\{1\}$ is open in $\mathbb{S}$, hence $\mu^c_{\mathbb{S}}(\{1\}) = (\mathsf{T}\chi'_{\{1\}})^{-1}(c) = (\mathsf{T}s)^{-1}(c)$ must be open in $\mathsf{T}\mathbb{S}$. For the converse, let $\mathbb{X}$ be a topological space and $a \in \Omega\mathbb{X}$. We need to show that $\mu^c_{\mathbb{X}}(a)$ is open. Since a is open, the characteristic map $\chi_a : \mathbb{X} \to \mathbb{S}$ is continuous and hence $\chi'_a = s \circ \chi_a$. We have

$$\begin{aligned}
\mu^c_{\mathbb{X}}(a) &= (\mathsf{T}\chi'_a)^{-1}(c) && \text{(Definition of } \mu^c\text{)} \\
&= (\mathsf{T}(s \circ \chi_a))^{-1}(c) && (\chi'_a = s \circ \chi_a) \\
&= (\mathsf{T}s \circ \mathsf{T}\chi_a)^{-1}(c) && \text{(Definition of functors)} \\
&= (\mathsf{T}\chi_a)^{-1} \circ (\mathsf{T}s)^{-1}(c). && \text{(Definition of inverse)}
\end{aligned}$$

Since $\mathsf{T}\chi_a$ is continuous and $(\mathsf{T}s)^{-1}(c)$ is assumed to be open in $\mathsf{T}\mathbb{S}$, the set $\mu^c_{\mathbb{X}}(a)$ is open in $\mathsf{T}\mathbb{X}$. □

The following proposition gives two sufficient conditions on T for its open predicate liftings to be strong. For a full subcategory $\mathbf{C}$ of $\mathbf{Top}$ let $\mathbf{preC}$ denote the category of topological spaces in $\mathbf{C}$ and (not necessarily continuous) functions.

Proposition 3.14. *Let T be an endofunctor on $\mathbf{C}$ and suppose $\mathbb{2}, \mathbb{S} \in \mathbf{C}$.*

(1) *If T preserves injective functions then every open predicate lifting for T is strong.*
(2) *If T extends to $\mathbf{preC}$, then every open predicate lifting for T is strong.*

Proof. For the first item, let $c \in \Omega\mathsf{T}\mathbb{S}^n$ determine the n-ary open predicate lifting λ^c. Since s^n is injective, by assumption $\mathsf{T}s^n$ is as well, and hence $c = (\mathsf{U}\mathsf{T}s^n)^{-1}((\mathsf{U}\mathsf{T}s^n)[c])$. Proposition 3.13 now implies that $\mu^{(\mathsf{U}\mathsf{T}s^n)[c]}$ is a strong open predicate lifting. It is easy to see that $\mu^{(\mathsf{U}\mathsf{T}s^n)[c]}$ extends λ^c, hence the latter is strong.

For the second item we show that, under the assumption, T preserves injective functions. Let $f : \mathbb{X} \to \mathbb{Y}$ be an injective function in $\mathbf{C}$, then there exists a (not necessarily continuous) function $g : \mathbb{Y} \to \mathbb{X}$ satisfying $g \circ f = \mathrm{id}_{\mathbb{X}}$. Then $\mathsf{T}g \circ \mathsf{T}f = \mathsf{T}(g \circ f) = \mathsf{T}\,\mathrm{id}_{\mathbb{X}} = \mathrm{id}_{\mathsf{T}\mathbb{X}}$, so $\mathsf{T}f$ has a (set-theoretic) left-inverse, hence is injective. □

Monotone open predicate lifting (hence also Scott-continuous ones) for an endofunctor on $\mathbf{KHaus}$ are always strong:

Proposition 3.15. *Let T be an endofunctor on $\mathbf{KHaus}$ and Λ a monotone geometric modal signature for T. Then Λ is strongly monotone.*

Proof. Let $\lambda \in \Lambda$. We need to show that λ is the restriction of some strong monotone predicate lifting. Define

$$\widetilde{\lambda}_{\mathbb{X}} : \breve{\mathsf{P}}^n\mathsf{U}\mathbb{X} \to \breve{\mathsf{P}}\mathsf{U}\mathsf{T}\mathbb{X} : (b_1, \ldots, b_n) \mapsto \bigcap\{\lambda_{\mathbb{X}}(a_1, \ldots, a_n) \mid a_i \in \Omega\mathbb{X} \text{ and } a_i \supseteq b_i\}.$$

Monotonicity of $\lambda_{\mathbb{X}}$ ensures $\widetilde{\lambda}_{\mathbb{X}}(a) = \lambda_{\mathbb{X}}(a)$ for all $a \in \Omega\mathbb{X}$ and $\widetilde{\lambda}$ is monotone by construction. So we only need to show that $\widetilde{\lambda}$ is indeed a strong open predicate lifting, i.e. a natural transformation $\breve{\mathsf{P}}^n\mathsf{U}\mathbb{X} \to \breve{\mathsf{P}}\mathsf{U}\mathsf{T}\mathbb{X}$. We assume λ to be unary, the general case being similar.

For a continuous map $f : \mathbb{X} \to \mathbb{X}'$ between compact Hausdorff spaces we need to show that $\widetilde{\lambda}_{\mathbb{X}} \circ f^{-1} = (\mathsf{T}f)^{-1} \circ \widetilde{\lambda}_{\mathbb{X}'}$. Since, by naturality of λ, the right hand side is equal to $\bigcap\{\lambda_{\mathbb{X}}(f^{-1}(a')) \mid a' \in \Omega\mathbb{X}' \text{ and } b' \subseteq a'\}$, it suffices to show

$$\bigcap\{\lambda_{\mathbb{X}}(c) \mid c \in \Omega\mathbb{X} \text{ and } f^{-1}(b') \subseteq c\} = \bigcap\{\lambda_{\mathbb{X}}(f^{-1}(a')) \mid a' \in \Omega\mathbb{X}' \text{ and } b' \subseteq a'\}. \tag{3.1}$$

If a' is an open superset of b' then clearly $f^{-1}(b') \subseteq f^{-1}(a')$. So every element in the intersection of the right hand side is contained in the one on the left hand side and therefore we have $\subseteq$ in (3.1). For the converse, suppose $c \in \Omega\mathbb{X}$ and $f^{-1}(b') \subseteq c$. Then the set $a' = X' \setminus f[X \setminus c]$ is open, contains b', and satisfies $f^{-1}(b') \subseteq f^{-1}(a') \subseteq c$. Therefore $\lambda_{\mathbb{X}}(f^{-1}(a'))$ is one of the elements in the intersection on the left hand side of (3.1). Since $\lambda_{\mathbb{X}}(f^{-1}(a')) \subseteq \lambda_{\mathbb{X}}(c)$ this shows "$\supseteq$" in (3.1). $\square$

4. The monotone neighbourhood functor on **KHaus**

In this section we define the monotone neighbourhood functor on **Frm** and show that it (individually) preserves regularity and compactness. This functor is a variation of the Vietoris Locale [21, Section 1]. Subsequently, we give a functor on **KHaus** which is dual to the restriction of the monotone neighbourhood functor to **KRFrm**.

4.1. **The monotone neighbourhood frame.**

Definition 4.1. For a frame F, let $\mathsf{M}F$ be the frame generated by $\Box a, \Diamond a$, where a ranges over F, subject to the relations

(M_1) $\Box(a \wedge b) \leq \Box a$
(M_2) $\Box a \wedge \Diamond b = \bot$ whenever $a \wedge b = \bot$
(M_3) $\Box \bigvee^{\uparrow} A = \bigvee^{\uparrow}\{\Box a \mid a \in A\}$
(M_4) $\Diamond a \leq \Diamond(a \vee b)$
(M_5) $\Box a \vee \Diamond b = \top$ whenever $a \vee b = \top$
(M_6) $\Diamond \bigvee^{\uparrow} A = \bigvee^{\uparrow}\{\Diamond a \mid a \in A\}$,

where $a, b \in F$ and A is a directed subset of F. For a homomorphism $f : F \to F'$ define $\mathsf{M}f : \mathsf{M}F \to \mathsf{M}F'$ on generators by $\Box a \mapsto \Box f(a)$ and $\Diamond a \mapsto \Diamond f(a)$. The assignment M defines a functor on **Frm**.

The proof of the following proposition closely resembles that of Proposition III4.3 in [20]. In a similar manner one can show that M preserves complete regularity and zero-dimensionaity.

Proposition 4.2. *If F is a regular frame, then so is $\mathsf{M}F$.*

Proof. We need to show that for all $c \in \mathsf{M}F$ we have $c = \bigvee\{d \in \mathsf{M}F \mid d \lessdot c\}$. It follows from Lemma 2.6 that it suffices to focus on the generators of $\mathsf{M}F$. Let $a \in F$, then we know $\bigvee\{d \in \mathsf{M}F \mid d \lessdot \Box a\} \leq \Box a$. Suppose $b \lessdot a$ in F, then by Lemma 2.5 ${\sim}b \vee a = \top$ and hence $\Diamond{\sim}b \vee \Box a \geq \top$. Also ${\sim}b \wedge b = \bot$ so it follows from (M_2) that $\Diamond{\sim}b \wedge \Box b = \bot$. This proves $\Box b \lessdot \Box a$, because the element $\Diamond{\sim}b$ is such that $\Diamond{\sim}b \vee \Box a = \top$ and $\Diamond{\sim}b \wedge \Box b = \bot$. Since F is regular and $\{b \in F \mid b \lessdot a\}$ is directed, it follows that

$$\Box a = \Box \bigvee^{\uparrow}\{b \in F \mid b \lessdot a\} = \bigvee^{\uparrow}\{\Box b \in \mathsf{M}F \mid b \lessdot a\} \leq \bigvee\{d \in \mathsf{M}F \mid d \lessdot \Box a\} \leq \Box a$$

so $\Box a = \bigvee\{d \in \mathsf{M}F \mid d \lessdot \Box a\}$. In a similar fashion one may show that $\Diamond a = \bigvee\{d \in \mathsf{M}F \mid d \lessdot \Diamond a\}$. This proves the proposition. $\square$

We now prove that the functor M preserves compactness. We proceed in a similar manner as [38, Theorem 4.2]. This relies on an auxiliary Definition and Lemma (Definition 4.3 and Lemma 4.4), in which we give an alternative description of $\mathsf{M}F$. We then prove that this alternative description preserves compactness.

In [16, Corollary 3.42] we proved the same result by first giving a duality result between frames and topological spaces, and then proving preservation of compactness on the topological side. The main difference between that proof and the one we present here is that the current one is constructive.

Write P_ω of the finite powerset functor and recall that $\mathsf{Z} : \mathbf{Frm} \to \mathbf{Set}$ is the forgetful functor.

Definition 4.3. For a frame F define $\mathsf{M}'F$ to be the free frame generated by $\mathsf{P}_\omega\mathsf{Z}F \times \mathsf{P}_\omega\mathsf{Z}F$, qua join-semilattice (that is, the join in $\mathsf{M}'F$ is given by $(\gamma, \delta) \vee (\gamma', \delta') = (\gamma \cup \gamma', \delta \cup \delta')$), subject to

(M_1') $(\gamma \cup \{a \wedge b\}, \delta) \leq (\gamma \cup \{a\}, \delta)$

(M_2') $(\gamma \cup \{a\}, \delta) \wedge (\gamma, \delta \cup \{b\}) \leq (\gamma, \delta)$ if $a \wedge b = 0$

(M_3') $(\gamma \cup \{\bigvee^\uparrow A\}, \delta) \leq \bigvee^\uparrow{}_{a \in A}(\gamma \cup \{a\}, \delta)$

(M_4') $(\gamma, \delta \cup \{a\}) \leq (\gamma, \delta \cup \{a \vee b\})$

(M_5') $\top \leq (\gamma \cup \{a\}, \delta \cup \{b\})$ if $a \vee b = 1$

(M_6') $(\gamma, \{\bigvee^\uparrow A\} \cup \delta) \leq \bigvee^\uparrow{}_{a \in A}(\gamma, \{a\} \cup \delta)$

This results an a frame isomorphic to $\mathsf{M}F$:

Lemma 4.4. *Let F be a frame. Then $\mathsf{M}F \cong \mathsf{M}'F$.*

Proof. Define $\mathsf{M}'F \to \mathsf{M}F : (\gamma, \delta) \mapsto \bigvee_{c \in \gamma} \Box c \vee \bigvee_{d \in \delta} \Diamond d$ and

$$\mathsf{M}F \to \mathsf{M}'F : \begin{cases} \Box a \mapsto (\{a\}, \emptyset) \\ \Diamond a \mapsto (\emptyset, \{a\}) \end{cases}$$

Clearly these define a bijection. Furthermore it is straightforward to verify that these maps are well defined by checking that the images of generators satisfy relations of the respective frame. □

Theorem 4.5. *Suppose F is compact. Then $\mathsf{M}F$ is compact.*

Proof. The frame $\mathsf{M}D$ is compact iff there is a preframe homomorphism $\varphi : \mathsf{M}D \to 2$ that is right adjoint to the unique frame homomorphism $! : 2 \to \mathsf{M}D$, where $2 = \{0, 1\}$ is the two-element frame.

By Proposition 4.4 we have $\mathsf{M}D \cong \mathsf{M}'D$, and since all the relations in Definition 4.3 are join-stable, we can use the preframe coverage theorem [23, Theorem 5.1] to find that $\mathsf{M}'F$ viewed as a preframe is the preframe generated by $\mathsf{P}_\omega\mathsf{Z}F \times \mathsf{P}_\omega\mathsf{Z}F$ qua poset, subject to the the relations from Definition 4.3.

Define

$$\varphi : \mathsf{M}'F \to 2 : (\gamma, \delta) \mapsto \begin{cases} 1 & \text{iff there are } c \in \gamma \text{ such that } c \vee (\bigvee \delta) = 1 \\ 0 & \text{otherwise} \end{cases} .$$

First we check that φ is indeed a pre-frame homomorphism. Since φ is defined on generators, it suffices to show that it preservers the relations (M_1') to (M_6'), because if it does it can be lifted in a unique way to a preframe homomorphism $\mathsf{M}'F \to 2$. It is clear that φ is a monotone morphism (hence preserves the poset structure of the generators). We check that φ preserves the relations one by one.

(M_1') If $\varphi(\gamma \cup \{a\}, \delta) = 0$ then $c \vee \bigvee \delta = \top_F$ for all $c \in \gamma$ and $(a \wedge b) \vee \bigvee \delta \leq a \vee \bigvee \delta = \top_F$.

(M_2') Suppose $\varphi(\gamma \cup \{a\}, \delta) = 1$ and $\varphi(\gamma, \delta \cup \{b\}) = 1$. Then either there is some $c \in \gamma$ such that $c \vee \bigvee \delta = \top_F$, which implies $\varphi(\gamma, \delta) = 1$, or $a \vee \bigvee \delta = \top_F$. In the latter case, note that we also have some $c' \in \gamma$ such that $c' \vee \bigvee \delta \vee b = \top_F$, so that

$$c' \vee \bigvee \delta = c' \vee \bigvee \delta \vee (a \wedge b) = (a \vee \bigvee \delta \vee c') \wedge (c' \vee \bigvee \delta \vee b) = \top_F \wedge \top_F = \top_F.$$

The first equality holds because $a \wedge b = \bot_F$. Again we find $\varphi(\gamma, \delta) = 1$.

(M_3') Suppose $\varphi(\gamma \cup \{\bigvee^{\uparrow} A\}, \delta) = 1$, then either $c \vee (\bigvee \delta) = \top_F$ for some $c \in \gamma$, or $\top_F = (\bigvee^{\uparrow} A) \vee (\bigvee \delta) = \bigvee^{\uparrow}{}_{a \in A}(a \vee (\bigvee \delta))$ (note that the latter is indeed a directed set, because A is). By compactness of F this gives $a \vee (\bigvee \delta) = \top_F$ for some $a \in A$. So both cases yield $\varphi\big(\bigvee^{\uparrow}{}_{a \in A}(\gamma \cup \{a\}, \delta)\big) = 1$.

(M_4') If $\varphi(\gamma, \delta \cup \{a\}) = 1$, then $c \vee \bigvee(\delta \cup \{a\}) = \top_F$ for some $c \in \gamma$, so $c \vee \bigvee(\delta \cup \{a \vee b\}) \geq c \vee \bigvee(\delta \cup \{a\}) = \top_F$.

(M_5') If $a \vee b = \top_F$, then $a \vee \bigvee(\delta \cup \{b\}) = \top_F$ so $\varphi(\gamma \cup \{a\}, \delta \cup \{b\}) = 1$.

(M_6') Suppose $\varphi(\gamma, \{\bigvee^{\uparrow} A\} \cup \delta) = 1$, then, for some $c \in \gamma$, we have

$$\top_F = c \vee \bigvee(\{\bigvee^{\uparrow} A\} \cup \delta) = \bigvee^{\uparrow}(c \vee a \vee \bigvee \delta)$$

and by compactness we must have $c \vee \bigvee^{\uparrow}(\{a\} \cup \delta) = \top_F$ for one of the a. (The set $\{c \vee a \vee \bigvee \delta \mid a \in A\}$ is directed and by (M_4').)

Lastly, we need to verify that φ is right-adjoint to $! : 2 \to \mathsf{M}'L$ (defined by $1 \mapsto \top_{\mathsf{M}'F} = (\{\top_F\}, \{\top_F\})$, and $0 \mapsto \bot_{\mathsf{M}'F} = (\emptyset, \emptyset)$). It suffices to show that $\varphi(!(p)) \geq p$ and $!(\varphi(\gamma, \delta)) \leq (\gamma, \delta)$. For the first, suppose $p = 1$, then $!(p)$ is the equivalence class of $(\{\top_F\}, \{\top_F\})$ and $\varphi(!(p)) = 1$. For the second, if $\varphi(\gamma, \delta) = 1$, then there are $c \in \gamma$ such that $c \vee (\bigvee \delta) = \top_F$ (in particular $\delta \neq \emptyset$) and hence

$$\top_{\mathsf{M}'F} = (\{\top_F\}, \delta) = (\{c \vee (\bigvee \delta)\}, \delta) \leq (\{c\}, \delta) \leq (\gamma, \delta).$$

The first inequality follows from recalling that δ is a finite set and applying (M_6') repeatedly. This completes the proof. □

We now know that M restricts to an endofunctor on **KRFrm**. We write $\mathsf{M}_{\mathbf{kr}}$ for this restriction.

Remark 4.6. The category **Loc** of locales and locale morphisms is the opposite of **Frm**. Therefore, we can also view M as an endofunctor on locales and $\mathsf{M}A$ as the monotone neighbourhood locale, where A is a locale.

4.2. **Monotone neighbourhood functor on KHaus.** We now describe the topological manifestation of the monotone neighbourhood functor.

Definition 4.7. Let $\mathbb{X} = (X, \tau)$ be a compact Hausdorff space. Let $\mathsf{D}_{\mathbf{kh}}\mathbb{X}$ be the collection of sets $W \subseteq \mathsf{P}X$ such that $u \in W$ iff there exists a closed $c \subseteq u$ such that every open superset of c is in W. Endow $\mathsf{D}_{\mathbf{kh}}\mathbb{X}$ with the topology generated by the subbase

$$\boxplus a := \{W \in \mathsf{D}_{\mathbf{kh}}\mathbb{X} \mid a \in W\}, \quad \diamondplus a := \{W \in \mathsf{D}_{\mathbf{kh}}\mathbb{X} \mid X \setminus a \notin W\},$$

where a ranges over $\Omega\mathbb{X}$. For continuous functions $f : \mathbb{X} \to \mathbb{X}'$ define $\mathsf{D}_{\mathbf{kh}}f : \mathsf{D}_{\mathbf{kh}}\mathbb{X} \to \mathsf{D}_{\mathbf{kh}}\mathbb{X}' : W \mapsto \{a \in \mathsf{P}X \mid f^{-1}(a) \in W\}$.

Lemma 4.8. *If $f : \mathbb{X} \to \mathbb{X}'$ is a morphism in* **KHaus**, *then $\mathsf{D}_{\mathbf{kh}}f$ is a well-defined continuous function from $\mathsf{D}_{\mathbf{kh}}\mathbb{X}$ to $\mathsf{D}_{\mathbf{kh}}\mathbb{X}'$.*

Proof. $\mathsf{D}_{\mathbf{kh}}f$ *is well-defined.* Let $W \in \mathsf{D}_{\mathbf{kh}}\mathbb{X}$. We need to show that $\mathsf{D}_{\mathbf{kh}}f(W) \in \mathsf{D}_{\mathbf{kh}}\mathbb{X}'$. Suppose $a' \in \mathsf{D}_{\mathbf{kh}}f(W)$. Then $f^{-1}[a'] \in W$, so there exists a closed $c \subseteq f^{-1}[a']$ such that $c \in W$. Since $\mathbb{X}$ is compact and $\mathbb{X}'$ is Hausdorff, $f[c]$ is a closed set in $\mathbb{X}'$. In addition we have $f[c] \subseteq a'$. Suppose $f[c] \subseteq b$ for some open $b \in \Omega\mathbb{X}'$, then $c \subseteq f^{-1}[b]$ so $f^{-1}[b] \in W$ and hence $b \in \mathsf{D}_{\mathbf{kh}}f(W)$. So all open supersets of $f[c]$ are in $\mathsf{D}_{\mathbf{kh}}f(W)$, and therefore $f[c] \in \mathsf{D}_{\mathbf{kh}}f(W)$. Thus, for $a' \in \mathsf{D}_{\mathbf{kh}}f(W)$, there exists a closed subset (in this case $f[c]$) of a' with the property that every clopen superset is in $\mathsf{D}_{\mathbf{kh}}f(W)$.

$\mathsf{D}_{\mathbf{kh}}f$ *is continuous.* For continuity we need to show that both $(\mathsf{D}_{\mathbf{kh}}f)^{-1}(\boxplus a')$ and $(\mathsf{D}_{\mathbf{kh}}f)^{-1}(\diamondplus a')$ are open in $\mathsf{D}_{\mathbf{kh}}\mathbb{X}$, whenever $a' \in \Omega(\mathbb{X}')$. It follows from a straightforward computation that $(\mathsf{D}_{\mathbf{kh}}f)^{-1}(\boxplus a') = \boxplus f^{-1}(a')$, which is open in $\mathsf{D}_{\mathbf{kh}}\mathbb{X}$ by definition, and similarly $(\mathsf{D}_{\mathbf{kh}}f)^{-1}(\diamondplus a') = \diamondplus f^{-1}(a') \in \Omega\mathsf{D}_{\mathbf{kh}}\mathbb{X}$. □

For the time being, we regard $\mathsf{D}_{\mathbf{kh}}$ as a functor **KHaus** $\to$ **Top**, because we have no evidence yet that it restricts to an endofunctor on **KHaus**. We aim to prove that $\mathsf{D}_{\mathbf{kh}}$ is dual to the restriction of M to **KRFrm**. As a corollary, we then obtain that $\mathsf{D}_{\mathbf{kh}}$ indeed restricts to **KHaus**.

Theorem 4.9. *If $\mathbb{X}$ is a compact Hausdorff space then*

$$\mathsf{pt}(\mathsf{M}(\mathsf{opn}\mathbb{X})) \cong \mathsf{D}_{\mathbf{kh}}\mathbb{X}.$$

We temporarily fix a compact Hausdorff space $\mathbb{X}$ and define the two maps constituting a homeomorphism.

Definition 4.10. For a compact Hausdorff space $\mathbb{X}$, define $\zeta : \mathsf{pf} \circ \mathsf{M} \circ \mathsf{opn}\mathbb{X} \to \mathsf{D}_{\mathbf{kh}}\mathbb{X}$ by sending a prime filter p to

$$W_p := {\uparrow}\{X \setminus a \mid p(\Diamond a) = \bot\}.$$

We have $W_p \in \mathsf{D}_{\mathbf{kh}}\mathbb{X}$ because it is the up-set of a collection of closed sets; indeed, for each $b \in W_p$ there exists a closed subset $X \setminus a \subseteq b$ with $p(\Diamond a) = \bot$ and by definition all open supersets of $X \setminus a$ are in W_p. Therefore ζ is well defined. In the converse direction we define:

Definition 4.11. For a compact Hausdorff space $\mathbb{X}$, define

$$\theta : \mathsf{D}_{\mathbf{kh}}\mathbb{X} \to \mathsf{pf} \circ \mathsf{M} \circ \mathsf{opn}\mathbb{X} : W \mapsto p_W,$$

where p_W is given on generators by

$$p_W : \mathsf{M} \circ \mathsf{opn}\mathbb{X} \to 2 : \begin{cases} \Box a \mapsto \top & \text{iff } a \in W \\ \Diamond a \mapsto \bot & \text{iff } X \setminus a \in W \end{cases}$$

Lemma 4.12. *The assignment θ is well defined.*

Proof. Since p_W is a frame homomorphisms defined on generators, it suffices to check that the $p_W(\Box a)$ and $p_W(\Diamond a)$ (where the a range over $\Omega\mathbb{X}$) satisfy (M_1) through (M_6) from Definition 4.1. Let us check (M_1), (M_2) and (M_3), items (M_4), (M_5) and (M_6) being similar.

(M_1) If $p_W(\Box(a \cap b)) = \top$ then $a \cap b \in W$. Since W is upward closed $a \in W$, so $p_W(\Box a) = \top$.

(M_2) If $a \cap b = \emptyset$ then $a \subseteq X \setminus b$. Suppose $p_W(\Box a) = \top$ then $a \in W$ so $X \setminus b \in W$ so $p_W(\Diamond b) = \bot$ hence $p_W(\Box a) \wedge p_W(\Diamond b) = \bot$.

(M_3) We claim that for all $W \in \mathsf{D}_{\mathbf{kh}}\mathbb{X}$ and directed sets $A \subseteq \Omega\mathbb{X}$ we have $\bigcup^{\uparrow} A \in W$ iff there is $a \in A$ with $a \in W$. The direction from right to left follows from the fact that W is upwards closed. Conversely, suppose $\bigcup^{\uparrow} A \in W$, then there is a closed set $k \subseteq \bigcup^{\uparrow} A$ with $k \in W$. The elements of A now cover the closed therefore compact set k, so there is a finite $A' \subseteq A$ with $k \subseteq \bigcup A'$ and since A is directed there is $a \in A$ with $\bigcup A' \subseteq a$. As $k \in W$ and $k \subseteq a$ it follows that $a \in W$.

Now we have $p_W(\Box\bigcup^{\uparrow} A) = \top$ iff $\bigcup^{\uparrow} A \in W$ iff there is $a \in A$ with $a \in W$ iff $\bigvee^{\uparrow}\{p_W(\Box a) \mid a \in A\} = \top$. □

The following lemma is key for proving that ζ and θ are continuous and each other's inverses.

Lemma 4.13. *For all $p \in \mathsf{pt} \circ \mathsf{M} \circ \mathsf{opn}\mathbb{X}$ we have $X \setminus a \in W_p$ iff $p(\Diamond a) = \bot$ and $a \in W_p$ iff $p(\Box a) = \top$.*

Proof. If $p(\Diamond a) = \bot$ then $X \setminus a \in W_p$. Conversely, Suppose $X \setminus a \in W_p$, then there is some b with $p(\Diamond b) = \bot$ and $X \setminus b \subseteq X \setminus a$. Therefore $a \subseteq b$ and $p(\Diamond a) \leq p(\Diamond b) = \bot$. This proves $X \setminus a \in W_p$ iff $p(\Diamond a) = \bot$.

If $a \in W_p$ then there is $X \setminus b \subseteq a$ in W_p, so $p(\Diamond b) = \bot$. Then $a \cup b = X$, so it follows from (M_5) of Definition 4.1 that $p(\Box a) = \top$. If $a \notin W_p$ and $a' \leqslant a$, then there exists b with $b \cap a' = \emptyset$ and $b \cup a = \mathbb{X}$. Since $X \setminus b \subseteq a$, set set $X \setminus b$ is not in W_p and hence we must have $p(\Diamond b) = \top$. As $a' \cap a = \emptyset$ it follows from (M_2) that $p(\Box a') = p(\emptyset) = \bot$. Now we use ($M_3$) and the fact that $a = \bigvee^{\uparrow}\{a' \mid a' \leqslant a\}$ (this is true because $\mathbb{X}$ is assumed to be compact Hausdorff so $\mathsf{opn}\mathbb{X}$ is compact regular) to find

$$p(\Box a) = \bigvee\nolimits^{\uparrow}\{p(\Box a') \mid a' \leqslant a\} = \bigvee\nolimits^{\uparrow}\{\bot \mid a' \leqslant a\} = \bot.$$

It follows that $a \in W_p$ iff $p(\Box a) = \top$. □

We have now aquired sufficient knowledge to prove Theorem 4.9.

Proof of Theorem 4.9. We claim that the maps ζ and θ define a homeomorphism between $\mathsf{D}_{\mathbf{kh}}\mathbb{X}$ and $\mathsf{pt}(\mathsf{M}(\mathsf{opn}\mathbb{X}))$. First we prove that they are each other's inverses, by showing that for all $p \in \mathsf{pt}(\mathsf{M}(\mathsf{opn}\mathbb{X}))$ and $W \in \mathsf{D}_{\mathbf{kh}}\mathbb{X}$ we have $p_{W_p} = p$ and $W_{p_W} = W$.

In order to prove that (the frame homomorphisms) p and p_{W_p} coincide, it suffices to show that they coincide on the generators of $\mathsf{M}(\mathsf{opn}\mathbb{X})$. By Definition 4.11 and Lemma 4.13 have

$$p(\Box a) = \top \quad \text{iff} \quad a \in W_p \quad \text{iff} \quad p_{W_p}(\Box a) = \top$$

and

$$p(\Diamond a) = \bot \quad \text{iff} \quad X \setminus a \notin W_p \quad \text{iff} \quad p_{W_p}(\Diamond a) = \bot.$$

In order to show that $W = W_{p_W}$ it suffices to show that $X \setminus a \in W$ iff $X \setminus a \in W_{p_W}$ for all open sets a, because elements of $\mathsf{D}_{\mathbf{kh}}\mathbb{X}$ are uniquely determined by the closed sets they contain. This follows immediately from the definitions and Lemma 4.13 that

$$X \setminus a \in W \quad \text{iff} \quad p_W(\Diamond a) = \bot \quad \text{iff} \quad X \setminus a \in W_{p_W}.$$

Therefore $\zeta = \theta^{-1}$.

We complete the proof by showing that ζ and θ are continuous. The opens of $\mathsf{pt}(\mathsf{M}(\mathsf{opn}\mathbb{X}))$ are generated by $\widetilde{\Box a} = \{p \mid p(\Box a) = \top\}$ and $\widetilde{\Diamond a} = \{p \mid p(\Diamond a) = \top\}$, for $a \in \Omega\mathbb{X}$. We have

$$\theta^{-1}(\widetilde{\Box a}) = \theta^{-1}(\{p \mid p(\Box a) = \top\}) = \{W \in \mathsf{D}_{\mathbf{kh}}\mathbb{X} \mid a \in W\} = \boxplus a$$

and similarly $\theta^{-1}(\widetilde{\Diamond a}) = \diamondbar a$. Continuity of θ follows from the fact that $\boxbar a$ and $\diamondbar a$ are open in $\mathsf{D}_{\mathbf{kh}}\mathbb{X}$. Conversely, the opens of $\mathsf{D}_{\mathbf{kh}}\mathbb{X}$ are generated by $\boxbar a$ and $\diamondbar a$, where a ranges over $\Omega\mathbb{X}$. It is routine to see that $\zeta^{-1}(\boxbar a) = \widetilde{\Box a}$ and $\zeta^{-1}(\diamondbar a) = \widetilde{\Diamond a}$. This proves continuity of ζ.

We showed that θ is a continuous function with continuous inverse ζ, hence a homeomorphism. This completes the proof of the theorem. $\square$

Corollary 4.14. *The assignment $\mathsf{D}_{\mathbf{kh}}$ defines an endofunctor on* **KHaus**.

Theorem 4.9 yields a map $\mathsf{M}_{\mathbf{kr}}(\mathsf{opn}\mathbb{X}) \to \mathsf{opn}(\mathsf{D}_{\mathbf{kh}}\mathbb{X})$ for a compact Hausdorff space $\mathbb{X}$ given by

$$\mathsf{M}_{\mathbf{kr}}(\mathsf{opn}\mathbb{X}) \longrightarrow \mathsf{opn}(\mathsf{pt}(\mathsf{M}_{\mathbf{kr}}(\mathsf{opn}\mathbb{X}))) \xrightarrow{\mathsf{opn}\varphi} \mathsf{opn}(\mathsf{D}_{\mathbf{kh}}\mathbb{X}).$$

Unravelling the definitions shows that, on generators, it is given by $\Box a \mapsto \boxbar a$ and $\Diamond a \mapsto \diamondbar a$.

Definition 4.15. For every compact Hausdorff space $\mathbb{X}$ define $\eta_{\mathbb{X}} : \mathsf{M}_{\mathbf{kr}}(\mathsf{opn}\mathbb{X}) \to \mathsf{opn}(\mathsf{D}_{\mathbf{kh}}\mathbb{X})$ on generators by $\eta_{\mathbb{X}}(\Box a) = \boxbar a$ and $\eta_{\mathbb{X}}(\Diamond a) = \diamondbar a$. By the preceding discussion $\eta_{\mathbb{X}}$ is a well-defined frame isomorphism.

It turns out that the maps $\eta_{\mathbb{X}}$ constitute a natural isomorphism.

Lemma 4.16. *The collection $\eta = (\eta_{\mathbb{X}})_{\mathbb{X} \in \mathbf{KHaus}}$ forms a natural isomorphism.*

Proof. It follows from Theorem 4.9 that each of the $\eta_{\mathbb{X}}$ is an isomorphism, so we only need to show naturality. That is, for any morphism $f : \mathbb{X} \to \mathbb{X}'$ in **KHaus**, the following diagram commutes,

$$\begin{tikzcd}
\mathsf{M}_{\mathbf{kr}}(\mathsf{opn}\mathbb{X}) \arrow[d, "\eta_{\mathbb{X}}"'] & \mathsf{M}_{\mathbf{kr}}(\mathsf{opn}\mathbb{X}') \arrow[l, "\mathsf{M}_{\mathbf{kr}}(\mathsf{opn}f)"'] \arrow[d, "\eta_{\mathbb{X}'}"] \\
\mathsf{opn}(\mathsf{D}_{\mathbf{kh}})\mathbb{X} & \mathsf{opn}(\mathsf{D}_{\mathbf{kh}}\mathbb{X}') \arrow[l, "\mathsf{opn}(\mathsf{D}_{\mathbf{kh}}f)"]
\end{tikzcd}$$

(Since opn is a contravariant functor, the horizontal arrows are reversed.) For this, suppose $\Box a'$ is a generator of $\mathsf{M}_{\mathbf{kr}}(\mathsf{opn}\mathbb{X}')$. Then

$$\begin{aligned}
\mathsf{opn}(\mathsf{D}_{\mathbf{kh}}f) \circ \eta_{\mathbb{X}'}(\Box a) &= \mathsf{opn}(\mathsf{D}_{\mathbf{kh}}f)(\boxbar a) && \text{(Definition 4.15)} \\
&= (\mathsf{D}_{\mathbf{kh}}f)^{-1}(\boxbar a) && \text{(Definition of } \mathsf{opn}) \\
&= \boxbar f^{-1}(a) && \text{(Lemma 4.8)} \\
&= \eta_{\mathbb{X}}(\Box f^{-1}(a)) && \text{(Definition 4.15)} \\
&= \eta_{\mathbb{X}} \circ \mathsf{M}_{\mathbf{kr}}(f^{-1}(\Box a)) && \text{(Definition of } \mathsf{M}) \\
&= \eta_{\mathbb{X}} \circ \mathsf{M}_{\mathbf{kr}}(\mathsf{opn}f)(\Box a). && \text{(Definition of } \mathsf{opn})
\end{aligned}$$

and by analogous reasoning $\Omega\mathsf{D}_{\mathbf{kh}}f \circ \eta_{\mathbb{X}'}(\Diamond a) = \eta_{\mathbb{X}} \circ \mathsf{M}_{\mathbf{kr}}(\mathsf{opn}f)(\Diamond a)$. This proves that the diagram commutes. $\square$

As an immediate corollary of Lemma 4.16 we obtain:

Theorem 4.17. *There is a dual equivalence*

$$\mathbf{Alg}(\mathsf{M}_{\mathbf{kr}}) \equiv^{\mathrm{op}} \mathbf{Coalg}(\mathsf{D}_{\mathbf{kh}}).$$

4.3. **Logic for the monotone neighbourhood functor.** We give predicate liftings for $\mathsf{D}_{\mathbf{kh}}$ that give rise to *monotone modal geometric logic.* Define $\lambda^{\Box}, \lambda^{\Diamond} : \Omega \to \Omega \circ \mathsf{D}_{\mathbf{kh}}$ by

$$\lambda^{\Box}_{\mathbb{X}}(a) = \{W \in \mathsf{D}_{\mathbf{kh}}\mathbb{X} \mid a \in W\}, \qquad \lambda^{\Diamond}_{\mathbb{X}}(a) = \{W \in \mathsf{D}_{\mathbf{kh}}\mathbb{X} \mid \mathbb{X} \setminus a \notin W\}.$$

It is easy to see that these are strongly monotone.

Write $\Box$ and $\Diamond$ for the corresponding modal operators and let $(\mathbb{X}, \gamma, V)$ be a $\mathsf{D}_{\mathbf{kh}}$-model. Then we have

$$x \Vdash \Box\varphi \quad \text{iff} \quad \gamma(x) \in \lambda^{\Box}_{\mathbb{X}}(\llbracket\varphi\rrbracket) \quad \text{iff} \quad \llbracket\varphi\rrbracket \in \gamma(x)$$

and similarly $x \Vdash \Diamond\varphi$ if $\mathbb{X} \setminus \llbracket\varphi\rrbracket \notin \gamma(x)$. This is the same as neighbourhood semantics for monotone modal logic over a classical base [10, 17].

Remark 4.18. We will see in Example 6.6 that the functor $\mathsf{D}_{\mathbf{kh}}$ on **KHaus** can be generalised to an endofunctor of **Top** which restricts to **Sob**.

5. Axioms, Soundness and completeness

We define the notion of *one-step axioms* (similar to [25, Definition 3.8]) and *one-step rules* for a collection of predicate liftings. These give rise to axioms and rules for the logic $\mathrm{GML}(\Phi, \Lambda)$ from Definition 3.4, and to an endofunctor L on the category of frames. As in Section 3 we let $\mathbf{C}$ be some full subcategory of **Top**, T an endofunctor on $\mathbf{C}$, and we view Ω as a functor $\mathbf{C} \to \mathbf{Frm}$.

At the end of Subsection 5.2, in order to derive a general completeness result, we restrict our attention to a language *without* proposition letters. This need not be problematic: proposition letters can be introduced via (nullary) predicate liftings. In particular, this means that the category of T-models is the same as the category of T-coalgebras.

Ultimately, using the duality proved in Lemma 4.16, we derive that monotone modal geometric logic without proposition letters is sound and complete with respect to $\mathsf{D}_{\mathbf{kh}}$-coalgebras.

5.1. **Axioms and algebraic semantics.** Let Λ be a collection of predicate liftings for an endofunctor T on $\mathbf{C}$. Recall that Φ denotes a set of propositional variables. The *zero-step formulas* of $\mathrm{GML}(\Phi, \Lambda)$ is simply the subcollection $\mathrm{GL}(\Phi)$ of $\mathrm{GML}(\Phi, \Lambda)$. The *one-step formulas* in $\mathrm{GML}(\Phi, \Lambda)$ are given recursively by

$$\varphi ::= \top \mid \bot \mid \varphi \wedge \varphi \mid \bigvee_{i \in I} \varphi_i \mid \heartsuit^{\lambda}(\pi_1, \ldots, \pi_n),$$

where $\lambda \in \Lambda$ is n-ary and $\pi_1, \ldots, \pi_n \in \mathrm{GL}(\Phi)$.

We define the notions of a one-step axiom and a one-step rule for $\mathrm{GML}(\Phi, \Lambda)$:

Definition 5.1. A *one-step axiom* for $\mathrm{GML}(\Phi, \Lambda)$ is a consequence pair $\alpha \lhd \beta$, where α, β are one-step formulas. A *one-step rule* is an expression of the form

$$\frac{a_i \lhd b_i \quad i \in I}{\alpha \lhd \beta}, \tag{5.1}$$

where I is some index-class, a_i, b_i are zero-step formulas for $i \in I$ and α, β are one-step formulas.

First let us investigate how one-step axioms and rules give rise to an (equationally defined) endofunctor $\mathsf{L}_{(\Lambda,\mathrm{Ax})}$ on **Frm** – when no confusion is likely we drop the subscript and simply write L. Given a frame F, the frame $\mathsf{L}F$ can be presented as follows. As *generators* we take the collection

$$\Lambda(F) = \{\underline{\lambda}(v_1, \dots, v_n) \mid \lambda \in \Lambda \ n\text{-ary}, \ v_i \in F\}.$$

The idea is now to *instantiate* the (meta)variables of the axiomatisation Ax with the elements of F. Zero-step formulas then naturally evaluate to elements of F. Consequently, an axiom $\alpha \triangleleft \beta$ gives rise to a relation $\alpha \leq \beta$, and a rule as in (5.1) yields the relation $\alpha \leq \beta$ *condtionally*, that is, we only consider the relation in those cases where $a_i \leq b_i$ for all i.

Definition 5.2. For a frame F, define $\mathsf{L}_{(\Lambda,\mathrm{Ax})}F = \mathsf{L}F$ to be the frame

$$\mathsf{Fr}(\Lambda(F))/R,$$

where R is the collection of relations that arises from substituting the metavariables from the schemata in Ax with elements from F. For a morphism $f : F \to F'$ define $\mathsf{L}f$ on objects by

$$\mathsf{L}f(\underline{\lambda}(a_1, \dots, a_n)) = \underline{\lambda}(f(a_1), \dots, f(a_n)).$$

Definition 5.3. A collection of axioms is called *sound* if the assignment $\rho : \mathsf{L} \circ \Omega \to \Omega \circ \mathsf{T}$, given for $X \in \mathbf{C}$ by

$$\rho_X : \mathsf{L} \circ \Omega X \to \Omega \circ \mathsf{T}X : \underline{\lambda}(a_1, \dots, a_n) \mapsto \lambda(a_1, \dots, a_n),$$

defines a natural transformation.

Example 5.4. Suppose $\mathsf{T} = \mathsf{D}_{\mathbf{kh}}$, the monotone neighbourhood functor on **KHaus**, and $\lambda^{\Box}, \lambda^{\Diamond}$ are given as in Subsection 4.3. The following collection of axioms and rules is sound:

$$(m_1)\ \frac{a \triangleleft b}{\underline{\lambda}^{\Box}(a) \triangleleft \underline{\lambda}^{\Box}(b)} \qquad (m_4)\ \frac{a \triangleleft b}{\underline{\lambda}^{\Diamond}(a) \triangleleft \underline{\lambda}^{\Diamond}(b)}$$

$$(m_2)\ \frac{a \wedge b \triangleleft \bot}{\underline{\lambda}^{\Box}a \wedge \underline{\lambda}^{\Diamond}b \triangleleft \bot} \qquad (m_5)\ \frac{\top \triangleleft a \vee b}{\top \triangleleft \underline{\lambda}^{\Box}a \vee \underline{\lambda}^{\Diamond}b}$$

$$(m_3)\ \underline{\lambda}^{\Box}\textstyle\bigvee^{\uparrow} A \triangleleft \bigvee\{\underline{\lambda}^{\Box}a \mid a \in A\} \qquad (m_6)\ \underline{\lambda}^{\Diamond}\textstyle\bigvee^{\uparrow} A \triangleleft \bigvee\{\underline{\lambda}^{\Diamond}a \mid a \in A\}$$

where A is a directed set of GL(Φ)-formulas (cf. Subsection 2.4). To be somewhat more precise, we can view both (m_3) and (m_6) to be the consequence of a rule, the premises of which are given by a set of consequence pairs witnessing the directedness of A.

To see, for example, that (m_3) is valid in a $\mathsf{D}_{\mathbf{kh}}$-coalgebra $(\mathbb{X}, \gamma)$, we need to show that

$$\boxminus \textstyle\bigcup^{\uparrow} A \subseteq \bigcup^{\uparrow}\{\boxminus a \mid a \in A\} \tag{5.2}$$

in $\mathsf{D}_{\mathbf{kh}}\mathbb{X}$, where A is a directed set of clopen subsets of $\mathbb{X}$. So suppose $W \in \boxminus\bigcup^{\uparrow} A$, then $\bigcup^{\uparrow} A \in W$. By definition there must be a closed $c \subseteq \bigcup^{\uparrow} A$ such that $c \in W$. Then $\bigcup^{\uparrow} A$ is an open cover of c and since c is closed, hence compact, there must be a finite subcover. But then there must be a single $a \in A$ such that $c \subseteq a$, because A is directed, and as W is up-closed under inclusions we have $a \in W$. This implies $W \in \boxminus a$, i.e. W is in the right hand side of (5.2).

The functor M from Definition 4.1 is obtained from the procedure of Definition 5.2.

Example 5.5. In a similar manner, one can find a collection of sound axioms and rules for the Vietoris functor on **KHaus** such that the procedure from Definition 5.2 yields the Vietoris locale from [21, Section 1].

For the remainder of this Section we work in the following setting:

Assumption 5.6. *Throughout the remainder of this section, we assume given a collection* Λ *of predicate liftings for an endofunctor* T *on* $\mathbf{C}$*, and a set* Ax *of axioms and rules for* GML(Λ) *which is sound in the sense of Definition 5.3. We write* L *for the endofunctor on* **Frm** *given by the procedure in Definition 5.2 and* ρ *is the associated natural transformation.*

Given a language $\mathrm{GML}(\Phi, \Lambda)$ and a collection of axiom and rule schemata, we write $\mathcal{GML}(\Phi, \Lambda, \mathrm{Ax})$ for the (minimal) collection of consequence pairs $\varphi \triangleleft \psi$ of formulas in $\mathrm{GML}(\Phi, \Lambda)$ which contains all axioms and is closed under the rules from geometric logic (cf. Subsection 2.4) and Ax. We write $\varphi \vdash_{\mathcal{GML}(\Phi,\Lambda,\mathrm{Ax})} \psi$ if the consequence pair $\varphi \triangleleft \psi$ is in $\mathcal{GML}(\Phi, \Lambda, \mathrm{Ax})$. If no confusion is likely, we omit the subscript from the turnstyle and simply write $\varphi \vdash \psi$.

Suppose given Φ, Λ and Ax, write L for the language $\mathrm{GML}(\Phi, \Lambda)$ modulo the equivalence relation $\dashv\vdash$ given by $\varphi \dashv\vdash \psi$ iff $\varphi \vdash \psi$ and $\psi \vdash \varphi$. Write $[\varphi]$ for the equivalence class in L of a formula $\varphi \in \mathrm{GML}(\Lambda)$. Then L carries a frame structure by setting $[\varphi] \vee [\psi] = [\varphi \vee \psi]$ and similar for the other connectives. Furthermore, we can define an L-algebra structure $\ell : \mathsf{L}L \to L$ via

$$\underline{\lambda}([\varphi_1], \ldots, [\varphi_n]) \mapsto [\heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n)].$$

Recall that $\mathbf{Alg}(\mathsf{L})$ denotes the category of L-algebras and L-algebra morphisms (see Subsection 2.2). We have:

Lemma 5.7. *The* L*-algebra* $\mathcal{L} = (L, \ell)$ *is initial in* $\mathbf{Alg}(\mathsf{L})$*.*

Moreover:

Lemma 5.8. *For any two formulas* φ, ψ *we have* $\varphi \vdash \psi$ *iff* $[\varphi] \leq [\psi]$ *in* $\mathcal{L}$*.*

The initial L-algebra $\mathcal{L}$ gives rise to an interpretation of $\mathrm{GML}(\Lambda)$ in every L-algebra: The interpretation of a formula φ in $\mathcal{A} = (A, \alpha) \in \mathbf{Alg}(\mathsf{L})$ is given by $(\!| \varphi |\!)_{\mathcal{A}} = i_{\mathcal{A}}([\varphi])$, where $i_{\mathcal{A}}$ is the unique L-algebra morphism $\mathcal{L} \to \mathcal{A}$. The interpretation is related to the semantics via the *complex algebra*:

Definition 5.9. The *complex* L*-algebra* of a T-coalgebra $\mathfrak{X} = (X, \gamma)$ is $\mathfrak{X}^+ = (\Omega X, \Omega\gamma \circ \rho_X)$, where ρ is the natural transformation from Definition 5.3.

We can view the interpretation of a formula φ in a T-coalgebra as an element of its complex algebra. Examination of the definitions shows that

$$[\![\varphi]\!]^{\mathfrak{X}} = (\!| \varphi |\!)_{\mathfrak{X}^+}. \tag{5.3}$$

Furthermore, we have $[\varphi] \leq [\psi]$ if and only if $(\!| \varphi |\!)_{\mathcal{A}} \leq (\!| \psi |\!)_{\mathcal{A}}$ for all L-algebras $\mathcal{A}$. Soundness of the logic now follows from soundness of the axioms: Suppose $\varphi \vdash \psi$, then $[\varphi] \leq [\psi]$ in $\mathcal{L}$ and hence $(\!| \varphi |\!)_{\mathcal{A}} \leq (\!| \psi |\!)_{\mathcal{A}}$ in any L-algebra $\mathcal{A}$. Therefore, by the observation from (5.3) we have $[\![\varphi]\!]^{\mathfrak{X}} \subseteq [\![\psi]\!]^{\mathfrak{X}}$ for every $\mathfrak{X} \in \mathbf{Mod}(\mathsf{T})$ hence $\varphi \Vdash_{\mathsf{T}} \psi$.

5.2. **Completeness.** We keep working within the assumptions of 5.6. In order to prove completeness with respect to $\mathbf{Coalg}(\mathsf{T})$, we want to show that

$$\varphi \Vdash_{\mathsf{T}} \psi \quad \text{implies} \quad \varphi \vdash \psi.$$

That is, if $\varphi \Vdash_{\mathsf{T}} \psi$ then $(\varphi, \psi) \in \mathcal{GML}(\Lambda, \mathrm{Ax})$. By Lemma 5.8 it suffices to show that $[\varphi] \leq [\psi]$ in the initial L-algebra $\mathcal{L} = (L, \ell)$ whenever $\varphi \Vdash_{\mathsf{T}} \psi$.

If $\varphi \Vdash_{\mathsf{T}} \psi$, then we know that $(\!|\varphi|\!)_{\mathfrak{X}^+} \leq (\!|\psi|\!)_{\mathfrak{X}^+}$ in every complex algebra $\mathfrak{X}^+$ for $\mathfrak{X} \in \mathbf{Mod}(\mathsf{T})$. However, there is no guarantee that $\mathcal{L}$ should be the complex algebra of some T-model, so we do not automatically get completeness.

The next proposition shows that in order to prove completeness it suffices to find a T-coalgebra $\mathfrak{X}$ and an L-algebra morphism $h : \mathfrak{X}^+ \to \mathcal{L}$.

Proposition 5.10. *If there exists a T-coalgebra $\mathfrak{X}$ and an L-algebra morphism $h : \mathfrak{X}^+ \to \mathcal{L}$, then*

$$\varphi \Vdash_{\mathsf{T}} \psi \quad \textit{implies} \quad \varphi \vdash \psi.$$

Proof. Write i for the unique L-algebra morphism $\mathcal{L} \to \mathfrak{X}^+$. Then initiality of $\mathcal{L}$ forces $h \circ i = \mathrm{id}_{\mathcal{L}}$. Suppose $\varphi \Vdash_{\mathsf{T}} \psi$, then we have $[\![\varphi]\!]^{\mathfrak{X}} \subseteq [\![\psi]\!]^{\mathfrak{X}}$, which in turn implies $(\!|\varphi|\!)_{\mathfrak{X}^+} \leq (\!|\psi|\!)_{\mathfrak{X}^+}$ by (5.3). It follows from monotonicity of h that

$$[\varphi] = (\!|\varphi|\!)_{\mathcal{L}} = h \circ i((\!|\varphi|\!)_{\mathcal{L}}) = h((\!|\varphi|\!)_{\mathfrak{X}^+}) \leq h((\!|\psi|\!)_{\mathfrak{X}^+}) = h \circ i((\!|\varphi|\!)_{\mathcal{L}}) = (\!|\varphi|\!)_{\mathcal{L}} = [\psi].$$

This proves the proposition. □

Ideally, one would use a duality of functors to establish that such an $\mathfrak{X}$ as in Proposition 5.10 exists. For example, this is how one can prove completeness for (classical) normal modal logic: The Vietoris functor on **Stone** is the Stone dual of the endofunctor on **BA** which determines the logic. However, since we do not start with a dual equivalence (like Stone duality) but rather with a dual *adjunction*, endofunctors on both categories cannot be dual.

To remedy this, we will make use of the fact that the dual adjunction between **Top** and **Frm** restricts to several dual equivalences (see Fact 2.10). Note that this does not yet guarantee that the initial L algebra from Lemma 5.7 is the complex algebra of some T-coalgebra! Indeed, its underlying frame need not be in the restricted dual equivalence. We will see that some of the dual equivalences are good enough to "imitate" frames that fall outside it.

We now investigate under which conditions we can use Proposition 5.10. As announced, we shall restrict our attention to the case where $\Phi = \emptyset$, i.e. we work in a language without proposition letters. This means that there is no need for having valuations, hence $\mathbf{Mod}(\mathsf{T})$ is simply (isomorphic to) $\mathbf{Coalg}(\mathsf{T})$. If $\Phi = \emptyset$, we shall write $\mathrm{GML}(\Lambda)$ instead of $\mathrm{GML}(\Phi, \Lambda)$ and $\mathcal{GML}(\Lambda, \mathrm{Ax})$ for $\mathcal{GML}(\Phi, \Lambda, \mathrm{Ax})$. The absence of proposition letters need not pose a big deficiency: proposition letters can simply be introduced via predicate liftings.

Specifically, we look for a subcategory $\mathbf{A}$ of $\mathbf{Alg}(\mathsf{L})$ such that:

(1) Every L-algebra is the codomain of an L-algebra morphism whose domain is in $\mathbf{A}$;
(2) Every algebra in $\mathbf{A}$ is the complex algebra corresponding to some T-coalgebra.

Clearly, if this is the case we can employ Proposition 5.10.

For the first item, it turns out useful to consider *coreflective subcategories* of **Frm**. We recall the definition from [2] (which is equivalent to the one in [29, Section IV.3]).

Definition 5.11 ([2], Definition 4.25)**.** A subcategory $\mathbf{B}$ of a category $\mathbf{A}$ is called *coreflective* if for every $A \in \mathbf{A}$ there exists a $B \in \mathbf{B}$ and a morphism $c : B \to A$ (called a coreflection) such that for every morphism $f : B' \to A$ in $\mathbf{A}$ with $B' \in \mathbf{B}$, there exists a unique $f' : B' \to B$ such that

$$\begin{tikzcd} B' \arrow[d, "f"'] \arrow[dr, "f"] & \\ B \arrow[r, "c"] & A \end{tikzcd}$$

commutes.

We can now formulate simple conditions that guarantee item (1) to hold.

Lemma 5.12. *Let $\mathbf{F}$ be a coreflective subcategory of* **Frm** *and suppose L restricts to an endofunctor L' on $\mathbf{F}$. Then for each L-algebra (A, α) there exists an L-algebra (B, β) with $B \in \mathbf{F}$ and an L-algebra morphism $(B, \beta) \to (A, \alpha)$.*

Proof. Let $c : B \to A$ be the coreflection of A in $\mathbf{F}$. Then we have a diagram

$$\begin{tikzcd} \mathsf{L}B \arrow[r, "\mathsf{L}c"] \arrow[d, dashed, "\beta"] & \mathsf{L}A \arrow[d, "\alpha"] \\ B \arrow[r, "c"] & A \end{tikzcd}$$

where $\mathsf{L}B$ is in $\mathbf{F}$ by assumption. By definition there exists $\beta : \mathsf{L}B \to B$ making the diagram commute. □

Thus, if $\mathbf{F}$ is a coreflective subcategory of **Frm** and L restricts to $\mathbf{F}$, then item (1) above is satisfied. Examples of such coreflective subcategories are **RFrm** [33, Section 4.2] and **KRFrm** [4, Proposition 3].

Now suppose $\mathbf{F}$ is spatial and write $\mathbf{T}$ for the subcategory of **Top** which is dually equivalent to $\mathbf{F}$. Furthermore, assume that $\mathbf{T}$ is a subcategory of $\mathbf{C}$ (for otherwise T is not defined for every space in $\mathbf{T}$).

If the restriction L' of L is dual to the restriction T' of T to $\mathbf{T}$, then we know that $\mathbf{Coalg}(\mathsf{T}') \equiv^{\mathrm{op}} \mathbf{Alg}(\mathsf{L}')$. In particular, this means that every element of $\mathbf{Alg}(\mathsf{L}')$ is the complex algebra of some T'-coalgebra (hence of some T-coalgebra), i.e. item (2) is satisfied. Summarising:

Theorem 5.13. *Suppose there exists a coreflective subcategory $\mathbf{F}$ of* **Frm** *such that*
- *The dual $\mathbf{T}$ of $\mathbf{F}$ is a subcategory of $\mathbf{C}$;*
- *L restricts to an endofunctor L' on $\mathbf{F}$;*
- *T restricts to an endofunctor T' on $\mathbf{F}^{\mathrm{op}}$;*
- *L' is dual to T';*

Then $\mathcal{GML}(\Lambda, \mathrm{Ax})$ is complete with respect to $\mathbf{Coalg}(\mathsf{T})$, in the sense that for for every consequence pair $\varphi \triangleleft \psi$ of closed GML*-formulas we have that*

$$\varphi \Vdash_{\mathsf{T}} \psi \quad \textit{implies} \quad \varphi \triangleleft \psi \in \mathcal{GML}(\Lambda, \mathrm{Ax}).$$

Let us apply this theorem to normal and monotone modal geometric logic.

Example 5.14. We denote by $\mathcal{M}$ the smallest collection of consequence pairs closed under the axioms and rules from geometric logic (see Subsection 2.4) and the ones presented in Example 5.4. (Note that the congruence rules follow from the monotonicity rules.) It follows from the duality between $\mathsf{D}_{\mathbf{kh}}$ and $\mathsf{M}_{\mathbf{kr}}$, the fact that **KRFrm** is a coreflective subcategory of **Frm**, and Theorem 5.13 that $\mathcal{M}$ is (sound and) complete with respect to $\mathbf{Coalg}(\mathsf{D}_{\mathbf{kh}})$.

Example 5.15. Similar to Example 5.14, one can prove that *normal* geometric modal logic $\mathcal{N}$ is sound and complete with respect to $\mathbf{Coalg}(\mathsf{V}_{\mathbf{kh}})$. In this case, the axioms and rules of $\mathcal{N}$ are the ones from geometric logic, those introduced for positive modal logic in [13, Section 2], and Scott-continuity. We leave the details to the reader.

6. Lifting logics from **Set** to **Top**

In [26, Section 4] the authors give a method to lift a **Set**-functor $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$, together with a collection of predicate liftings Λ for T, to an endofunctor on **Stone**. We adapt their approach to obtain an endofunctor $\widehat{\mathsf{T}}_\Lambda$ on **Top**, and a collection of Scott-continuous open predicate liftings $\widehat{\Lambda}$ for $\widehat{\mathsf{T}}_\Lambda$. In this section the notation $\bigvee^\uparrow$ is used for directed joins, i.e. joins over directed sets.

6.1. **Lifting functors from Set to Top.** Let T be an endofunctor on **Set** and Λ a collection of predicate liftings for T. To define the action of $\widehat{\mathsf{T}}_\Lambda$ on a topological space $\mathbb{X}$ we take the following steps:

Step 1. Construct a frame $\dot{\mathsf{F}}_\Lambda\mathbb{X}$ of the images of predicate liftings applied to the open sets of $\mathbb{X}$ (viewed simply as subsets of $\mathsf{T}(\mathsf{U}\mathbb{X})$);

Step 2. Quotient $\dot{\mathsf{F}}_\Lambda\mathbb{X}$ with a suitable relation that ensures $\bigvee^\uparrow{}_{b\in B}\lambda(b) = \lambda(\bigvee^\uparrow B)$ whenever λ is monotone;

Step 3. Employ the functor $\mathsf{pt} : \mathbf{Frm} \to \mathbf{Top}$ to obtain a (sober) topological space.

This is the content of Definitions 6.1, 6.3 and 6.5. Recall that $\mathsf{U} : \mathbf{Top} \to \mathbf{Set}$ is the forgetful functor and that Q is the contravariant functor sending a set to its Boolean powerset algebra.

Definition 6.1. Let $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$ be a functor and Λ a collection of predicate liftings for T. We define a contravariant functor $\dot{\mathsf{F}}_\Lambda : \mathbf{Top} \to \mathbf{Frm}$. For a topological space $\mathbb{X}$ let $\dot{\mathsf{F}}_\Lambda\mathbb{X}$ be the subframe of $\mathsf{Q}(\mathsf{T}(\mathsf{U}\mathbb{X}))$ generated by the set

$$\{\lambda_{\mathsf{U}\mathbb{X}}(a_1, \ldots, a_n) \mid \lambda \in \Lambda \text{ } n\text{-ary}, \, a_1, \ldots, a_n \in \Omega\mathbb{X}\}.$$

That is, we close this set under finite intersections and arbitrary unions in $\mathsf{Q}(\mathsf{T}(\mathsf{U}\mathbb{X}))$. For a continuous map $f : \mathbb{X} \to \mathbb{X}'$ let $\dot{\mathsf{F}}_\Lambda f : \dot{\mathsf{F}}_\Lambda\mathbb{X}' \to \dot{\mathsf{F}}_\Lambda\mathbb{X}$ be the restriction of $\mathsf{Q}(\mathsf{T}(\mathsf{U}f))$ to $\dot{\mathsf{F}}_\Lambda\mathbb{X}'$.

Lemma 6.2. *The assignment $\dot{\mathsf{F}}_\Lambda$ defines a contravariant functor.*

Proof. We need to show that $\dot{\mathsf{F}}_\Lambda$ is well defined on morphisms and that it is functorial. To show that the action of $\dot{\mathsf{F}}_\Lambda$ on morphisms is well-defined, it suffices to show that $(\dot{\mathsf{F}}_\Lambda f)(\lambda_{\mathsf{U}\mathbb{X}}(a_1', \ldots, a_n')) \in \dot{\mathsf{F}}_\Lambda(\mathbb{X})$ for all generators $\lambda_{\mathsf{U}\mathbb{X}}(a_1', \ldots, a_n')$ of $\dot{\mathsf{F}}_\Lambda\mathbb{X}'$, because frame homomorphisms preserve finite meets and all joins. This holds by naturality of λ:

$$(\dot{\mathsf{F}}_\Lambda f)(\lambda_{\mathsf{U}\mathbb{X}'}(a_1, \ldots, a_n)) = (\mathsf{T}f)^{-1}(\lambda_{\mathsf{U}\mathbb{X}'}(a_1, \ldots, a_n)) = \lambda_{\mathsf{U}\mathbb{X}}(f^{-1}(a_1), \ldots, f^{-1}(a_n)).$$

By continuity of f we have $f^{-1}(a_i) \in \Omega\mathbb{X}$ so the latter is indeed in $\dot{\mathsf{F}}_\Lambda\mathbb{X}$. Functoriality of $\dot{\mathsf{F}}_\Lambda$ follows from functoriality of $\mathsf{Q} \circ \mathsf{T} \circ \mathsf{U}$. $\square$

Definition 6.3. Let Λ be a collection of predicate liftings for a set functor T. For $\mathbb{X} \in \mathbf{Top}$, let $\widehat{\mathsf{F}}_\Lambda\mathbb{X}$ be the quotient of $\dot{\mathsf{F}}_\Lambda\mathbb{X}$ with respect to the congruence $\sim$ generated by

$$\textstyle\bigvee^\uparrow{}_{b\in B}\lambda(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \sim \lambda(a_1, \ldots, a_{i-1}, \bigvee^\uparrow B, a_{i+1}, \ldots, a_n)$$

for all $a_i \in \Omega\mathbb{X}$, $B \subseteq \Omega\mathbb{X}$ directed, and $\lambda \in \Lambda$ monotone in its i-th argument. Write $q_{\mathbb{X}} : \dot{\mathsf{F}}_\Lambda\mathbb{X} \to \widehat{\mathsf{F}}_\Lambda\mathbb{X}$ for the quotient map and $[x]$ for the equivalence class in $\widehat{\mathsf{F}}_\Lambda\mathbb{X}$ of an element $x \in \dot{\mathsf{F}}_\Lambda\mathbb{X}$. For a continuous function $f : \mathbb{X} \to \mathbb{X}'$ define $\widehat{\mathsf{F}}_\Lambda f : \widehat{\mathsf{F}}_\Lambda\mathbb{X}' \to \widehat{\mathsf{F}}_\Lambda\mathbb{X} : [\lambda_{\mathsf{U}\mathbb{X}}(a_1, \dots, a_n)] \mapsto [\dot{\mathsf{F}}_\Lambda(\lambda_{\mathsf{U}\mathbb{X}}(a_1, \dots, a_n))]$.

Quotienting by the congruence from Definition 6.3 ensures that the lifted versions of monotone predicate liftings are Scott-continuous, see Proposition 6.11 below. This is useful when constructing a final model, because Scott-continuity ensures that the the collection of formulas modulo so-called semantic equivalence is set-sized (see Lemma 7.3 below), and consequently the final model aids in comparing several equivalence notions in Section 8.

Lemma 6.4. *The assignment $\widehat{\mathsf{F}}_\Lambda$ defines a contravariant functor.*

Proof. We need to prove functoriality of $\widehat{\mathsf{F}}_\Lambda$ and that $\widehat{\mathsf{F}}_\Lambda f$ is well defined for every continuous map $f : \mathbb{X} \to \mathbb{X}'$. In order to show that $\widehat{\mathsf{F}}_\Lambda$ is well defined, it suffices to show that $\dot{\mathsf{F}}_\Lambda f$ is invariant under the congruence $\sim$. If $f : \mathbb{X} \to \mathbb{X}'$ is a continuous, then

$$\begin{aligned}
&\textstyle\bigvee^{\uparrow}_{b' \in B}(\dot{\mathsf{F}}_\Lambda f)(\lambda_{\mathsf{U}\mathbb{X}'}(a'_1, \dots, a'_{i-1}, b', a'_{i+1}, \dots, a'_n)) \\
&\qquad = \textstyle\bigvee^{\uparrow}_{b' \in B}(\mathsf{T} f)^{-1}(\lambda_{\mathsf{U}\mathbb{X}'}(a'_1, \dots, a'_{i-1}, b', a'_{i+1}, \dots, a'_n)) \\
&\qquad = \textstyle\bigvee^{\uparrow}_{b' \in B}\lambda_{\mathsf{U}\mathbb{X}}(f^{-1}(a'_1), \dots, f^{-1}(a'_{i-1}), f^{-1}(b'), f^{-1}(a'_{i+1}), \dots, f^{-1}(a'_n)) \\
&\qquad \sim \lambda_{\mathsf{U}\mathbb{X}}(f^{-1}(a'_1), \dots, f^{-1}(a'_{i-1}), f^{-1}(\textstyle\bigvee^{\uparrow} B), f^{-1}(a'_{i+1}), \dots, f^{-1}(a'_n)) \\
&\qquad = \dot{\mathsf{F}}_\Lambda f(\lambda_{\mathsf{U}\mathbb{X}}(a'_1, \dots, a'_{i-1}, \textstyle\bigvee^{\uparrow} B, a'_{i+1}, \dots, a'_n))
\end{aligned}$$

so $\dot{\mathsf{F}}_\Lambda f$ is invariant under the congruence. In the $\sim$-step we use the fact that $\{f^{-1}(b') \mid b' \in B\}$ is directed in $\Omega\mathbb{X}$. Functoriality of $\widehat{\mathsf{F}}_\Lambda f$ follows from functoriality of $\mathsf{Q} \circ \mathsf{T} \circ \mathsf{U}$. $\square$

We are now ready to define the topological Kupke-Kurz-Pattinson lift of a functor on **Set** together with a collection of predicate liftings, to a functor on **Top**.

Definition 6.5. Define the *topological Kupke-Kurz-Pattinson lift* (KKP lift for short) of T with respect to Λ to be the functor

$$\widehat{\mathsf{T}}_\Lambda = \mathsf{pt} \circ \widehat{\mathsf{F}}_\Lambda.$$

This is a functor **Top** $\to$ **Top** and since pt lands in **Sob** it restricts to an endofunctor on **Sob**.

Let us put our theory to action. As stated in Section 4 we can generalise the monotone functor $\mathsf{D}_{\mathbf{kh}}$ on **KHaus** from Definition 4.7 to an endofunctor on **Top**. We will show that lifting the monotone set-functor D with respect to the predicate liftings for box and diamond from Example 2.4 gives rise to a functor on **Top** which restricts to $\mathsf{D}_{\mathbf{kh}}$.

Example 6.6 (The monotone functor)**.** Recall the set functor D from Example 2.2: $\mathsf{D} : X \to \{W \in \breve{\mathsf{P}}\breve{\mathsf{P}}X \mid W \text{ is up-closed under inclusion order}\}$. The box and diamond are given by the predicate liftings $\lambda^{\Box}, \lambda^{\Diamond} : \breve{\mathsf{P}} \to \breve{\mathsf{P}} \circ \mathsf{D}$ defined by

$$\lambda^{\Box}_X(a) := \{W \in \mathsf{D}X \mid a \in W\}, \qquad \lambda^{\Diamond}_X(a) := \{W \in \mathsf{D}X \mid (X \setminus a) \notin W\},$$

where $X \in$ **Set**. Furthermore recall from Definition 4.7 that for a compact Hausdorff space $\mathbb{X}$ the space $\mathsf{D}_{\mathbf{kh}}\mathbb{X}$ is the subset of $\mathsf{D}(\mathsf{U}\mathbb{X})$ of collections of sets W satisfying for all $u \subseteq \mathsf{U}\mathbb{X}$

that $u \in W$ iff there exists a closed $c \subseteq u$ such that every open superset of c is in W. In particular this means $\mathsf{U}(\mathsf{D}_{\mathbf{kh}}\mathbb{X}) \subseteq \mathsf{D}(\mathsf{U}\mathbb{X})$. The set $\mathsf{D}_{\mathbf{kh}}\mathbb{X}$ is topologised by the subbase

$$\boxbar a := \{W \in \mathsf{D}_{\mathbf{kh}}\mathbb{X} \mid a \in W\}, \quad \diamondbar a := \{W \in \mathsf{D}_{\mathbf{kh}}\mathbb{X} \mid (\mathbb{X} \setminus a) \notin W\}.$$

By Theorem 4.9 the functor $\mathsf{M} : \mathbf{Frm} \to \mathbf{Frm}$ from Definition 4.1 is such that $\mathsf{M}(\mathsf{opn}\mathbb{X}) \cong \mathsf{opn}(\mathsf{D}_{\mathbf{kh}}\mathbb{X})$ whenever $\mathbb{X}$ is a compact Hausdorff space.

We claim that

$$\mathsf{D}_{\mathbf{kh}} = (\widehat{\mathsf{D}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}})_{\restriction \mathbf{KHaus}} \tag{6.1}$$

and to prove this we will show that $\widehat{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\mathbb{X} = \mathsf{opn}(\mathsf{D}_{\mathbf{kh}}\mathbb{X})$ for every compact Hausdorff space $\mathbb{X}$. Define a map $\varphi : \mathsf{M}(\mathsf{opn}\mathbb{X}) \to \widehat{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\mathbb{X}$ on generators by $\Box a \mapsto [\lambda^{\Box}(a)]$ and $\Diamond a \mapsto [\lambda^{\Diamond}(a)]$. This is well defined because the $[\lambda^{\Box}(a)], [\lambda^{\Diamond}(a)]$ satisfy relations (M_1) – (M_6) from Definition 4.1 and it is surjective because the image of φ contains the generators of $\widehat{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\mathbb{X}$.

So we only need to show injectivity of φ. Our strategy to prove this is to define a map $\psi : \widehat{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\mathbb{X} \to \mathsf{opn}(\mathsf{D}_{\mathbf{kh}}\mathbb{X})$ and show that it is inverse to φ on the level of sets. Since a set-theoretic inverse suffices we do not need to prove that ψ is a homomorphism; we just want it to be well defined. Instead of defining $\psi : \widehat{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\mathbb{X} \to \mathsf{opn}(\mathsf{D}_{\mathbf{kh}}\mathbb{X})$ directly, we will give a well-defined map $\psi' : \dot{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\mathbb{X} \to \mathsf{opn}(\mathsf{D}_{\mathbf{kh}}\mathbb{X})$ whose kernel contains the kernel of the quotient map $q_{\mathbb{X}} : \dot{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\mathbb{X} \to \widehat{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\mathbb{X}$. This in turn yields the map ψ we require. In a diagram:

$$\begin{tikzcd}
\dot{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\mathbb{X} \arrow[rr, "\psi'"] \arrow[dr, "q_{\mathbb{X}}"'] & & \mathsf{opn}(\mathsf{D}_{\mathbf{kh}}\mathbb{X}) \\
& \widehat{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\mathbb{X} \arrow[ur, dashed, "\psi"'] &
\end{tikzcd} \tag{6.2}$$

Define $\psi' : \dot{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\mathbb{X} \to \mathsf{M}(\mathsf{opn}\mathbb{X})$ on generators by $\lambda^{\Box}(a) \mapsto \boxbar a$ and $\lambda^{\Diamond}(a) \mapsto \diamondbar a$. In order to show that this assignments yields a well-defined map (hence extends to a frame homomorphism by Remark 2.8) we need to show that the presentation of an element in $\dot{\mathsf{F}}_{\{\lambda^{\Box},\lambda^{\Diamond}\}}\mathbb{X}$ does not affect its image under ψ'. That is, if

$$\bigcup_{i\in I}\Big(\bigcap_{j\in J_i}\lambda^{\Box}(a_{i,j}) \cap \bigcap_{j'\in J'_i}\lambda^{\Diamond}(a_{i,j'})\Big) = \bigcup_{k\in K}\Big(\bigcap_{\ell\in L_k}\lambda^{\Box}(a_{k,\ell}) \cap \bigcap_{\ell'\in L'_k}\lambda^{\Diamond}(a_{k,\ell'})\Big), \tag{6.3}$$

where J_i, J'_i, L_k and L'_k are finite index sets, then

$$\bigcup_{i\in I}\Big(\bigcap_{j\in J}\psi'(\lambda^{\Box}(a_{i,j})) \cap \bigcap_{j'\in J'}\psi'(\lambda^{\Diamond}(a_{i,j'}))\Big) = \bigcup_{k\in K}\Big(\bigcap_{\ell\in L}\psi'(\lambda^{\Box}(a_{k,\ell})) \cap \bigcap_{\ell'\in L'}\psi'(\lambda^{\Diamond}(a_{k,\ell'}))\Big).$$

As stated we have $\mathsf{U}(\mathsf{D}_{\mathbf{kh}}\mathbb{X}) \subseteq \mathsf{D}(\mathsf{U}\mathbb{X})$. Observe

$$\psi'(\lambda^{\Box}(a)) = \boxbar a = \{W \in \mathsf{D}(\mathsf{U}\mathbb{X}) \mid a \in W\} \cap \mathsf{U}(\mathsf{D}_{\mathbf{kh}}\mathbb{X}) = \lambda^{\Box}(a) \cap \mathsf{U}(\mathsf{D}_{\mathbf{kh}}\mathbb{X}),$$

and similarly $\psi'(\lambda^{\Diamond}(a)) = \lambda^{\Diamond}(a) \cap \mathsf{U}(\mathsf{D}_{\mathbf{kh}}\mathbb{X})$. Suppose the identity in (6.3) holds, then we have

$$\begin{aligned}
&\bigcup_{i \in I} \Big(\bigcap_{j \in J} \psi'(\lambda^{\Box}(a_{i,j})) \cap \bigcap_{j' \in J'} \psi'(\lambda^{\Diamond}(a_{i,j'})) \Big) \\
&\quad = \bigcup_{i \in I} \Big(\bigcap_{j \in J} (\lambda^{\Box}(a_{i,j}) \cap \mathsf{U}(\mathsf{D}_{\mathbf{kh}}\mathbb{X})) \cap \bigcap_{j' \in J'} (\lambda^{\Diamond}(a_{i,j'}) \cap \mathsf{U}(\mathsf{D}_{\mathbf{kh}}\mathbb{X})) \Big) \\
&\quad = \bigcup_{i \in I} \Big(\mathsf{U}(\mathsf{D}_{\mathbf{kh}}\mathbb{X}) \cap \bigcap_{j \in J} \lambda^{\Box}(a_{i,j}) \cap \bigcap_{j' \in J'} \lambda^{\Diamond}(a_{i,j'}) \Big) \\
&\quad = \mathsf{U}(\mathsf{D}_{\mathbf{kh}}\mathbb{X}) \cap \bigcup_{i \in I} \Big(\bigcap_{j \in J} \lambda^{\Box}(a_{i,j}) \cap \bigcap_{j' \in J'} \lambda^{\Diamond}(a_{i,j'}) \Big) \\
&\quad = \mathsf{U}(\mathsf{D}_{\mathbf{kh}}\mathbb{X}) \cap \bigcup_{k \in K} \Big(\bigcap_{\ell \in L} \lambda^{\Box}(a_{k,\ell}) \cap \bigcap_{\ell' \in L'} \lambda^{\Diamond}(a_{k,\ell'}) \Big) \\
&\quad = \bigcup_{k \in K} \Big(\bigcap_{\ell \in L} \psi'(\lambda^{\Box}(a_{k,\ell})) \cap \bigcap_{\ell' \in L'} \psi'(\lambda^{\Diamond}(a_{k,\ell'})) \Big).
\end{aligned}$$

So ψ' is well defined.

It is easy to see that $\bigvee^{\uparrow}{}_{b \in B}\lambda(b) \sim \lambda(\bigvee^{\uparrow} B)$ implies $\big(\bigvee^{\uparrow}{}_{b \in B}\lambda(b), \lambda(\bigvee^{\uparrow} B)\big) \in \ker(\psi)$ for $\lambda \in \{\lambda^{\Box}, \lambda^{\Diamond}\}$. Since these pairs generate the congruence of Definition 6.3, we have $\sim = \ker(q_{\mathbb{X}}) \subseteq \ker(\psi')$ and hence there exists a map $\psi : \widehat{\mathsf{F}}_{\{\lambda^{\Box}, \lambda^{\Diamond}\}}\mathbb{X} \to \mathsf{opn}(\widehat{\mathsf{T}}\mathbb{X})$ such that the diagram in (6.2) commutes. Therefore ψ is (well) defined on generators by $[\lambda^{\Box}(a)] \mapsto \Box a$ and $[\lambda^{\Diamond}(a)] \mapsto \Diamond a$. One can easily check that $\psi \circ \varphi = \mathrm{id}$ and $\varphi \circ \psi = \mathrm{id}$ by looking at the action on the generators. It follows that φ is injective.

This entails that for compact Hausdorff spaces $\mathbb{X}$,

$$\widehat{\mathsf{D}}_{\{\lambda^{\Box}, \lambda^{\Diamond}\}}\mathbb{X} = \mathsf{D}_{\mathbf{kh}}\mathbb{X},$$

Furthermore, it can be seen that for continuous maps $f : \mathbb{X} \to \mathbb{X}'$ we have $\mathsf{F}_{\{\lambda^{\Box}, \lambda^{\Diamond}\}} f = \mathsf{opn}(\mathsf{D}_{\mathbf{kh}} f)$. As a consequence, when restricted to **KHaus** we have (6.1) indeed. That is, lifting the monotone functor on **Set** with respect to the box/diamond lifting yields a generalisation of the monotone functor on **KHaus** from Definition 4.7.

Example 6.7. Using similar techniques as in the previous example, one can show that, when restricted to **KHaus**, the topological Kupke-Kurz-Pattinson lift of P with respect to the usual box and diamond lifting coincides with the Vietoris functor. (An algebraic description similar to the one in Theorem 4.9 is given in [20, Proposition III4.6].)

Example 6.8. Not every endofunctor on **Top** can be obtained as the lift of a **Set**-functor with respect to a (cleverly) chosen set of predicate liftings in the sense of Definition 6.5. A trivial counterexample is the functor $\mathsf{F} : \mathbf{Top} \to \mathbf{Top}$ from Example 3.9. For every topological space $\mathbb{X}$ we have $\mathsf{F}\mathbb{X} = \mathbb{2}$, which is not a T_0 space, hence not a sober space. Therefore F does not preserve sobriety, while every lifted functor automatically preserves sobriety. Thus F is not the lift of a **Set**-functor.

6.2. **Lifting predicate liftings.** We describe how to lift a predicate lifting to an open predicate lifting. Recall that $\mathsf{Z} : \mathbf{Frm} \to \mathbf{Set}$ is the forgetful functor which sends a frame to its underlying set.

Definition 6.9. Let Λ be a collection of predicate liftings for a functor $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$. A predicate lifting $\lambda : \breve{\mathsf{P}}^n \to \breve{\mathsf{P}} \circ \mathsf{T}$ in Λ induces an open predicate lifting $\widehat{\lambda} : \Omega^n \to \Omega \circ \widehat{\mathsf{T}}_\Lambda$ for $\widehat{\mathsf{T}}_\Lambda$ via

$$\Omega^n\mathbb{X} \xrightarrow{\lambda_{\mathsf{U}\mathbb{X}}} \mathsf{Z}(\dot{\mathsf{F}}_\Lambda\mathbb{X}) \xrightarrow{\mathsf{Z}q_{\mathbb{X}}} \mathsf{Z}(\widehat{\mathsf{F}}_\Lambda\mathbb{X}) \xrightarrow{\mathsf{Z}k_{\widehat{\mathsf{F}}_\Lambda\mathbb{X}}} \Omega(\mathsf{pt}(\widehat{\mathsf{F}}_\Lambda\mathbb{X})) = \Omega(\widehat{\mathsf{T}}_\Lambda\mathbb{X}).$$

By $\lambda_{\mathsf{U}\mathbb{X}}$ we actually mean the restriction of $\lambda_{\mathsf{U}\mathbb{X}}$ to $\Omega^n\mathbb{X} \subseteq \breve{\mathsf{P}}(\mathsf{U}\mathbb{X})$. The map $k_{\mathsf{F}\mathbb{X}}$ is the frame homomorphism given by $a \mapsto \{p \in \mathsf{pt}(\mathsf{F}_\Lambda\mathbb{X}) \mid p(a) = 1\}$. Then $\widehat{\Lambda} := \{\widehat{\lambda} \mid \lambda \in \Lambda\}$ is a geometric modal signature for $\widehat{\mathsf{T}}_\Lambda$.

Lemma 6.10. *The assignment $\widehat{\lambda}$ is a natural transformation.*

Proof. For a continuous function $f : \mathbb{X} \to \mathbb{X}'$ the following diagram commutes in $\mathbf{Set}$:

$$\begin{tikzcd}
\Omega^n\mathbb{X}' \arrow[r, "\lambda_{\mathsf{U}\mathbb{X}'}"] \arrow[d, "(f^{-1})^n"'] & \mathsf{Z}(\dot{\mathsf{F}}_\Lambda\mathbb{X}') \arrow[r, "\mathsf{Z}q_{\mathbb{X}'}"] \arrow[d, "(\mathsf{T}f)^{-1}"] & \mathsf{Z}(\widehat{\mathsf{F}}_\Lambda\mathbb{X}') \arrow[r, "\mathsf{Z}k_{\widehat{\mathsf{F}}_\Lambda\mathbb{X}'}"] \arrow[d, "(\mathsf{T}f)^{-1}"] & \Omega(\mathsf{pt}(\widehat{\mathsf{F}}_\Lambda\mathbb{X}')) \arrow[d, "\Omega(\mathsf{pt}((\mathsf{T}f)^{-1}))"] \\
\Omega^n\mathbb{X} \arrow[r, "\lambda_{\mathsf{U}\mathbb{X}}"'] & \mathsf{Z}(\dot{\mathsf{F}}_\Lambda\mathbb{X}) \arrow[r, "\mathsf{Z}q_{\mathbb{X}}"'] & \mathsf{Z}(\widehat{\mathsf{F}}_\Lambda\mathbb{X}) \arrow[r, "\mathsf{Z}k_{\widehat{\mathsf{F}}_\Lambda\mathbb{X}}"'] & \Omega(\mathsf{pt}(\widehat{\mathsf{F}}_\Lambda\mathbb{X}))
\end{tikzcd}$$

Commutativity of the left square follows from naturality of λ, commutativity of the middle square follows from the proof of Lemma 6.4 and commutativity of the right square can be seen as follows: let $a_1', \dots, a_n' \in \Omega\mathbb{X}'$, then

$$\begin{aligned}
&\Omega(\mathsf{pt}((\mathsf{T}f)^{-1})) \circ \mathsf{Z}k_{\mathsf{F}_\Lambda\mathbb{X}'}(\lambda_{\mathsf{U}\mathbb{X}'}(a_1', \dots, a_n')) \\
&\qquad = \{q \in \mathsf{pt}(\mathsf{F}_\Lambda\mathbb{X}) \mid q \circ (\mathsf{T}f)^{-1}(\lambda_{\mathsf{U}\mathbb{X}'}(a_1', \dots, a_n')) = 1\} \\
&\qquad = \mathsf{Z}k_{\mathsf{F}_\Lambda\mathbb{X}}((\mathsf{T}f)^{-1}(\lambda_{\mathsf{U}\mathbb{X}'}(a_1', \dots, a_n'))).
\end{aligned}$$

So $\widehat{\lambda}$ is an open predicate lifting. □

The nature of the definitions of $\widehat{\mathsf{T}}_\Lambda$ and $\widehat{\Lambda}$ yields the following desirable results.

Proposition 6.11.

(1) *Let $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$ be a functor and Λ a collection of predicate liftings for T. Then $\widehat{\Lambda}$ is characteristic for $\widehat{\mathsf{T}}_\Lambda$.*
(2) *If $\lambda \in \Lambda$ are monotone, then $\widehat{\lambda} \in \widehat{\Lambda}$ is Scott-continuous.*

Proof. Let $\mathbb{X}$ be a topological space. For the first item, we need to show that the collection

$$\{\widehat{\lambda}(a_1, \dots, a_n) \mid \lambda \in \Lambda \text{ } n\text{-ary}, a_i \in \Omega\mathbb{X}\} \tag{6.4}$$

forms a subbase for the topology on $\widehat{\mathsf{T}}_\Lambda\mathbb{X}$. An arbitrary nonempty open set of $\widehat{\mathsf{T}}_\Lambda\mathbb{X}$ is of the form $\widetilde{x} = \{p \in \mathsf{pt}(\widehat{\mathsf{F}}_\Lambda\mathbb{X}) \mid p(x) = 1\}$, for $x \in \widehat{\mathsf{F}}_\Lambda\mathbb{X}$. An arbitrary element of $\widehat{\mathsf{F}}_\Lambda\mathbb{X}$ is the equivalence class of an arbitrary union of finite intersections of elements of the form $\lambda_{\mathsf{U}\mathbb{X}}(a_1, \dots, a_n)$, for $\lambda \in \Lambda$ and $a_1, \dots, a_n \in \Omega\mathbb{X}$. So we may write $x =$

$\bigcup_{i\in I}(\bigcap_{j\in J_i}[\lambda^{i,j}_{\mathsf{U}\mathbb{X}}(a^{i,j}_1,\dots,a^{i,j}_{n_{i,j}})])$ for some index set I, finite index sets J_i, $\lambda^{i,j}\in\Lambda$ and open sets $a^{i,j}_k\in\Omega\mathbb{X}$. We get

$$\widetilde{x}=\bigcup_{i\in I}\Big(\bigcap_{j\in J_i}\widetilde{[\lambda^{i,j}_{\mathsf{U}\mathbb{X}}(a^{i,j}_1,\dots,a^{i,j}_{n_{i,j}})]}\Big)=\bigcup_{i\in I}\Big(\bigcap_{j\in J_i}\widehat{\lambda}^{i,j}_{\mathbb{X}}(a^{i,j}_1,\dots,a^{i,j}_{n_{i,j}})\Big).$$

The second equality follows from Definition 6.9. This shows that the open sets in (6.4) indeed form a subbase for the open sets of $\widehat{\mathsf{T}}_\Lambda\mathbb{X}$.

The second item follows immediately from the definitions. □

Example 6.12. Let $\lambda^{\Box}$ be the box-predicate lifting for the monotone functor D on **Set** from Example 2.4. Then the procedure from Definition 6.9 sends an open a in a compact Hausdorff space $\mathbb{X}$ to $[\lambda^{\Box}_{\mathsf{U}\mathbb{X}}(a)]$ in $\widehat{\mathsf{F}}_\Lambda\mathbb{X}$. We know from Example 6.6 that $\widehat{\mathsf{F}}_\Lambda\mathbb{X}$ is isomorphic to $\mathsf{M}(\mathsf{opn}\mathbb{X})$ (provided $\mathbb{X}$ is compact Hausdorff) and the element $[\lambda^{\Box}_{\mathsf{U}\mathbb{X}}(a)]$ corresponds to $\Box a\in\mathsf{M}(\mathsf{opn}\mathbb{X})$. Therefore

$$\widehat{\lambda}^{\Box}_{\mathbb{X}}(a)=\widetilde{\Box a}\in\Omega\circ\mathsf{pt}\circ\mathsf{M}\circ\mathsf{opn}\mathbb{X}=\Omega\circ\mathsf{pt}\circ\widehat{\mathsf{F}}_\Lambda\mathbb{X}$$

We know from Theorem 4.9 that $\mathsf{pt}\circ\mathsf{M}\circ\mathsf{opn}\cong\mathsf{D}_{\mathbf{kh}}$ and under this isomorphism $\widetilde{\Box a}\in\Omega\circ\mathsf{pt}\circ\mathsf{M}\circ\mathsf{opn}\mathbb{X}$ is sent to $\boxplus a\in\Omega(\mathsf{D}_{\mathbf{kh}}\mathbb{X})$. Therefore

$$\widehat{\lambda}^{\Box}_{\mathbb{X}}(a)=\boxplus a,$$

which yields to open predicate liftings defined in Subsection 4.3. Similarly, the predicate lifting $\lambda^{\Diamond}$ from Example 2.4 lifts the similarly-named open predicate lifting for $\mathsf{D}_{\mathbf{kh}}$ from Subsection 4.3.

Example 6.13. Similar to Example 6.12, the predicate liftings from Example 2.3 for the powerset functor lift to the open predicate liftings from Example 3.10 for the Vietoris functor on **KHaus**.

7. A final modal construction

We construct a final model in $\mathbf{Mod}(\mathsf{T})$ for a functor T where either T is an endofunctor on **Sob**, or T is an endofunctor on **Top** which preserves sobriety. This assumption need not be problematic: If a functor on **Top** does not preserve sobriety we can look at its sobrification. Topological functors which arise as lifts from set functors using the procedure in Section 6 automatically preserve sobriety.

Assumption 7.1. *Throughout this section, fix an endofunctor T on* **Top** *which preserves sobriety, and a Scott-continuous characteristic geometric modal signature Λ for T. Recall that Φ is a set of proposition letters.*

Definition 7.2. Call two formulas φ and ψ *equivalent* in $\mathbf{Mod}(\mathsf{T})$ with respect to Λ, notation: $\varphi\equiv_{\mathsf{T},\Lambda}\psi$, if $\mathfrak{X},x\Vdash\varphi$ iff $\mathfrak{X},x\Vdash\psi$ for all $\mathfrak{X}\in\mathbf{Mod}(\mathsf{T})$ and $x\in\mathfrak{X}$. Denote the equivalence class of φ in $\mathrm{GML}(\Phi,\Lambda)$ by $[\varphi]$. Let $\mathbf{E}=\mathbf{E}(\mathsf{T},\Phi,\Lambda)$ be the collection of formulas modulo $\equiv_{\mathsf{T},\Lambda}$.

Recall that a coherent formula is one which does not involve arbitrary disjunctions.

Lemma 7.3 (Normal form)**.** *Under the assumption, every formula is equivalent to a formula of the form $\bigvee_{i\in I}\varphi_i$, where all the φ_i are coherent.*

Proof. The proof proceeds by induction on the complexity of the formula. Suppose $\varphi = \varphi_1 \vee \varphi_2$. By induction we may assume that $\varphi_1 \equiv_{\mathsf{T},\Lambda} \bigvee_{i\in I} \psi_i$ and $\varphi_2 \equiv_{\mathsf{T},\Lambda} \bigvee_{j\in J} \psi_j$, where all the ψ_i and ψ_j are coherent, and we have $\varphi \equiv_{\mathsf{T},\Lambda} \bigvee_{i\in I\cup J} \psi_i$, as desired. If $\varphi = \varphi_1 \wedge \varphi_2$, then $\varphi \equiv_{\mathsf{T},\Lambda} (\bigvee_{i\in I} \psi_i) \wedge (\bigvee_{j\in J} \psi_j) \equiv_{\mathsf{T},\Lambda} \bigvee_{(i,j)\in I\times J} \psi_i \wedge \psi_j$. Lastly, suppose $\varphi = \heartsuit^{\lambda}(\bigvee_{i\in I} \psi_i)$, where all the ψ_i are coherent. Then we have $\bigvee_{i\in I} \psi_i = \bigvee\{\bigvee_{i\in I'} \psi_i \mid I' \subseteq I \text{ finite}\}$ and by construction the set $\{[\![\bigvee_{i\in I'} \varphi_i]\!]^{\mathfrak{X}} \mid I' \subseteq I, I' \text{ finite}\}$ is directed for every T-model $\mathfrak{X} = (\mathbb{X}, \gamma, V)$. Hence by Scott-continuity of λ we obtain

$$\lambda_{\mathbb{X}}([\![\bigvee_{i\in I} \psi_i]\!]^{\mathfrak{X}}) = \lambda_{\mathbb{X}}(\bigcup\{[\![\bigvee_{i\in I'} \psi_i]\!]^{\mathfrak{X}} \mid I' \subseteq I \text{ finite}\}) = \bigcup\{\lambda_{\mathbb{X}}([\![\bigvee_{i\in I'} \psi_i]\!]^{\mathfrak{X}} \mid I' \subseteq I \text{ finite}\}.$$

Therefore $\varphi \equiv_{\mathsf{T},\Lambda} \bigvee\{\heartsuit^{\lambda}(\bigvee_{i\in I} \psi_i) \mid I' \subseteq I \text{ finite}\}$, i.e. φ is equivalent to an arbitrary disjunction of coherent formulas. The case for n-ary modalities is similar. □

Corollary 7.4. *The collection* $\mathbf{E}$ *from Definition 7.2 is a set.*

Proof. This follows immediately from Lemma 7.3 and the fact that the collection of coherent formulas is a set. □

Definition 7.5. Define top, bottom, disjunction and arbitrary conjunction on $\mathbf{E}$ by $\top_{\mathbf{E}} = [\top]$, $\bot_{\mathbf{E}} = [\bot]$, $[\varphi] \wedge [\psi] := [\varphi \wedge \psi]$ and $\bigvee_{i\in I}[\varphi_i] := [\bigvee_{i\in I} \varphi_i]$.

It is easy to check that $\mathbf{E}$ now forms a frame. The theory of a point x in a geometric T-model $\mathfrak{X}$ is the collection of formulas that are true at x. The theory of x defines a completely prime filter in $\mathbf{E}$. This motivates the next definition.

Definition 7.6. Let $\mathbb{Z} = \mathsf{pt}\mathbf{E}$. For every geometric T-model $\mathfrak{X} = (\mathbb{X}, \gamma, V)$ define the theory map by

$$\mathrm{th}_{\mathfrak{X}} : \mathbb{X} \to \mathbb{Z} : x \mapsto \{[\varphi] \in \mathbf{E} \mid \mathfrak{X}, x \Vdash \varphi\}.$$

The space $\mathbb{Z}$ will turn out to be the state space of a final model in $\mathbf{Mod}(\mathsf{T})$ and we will see in Proposition 7.13 that the theory maps are T-model morphisms.

Definition 7.7. Set $\mathsf{L} = \mathsf{opn} \circ \mathsf{T} \circ \mathsf{pt} : \mathbf{Frm} \to \mathbf{Frm}$. This functor restricts to an endofunctor on $\mathbf{SFrm}$ which is dual to the restriction of T to $\mathbf{Sob}$. Since Λ is characteristic, the frame $\mathsf{L}\mathbf{E}$ is generated by $\{\lambda_{\mathbb{X}}(\widetilde{[\varphi_1]}, \dots, \widetilde{[\varphi_n]}) \mid \lambda \in \Lambda, \varphi_i \in \mathrm{GML}(\Phi, \Lambda)\}$. Define an L-algebra structure $\delta : \mathsf{L}\mathbf{E} \to \mathbf{E}$ on generators by

$$\delta : \mathsf{L}\mathbf{E} \to \mathbf{E} : \lambda_{\mathsf{pt}\mathbf{E}}(\widetilde{[\varphi_1]}, \dots, \widetilde{[\varphi_n]}) \mapsto [\heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n)].$$

We need to show that δ is well defined. For this purpose it suffices to show that the images of the generators of $\mathbf{E}$ satisfy the same relations that they satisfy in $\mathsf{L}\mathbf{E}$. Recall $\mathbb{Z} = \mathsf{pt}\mathbf{E}$, then $\mathsf{L}\mathbf{E} = \mathsf{opn}(\mathsf{T}\mathbb{Z})$.

Lemma 7.8. *If*

$$\bigcup_{i\in I} \Big(\bigcap_{j\in J_i} \lambda_{\mathbb{Z}}^{i,j}(\widetilde{\varphi}_1^{i,j}, \dots, \widetilde{\varphi}_{n_{i,j}}^{i,j}) \Big) = \bigcup_{k\in K} \Big(\bigcap_{\ell\in L_k} \lambda_{\mathbb{Z}}^{k,\ell}(\widetilde{\varphi}_1^{k,\ell}, \dots, \widetilde{\varphi}_{n_{k,\ell}}^{k,\ell}) \Big) \tag{7.1}$$

then

$$\bigvee_{i\in I} \Big(\bigwedge_{j\in J_i} \heartsuit^{\lambda^{i,j}}(\varphi_1^{i,j}, \dots, \varphi_{n_{i,j}}^{i,j}) \Big) \equiv_{\mathsf{T},\Lambda} \bigvee_{k\in K} \Big(\bigwedge_{\ell\in L_k} \heartsuit^{\lambda^{k,\ell}}(\varphi_1^{k,\ell}, \dots, \varphi_{n_{k,\ell}}^{k,\ell}) \Big), \tag{7.2}$$

where the J_i *and* L_k *are finite index sets and* I *and* K *are index sets of arbitrary size.*

Proof. We will see that this follows from naturality of λ. Our strategy is to show that the truth sets of the left hand side and right hand side of (7.2) coincide in every geometric T-model $\mathfrak{X} = (\mathbb{X}, \gamma, V)$.

Observe that the map $\mathrm{th}_{\mathfrak{X}} : \mathbb{X} \to \mathbb{Z}$, which sends a point to its theory, is continuous because

$$\mathrm{th}_{\mathfrak{X}}^{-1}(\widetilde{\varphi}) = [\![\varphi]\!]^{\mathfrak{X}}, \tag{7.3}$$

which is open in $\mathbb{X}$ for all formulas φ. Compute

$$\begin{aligned}
&\bigcup_{i\in I}\Big(\bigcap_{j\in J_i}\lambda^{i,j}_{\mathbb{X}}([\![\varphi^{i,j}_1]\!]^{\mathfrak{X}},\dots,[\![\varphi^{i,j}_{n_{i,j}}]\!]^{\mathfrak{X}})\Big)\\
&\qquad= \bigcup_{i\in I}\Big(\bigcap_{j\in J_i}\lambda^{i,j}_{\mathbb{X}}(\mathrm{th}_{\mathfrak{X}}^{-1}(\widetilde{\varphi}^{i,j}_1),\dots,\mathrm{th}_{\mathfrak{X}}^{-1}(\widetilde{\varphi}^{i,j}_{n_{i,j}}))\Big) && \text{(By (7.3))}\\
&\qquad= \bigcup_{i\in I}\Big(\bigcap_{j\in J_i}(\mathsf{T}\,\mathrm{th}_{\mathfrak{X}})^{-1}\big(\lambda^{i,j}_{\mathbb{Z}}(\widetilde{\varphi}^{i,j}_1,\dots,\widetilde{\varphi}^{i,j}_{n_{i,j}})\big)\Big) && \text{(Naturality of } \lambda)\\
&\qquad= (\mathsf{T}\,\mathrm{th}_{\mathfrak{X}})^{-1}\Big(\bigcup_{i\in I}\Big(\bigcap_{j\in J_i}\lambda^{i,j}_{\mathbb{Z}}(\widetilde{\varphi}^{i,j}_1,\dots,\widetilde{\varphi}^{i,j}_{n_{i,j}})\Big)\Big) && (\star)\\
&\qquad= (\mathsf{T}\,\mathrm{th}_{\mathfrak{X}})^{-1}\Big(\bigcup_{k\in K}\Big(\bigcap_{\ell\in L_k}\lambda^{k,\ell}_{\mathbb{Z}}(\widetilde{\psi}^{k,\ell}_1,\dots,\widetilde{\psi}^{k,\ell}_{n_{k,\ell}})\Big)\Big) && \text{(Assumption (7.1))}\\
&\qquad= \bigcup_{k\in K}\Big(\bigcap_{\ell\in L_k}(\mathsf{T}\,\mathrm{th}_{\mathfrak{X}})^{-1}\big(\lambda^{k,\ell}_{\mathbb{Z}}(\widetilde{\psi}^{k,\ell}_1,\dots,\widetilde{\psi}^{k,\ell}_{n_{k,\ell}})\big)\Big) && (\star)\\
&\qquad= \bigcup_{k\in K}\Big(\bigcap_{\ell\in L_k}\lambda^{k,\ell}_{\mathbb{X}}(\mathrm{th}_{\mathfrak{X}}^{-1}(\psi^{k,\ell}_1),\dots,\mathrm{th}_{\mathfrak{X}}^{-1}(\psi^{k,\ell}_{n_{k,\ell}}))\Big) && \text{(Naturality of } \lambda)\\
&\qquad= \bigcup_{k\in K}\Big(\bigcap_{\ell\in L_k}\lambda^{k,\ell}_{\mathbb{X}}([\![\psi^{k,\ell}_1]\!]^{\mathfrak{X}},\dots,[\![\psi^{k,\ell}_{n_{k,\ell}}]\!]^{\mathfrak{X}})\Big). && \text{(By (7.3))}
\end{aligned}$$

The steps with $(\star)$ hold because inverse images of functions preserve all unions and intersections. This entails that for all geometric T-models and all states x in $\mathfrak{X}$ we have

$$\mathfrak{X}, x \Vdash \bigvee_{i\in I}\Big(\bigwedge_{j\in J_i}\heartsuit^{\lambda^{i,j}}(\varphi^{i,j}_1,\dots,\varphi^{i,j}_{n_{i,j}})\Big) \quad\text{iff}\quad \mathfrak{X}, x \Vdash \bigvee_{k\in K}\Big(\bigwedge_{\ell\in L_k}\heartsuit^{\lambda^{k,\ell}}(\varphi^{k,\ell}_1,\dots,\varphi^{k,\ell}_{n_{k,\ell}})\Big),$$

and hence (7.2) holds. Therefore δ is well defined. □

The algebra structure on $\mathbf{E}$ entails a coalgebra structure on $\mathbb{Z}$.

Definition 7.9. Let $\zeta : \mathbb{Z} \to \mathsf{T}\mathbb{Z}$ be the composition

$$\mathsf{pt}\mathbf{E} \xrightarrow{\mathsf{pt}\delta} \mathsf{pt}(\mathsf{L}\mathbf{E}) =\!=\!= \mathsf{pt}(\mathsf{opn}(\mathsf{T}(\mathsf{pt}\mathbf{E}))) \xrightarrow{k^{-1}_{\mathsf{T}(\mathsf{pt}\mathbf{E})}} \mathsf{T}(\mathsf{pt}\mathbf{E}).$$

Here $k_{\mathsf{T}(\mathsf{pt}\mathbf{E})} : \mathsf{T}(\mathsf{pt}\mathbf{E}) \to \mathsf{pt}(\mathsf{opn}(\mathsf{T}(\mathsf{pt}\mathbf{E})))$ is the isomorphism given in Remark 2.11. Since $\mathbb{Z} = \mathsf{pt}\mathbf{E}$ this indeed defines a map $\mathbb{Z} \to \mathsf{T}\mathbb{Z}$.

For an object $\Gamma \in \mathbb{Z}$, the element $(\mathsf{pt}\delta)(\Gamma)$ is the completely prime filter

$$F = \{\lambda(\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n) \in \mathsf{pt}(\mathsf{opn}(\mathsf{T}(\mathsf{pt}\mathbf{E}))) \mid [\heartsuit^{\lambda}(\varphi_1,\dots,\varphi_n)] \in \Gamma\}$$

in $\mathsf{pt}(\mathsf{opn}(\mathsf{T}(\mathsf{pt}\mathbf{E})))$. The element $\zeta(\Gamma)$ is the unique element in $\mathsf{T}(\mathsf{pt}\mathbf{E})$ corresponding to F under the isomorphism $k_{\mathsf{T}(\mathsf{pt}\mathbf{E})}$. By definition of $k_{\mathsf{T}(\mathsf{pt}\mathbf{E})}$, this is the unique element in the intersection of

$$\{\lambda(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n) \mid [\heartsuit^\lambda(\varphi_1, \dots, \varphi_n)] \in \Gamma\}.$$

Moreover, it follows from the definition of $k_{\mathsf{T}(\mathsf{pt}\mathbf{E})}$ that $[\heartsuit^\lambda(\varphi_1, \dots, \varphi_n)] \notin \Gamma$ implies $\zeta(\Gamma) \notin \lambda(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n)$.

Notation 7.10. If no confusion is likely to occur we will omit the square brackets that indicate equivalence classes of formulas in $\mathbf{E}$. That is, we shall write $\varphi \in \mathbf{E}$ instead of $[\varphi] \in \mathbf{E}$.

We can now endow the T-coalgebra $(\mathbb{Z}, \zeta)$ with a valuation. Thereafter we will show that $(\mathbb{Z}, \zeta)$ together with this valuation is final in $\mathbf{Mod}(\mathsf{T})$.

Definition 7.11. Let $V_{\mathbb{Z}} : \Phi \to \Omega\mathbb{Z}$ be the valuation $p \mapsto \widetilde{p}$.

The triple $\mathfrak{Z} = (\mathbb{Z}, \zeta, V_{\mathbb{Z}})$ is a geometric T-model, simply because it is a T-coalgebra with a valuation. We can prove a truth lemma for $\mathfrak{Z}$:

Lemma 7.12 (Truth lemma). *We have* $\mathfrak{Z}, \Gamma \Vdash \varphi$ *iff* $\varphi \in \Gamma$.

Proof. Use induction on the complexity of the formula. The propositional case follows immediately from the definition of $V_{\mathbb{Z}}$. The cases $\varphi = \varphi_1 \wedge \varphi_2$ and $\varphi = \bigvee_{i \in i} \varphi_i$ are routine. So suppose $\varphi = \heartsuit^\lambda(\varphi_1, \dots, \varphi_n)$. We have

$$\begin{aligned}
\mathfrak{Z}, \Gamma \Vdash \heartsuit^\lambda(\varphi_1, \dots, \varphi_n) \quad &\text{iff} \quad \zeta(\Gamma) \in \lambda_{\mathbb{Z}}([\![\varphi_1]\!]^{\mathfrak{Z}}, \dots, [\![\varphi_n]\!]^{\mathfrak{Z}}) && \text{(Definition of } \Vdash) \\
&\text{iff} \quad \zeta(\Gamma) \in \lambda_{\mathbb{Z}}(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n) && \text{(Induction)} \\
&\text{iff} \quad \heartsuit^\lambda(\varphi_1, \dots, \varphi_n) \in \Gamma. && \text{(Definition of } \zeta)
\end{aligned}$$

This proves the lemma. □

Proposition 7.13. *For every geometric* T*-model* $\mathfrak{X} = (\mathbb{X}, \gamma, V)$ *the map* $\mathrm{th}_{\mathfrak{X}} : \mathbb{X} \to \mathbb{Z}$ *is a* T*-model morphism.*

Proof. We need to show that $\mathrm{th}_{\mathfrak{X}}$ is a T-coalgebra morphism and that $\mathrm{th}_{\mathfrak{X}}^{-1} \circ V_{\mathfrak{Z}} = V$. The latter follows from the fact that for every proposition letter p we have

$$V(p) = \{x \in X \mid \mathfrak{X}, x \Vdash p\} = \mathrm{th}_{\mathfrak{X}}^{-1}(\widetilde{p}) = \mathrm{th}_{\mathfrak{X}}^{-1}(V_{\mathfrak{Z}}(p)).$$

In order to show that $\mathrm{th}_{\mathfrak{X}}$ is a T-coalgebra morphism, we have to show that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathrm{th}_{\mathfrak{X}}"] \arrow[d, "\gamma"'] & \mathbb{Z} \arrow[d, "\zeta"] \\
\mathsf{T}\mathbb{X} \arrow[r, "\mathsf{T}\,\mathrm{th}_{\mathfrak{X}}"'] & \mathsf{T}\mathbb{Z}
\end{tikzcd}$$

Let $x \in \mathbb{X}$. Since $\mathsf{T}\mathbb{Z}$ is sober, hence T_0, it suffices to show that $\mathsf{T}\,\mathrm{th}_{\mathfrak{X}}(\gamma(x))$ and $\zeta(\mathrm{th}_{\mathfrak{X}}(x))$ are in precisely the same opens of $\mathsf{T}\mathbb{Z}$. Moreover, we know that the open sets of $\mathsf{T}\mathbb{Z}$ are generated by the sets $\lambda_{\mathbb{Z}}(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n)$, so it suffices to show that for all $\lambda \in \Lambda$ and $\varphi_i \in \mathrm{GML}(\Phi, \Lambda)$ we have

$$\mathsf{T}\,\mathrm{th}_{\mathfrak{X}}(\gamma(x)) \in \lambda_{\mathbb{Z}}(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n) \quad \text{iff} \quad \zeta(\mathrm{th}_{\mathfrak{X}}(x)) \in \lambda_{\mathbb{Z}}(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n).$$

This follows from the following computation,

$$\begin{aligned}
&\mathsf{T}\,\mathrm{th}_{\mathfrak{X}}(\gamma(x)) \in \lambda_{\mathbb{Z}}(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n) \\
&\quad \text{iff} \quad \gamma(x) \in (\mathsf{T}\,\mathrm{th}_{\mathfrak{X}})^{-1}(\lambda_{\mathbb{Z}}(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n)) \\
&\quad \text{iff} \quad \gamma(x) \in \lambda_{\mathbb{X}}(\mathrm{th}_{\mathfrak{X}}^{-1}(\widetilde{\varphi}_1), \dots, \mathrm{th}_{\mathfrak{X}}^{-1}(\widetilde{\varphi}_n)) && \text{(Naturality of } \lambda) \\
&\quad \text{iff} \quad \gamma(x) \in \lambda_{\mathbb{X}}([\![\varphi_1]\!]^{\mathfrak{X}}, \dots, [\![\varphi_n]\!]^{\mathfrak{X}}) && \text{(By (7.3))} \\
&\quad \text{iff} \quad \mathfrak{X}, x \Vdash \heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n) && \text{(Definition of } \Vdash) \\
&\quad \text{iff} \quad \heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n) \in \mathrm{th}_{\mathfrak{X}}(x) && \text{(Definition of } \mathrm{th}_{\mathfrak{X}}) \\
&\quad \text{iff} \quad \zeta(\mathrm{th}_{\mathfrak{X}}(x)) \in \lambda_{\mathbb{Z}}(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n) && \text{(Definition of } \zeta)
\end{aligned}$$

This proves the proposition. □

The developed theory results in the following theorem.

Theorem 7.14. *Let T be an endofunctor on* **Top** *which preserves sobriety, and Λ a Scott-continuous characteristic geometric modal signature for T. Then the geometric T-model $\mathfrak{Z} = (\mathbb{Z}, \zeta, V_{\mathbb{Z}})$ is final in* $\mathbf{Mod}(\mathsf{T})$.

Proof. Proposition 7.13 states that for every geometric T-model $\mathfrak{X} = (\mathbb{X}, \gamma, V)$ there exists a T-coalgebra morphism $\mathrm{th}_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{Z}$, so we only need to show that this morphism is unique.

Let $f : \mathfrak{X} \to \mathfrak{Z}$ be any coalgebra morphism. We know from Proposition 3.7 that coalgebra morphisms preserve truth, so for all $x \in \mathbb{X}$ we have $\varphi \in f(x)$ iff $\mathfrak{Z}, f(x) \Vdash \varphi$ iff $\mathfrak{X}, x \Vdash \varphi$. Therefore we must have $f(x) = \mathrm{th}_{\mathfrak{X}}(x)$. □

As an immediate corollary we obtain the following theorem. Recall from Definition 3.8 that two states x and x' are behaviourally equivalent in $\mathbf{Mod}(\mathsf{T})$ if there are T-model morphisms f and f' with $f(x) = f'(x')$.

Theorem 7.15. *Under the assumptions of Theorem 7.14, we have* $\equiv_{\Lambda} = \simeq_{\mathbf{Mod}(\mathsf{T})}$.

Proof. If x and x' are behaviourally equivalent, then they are modally equivalent by Proposition 3.7. Conversely, if they are modally equivalent, then $\mathrm{th}_{\mathfrak{X}}(x) = \mathrm{th}_{\mathfrak{X}'}(x')$ by construction, so they are behaviourally equivalent. □

Remark 7.16. If T is an endofunctor on **Sob** instead of **Top**, then the same procedure yields a final model in $\mathbf{Mod}(\mathsf{T})$. In particular, T need not be the restriction of a **Top**-endofunctor. However, if T is an endofunctor on **KSob** or **KHaus** the procedure above does not guarantee a final coalgebra in $\mathbf{Mod}(\mathsf{T})$; indeed the state space $\mathbb{Z}$ of the final coalgebra $\mathfrak{Z}$ we just constructed need not be compact sober nor compact Hausdorff.

Of course, there may be a different way to attain similar results for **KSob** or **KHaus**. We leave this as an interesting open question. In Theorem 8.9 we prove an analog of Theorem 7.15 for endofunctors on **KSob**.

8. Bisimulations

This section is devoted to bisimulations and bisimilarity between coalgebraic geometric models. We compare two notions of bisimilarity, modal equivalence (from Definition 3.6) and behavioural equivalence (Definition 3.8). Again, where **C** is be a full subcategory of **Top** and T an endofunctor on **C**, we give definitions and propositions in this generality where possible. When necessary, we will restrict our scope to particular instances of **C**.

Definition 8.1. Let $\mathfrak{X} = (\mathbb{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathbb{X}', \gamma', V')$ be two geometric T-models. Let $\mathbb{B}$ be an object in $\mathbf{C}$ such that $\mathbb{B} \subseteq \mathbb{X} \times \mathbb{X}'$, with projections $\pi : \mathbb{B} \to \mathbb{X}$ and $\pi' : \mathbb{B} \to \mathbb{X}'$. Then $\mathbb{B}$ is called an *Aczel-Mendler bisimulation* between $\mathfrak{X}$ and $\mathfrak{X}'$ if for all $(x, x') \in \mathbb{B}$ we have $x \in V(p)$ iff $x' \in V'(p)$ and there exists a transition map $\beta : \mathbb{B} \to \mathsf{T}\mathbb{B}$ that makes π and π' coalgebra morphisms. That is, β is such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{X} \arrow[d, "\gamma"'] & \mathbb{B} \arrow[l, "\pi"'] \arrow[r, "\pi'"] \arrow[d, "\beta"] & \mathbb{X}' \arrow[d, "\gamma'"] \\
\mathsf{T}\mathbb{X} & \mathsf{T}\mathbb{B} \arrow[l, "\mathsf{T}\pi"] \arrow[r, "\mathsf{T}\pi'"'] & \mathsf{T}\mathbb{X}'
\end{tikzcd}$$

Two states $x \in \mathsf{U}\mathbb{X}, x' \in \mathsf{U}\mathbb{X}'$ are called *bisimilar*, notation $x \leftrightarrows x'$, if they are linked by a coalgebra bisimulation.

It follows from Proposition 3.7 that bisimilar states satisfy the same formulas. Furthermore, it easily follows by taking pushouts that Aczel-Mendler bisimilarity implies behavioural equivalence. If moreover T preserves weak pullbacks, the converse holds as well [34].

However, we do not wish to make this assumption on topological spaces, since few functors seem to preserve weak pullbacks. For example, the Vietoris functor does not preserve weak pullbacks [7, Corollary 4.3] and neither does the monotone functor from Definition 4.7. (To see the latter statement, consider the example given in Section 4 of [18] and equip the sets in use with the discrete topology.) Therefore we define Λ-bisimulations for **Top**-coalgebras as an alternative to Aczel-Mendler bisimulations. This notion is an adaptation of ideas in [3, 15]. Under some conditions on Λ, Λ-bisimilarity coincides with behavioural equivalence.

In the next definition we need the concept of coherent pairs: If X and X' are two sets and $B \subseteq X \times X'$ is a relation, then a pair $(a, a') \in \mathsf{P}X \times \mathsf{P}X'$ is called *B-coherent* if $B[a] \subseteq a'$ and $B^{-1}[a'] \subseteq a$. For details and properties see Section 2 in [19].

Definition 8.2. Let T be an endofunctor on $\mathbf{C}$, Λ a geometric modal signature for T and $\mathfrak{X} = (\mathbb{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathbb{X}', \gamma', V')$ two geometric T-models. A *Λ-bisimulation* between $\mathfrak{X}$ and $\mathfrak{X}'$ is a relation $B \subseteq \mathsf{U}\mathbb{X} \times \mathsf{U}\mathbb{X}'$ such that for all $(x, x') \in B$ and $p \in \Phi$ and all tuples of B-coherent pairs of opens $(a_i, a_i') \in \Omega\mathbb{X} \times \Omega\mathbb{X}'$ we have

$$x \in V(p) \quad \text{iff} \quad x' \in V'(p)$$

and

$$\gamma(x) \in \lambda_{\mathbb{X}}(a_1, \ldots, a_n) \quad \text{iff} \quad \gamma'(x') \in \lambda_{\mathbb{X}'}(a_1', \ldots, a_n'). \tag{8.1}$$

Two states are called Λ-bisimilar if there is a Λ-bisimulation linking them, notation: $x \leftrightarrows_\Lambda x'$.

We give an alternative characterisation of (8.1) to elucidate the connection with [3].

Remark 8.3. Let $B \subseteq \mathbb{X} \times \mathbb{X}'$ be a relation endowed with the subspace topology and let $\pi : B \to \mathbb{X}$ and $\pi' : B \to \mathbb{X}'$ be projections. Then $(a, a') \in \Omega\mathbb{X} \times \Omega\mathbb{X}'$ is B-coherent iff $\pi^{-1}(a) = (\pi')^{-1}(a')$.

Let P be the pullback of the cospan $\Omega\mathbb{X} \xrightarrow{\Omega\pi} \Omega B \xleftarrow{\Omega\pi'} \Omega\mathbb{X}'$ in **Frm** and let $p : P \to \mathbb{X}$ and $p' : P \to \mathbb{X}'$ be the corresponding projections. Then the B-coherent pairs are precisely $(p(x), p'(x))$, where x ranges over P. It follows from the definitions that equation (8.1) holds for all B-coherent pairs if and only if

$$\Omega\pi \circ \Omega\gamma \circ \lambda_{\mathbb{X}} \circ p^n = \Omega\pi' \circ \Omega\gamma' \circ \lambda_{\mathbb{X}'} \circ (p')^n,$$

where λ is n-ary.

As desired, Λ-bisimilar states satisfy the same formulas.

Proposition 8.4. *Let* T *be an endofunctor on* $\mathbf{C}$ *and* Λ *a geometric modal signature for* T*. Then* $\leftrightarroweq_\Lambda \subseteq \equiv_\Lambda$.

Proof. Let B be a Λ-bisimulation between geometric T-models $\mathfrak{X}$ and $\mathfrak{X}'$, and suppose xBx'. Using induction on the complexity of the formula, we show that $\mathfrak{X}, x \Vdash \varphi$ iff $\mathfrak{X}', x' \Vdash \varphi$ for all $\varphi \in \mathrm{GML}(\Phi, \Lambda)$. The propositional case is by definition, and $\wedge$ and $\bigvee$ are routine. Suppose $\mathfrak{X}, x \Vdash \heartsuit^\lambda(\varphi_1, \ldots, \varphi_n)$, then $\gamma(x) \in \lambda_{\mathbb{X}}([\![\varphi_1]\!]^{\mathfrak{X}}, \ldots, [\![\varphi_n]\!]^{\mathfrak{X}})$. By the induction hypothesis $([\![\varphi_i]\!]^{\mathfrak{X}}, [\![\varphi_i]\!]^{\mathfrak{X}'})$ is B-coherent for all i. Then $\gamma'(x') \in \lambda_{\mathbb{X}'}([\![\varphi_1]\!]^{\mathfrak{X}'}, \ldots, [\![\varphi_n]\!]^{\mathfrak{X}'})$ since B is a Λ-bisimulation, hence $\mathfrak{X}', x' \Vdash \heartsuit^\lambda(\varphi_1, \ldots, \varphi_n)$. The converse is proven symmetrically. □

Proposition 8.5. *Let* T *be an endofunctor on* $\mathbf{C}$ *and* Λ *a geometric modal signature for* T*. Then* $\leftrightarroweq \subseteq \leftrightarroweq_\Lambda$.

Proof. It suffices to show that every Aczel-Mendler bisimulation is a Λ-bisimulation. Suppose B is an Aczel-Mendler bisimulation and let β be the map that turns B into a coalgebra, then the following diagram commutes:

$$\begin{tikzcd}
\mathbb{X} \arrow[d, "\gamma"'] & B \arrow[l, "\pi"'] \arrow[r, "\pi'"] \arrow[d, "\beta"] & \mathbb{X}' \arrow[d, "\gamma'"] \\
\mathsf{T}\mathbb{X} & \mathsf{T}B \arrow[l, "\mathsf{T}\pi"] \arrow[r, "\mathsf{T}\pi'"'] & \mathsf{T}\mathbb{X}'
\end{tikzcd} \tag{8.2}$$

We will show that B is a Λ-bisimulation. By definition $x \in V(p)$ iff $x' \in V'(p)$ whenever xBx'. We prove the forth condition from Definition 8.2. Let $\lambda \in \Lambda$ and $(x, x') \in B$. Suppose $(a_1, a_1'), \ldots, (a_n, a_n')$ are B-coherent pairs of opens and $\gamma(x) \in \lambda_{\mathbb{X}}(a_1, \ldots, a_n)$. Then we have

$$\begin{aligned}
\beta(x, x') &\in (\mathsf{T}\pi)^{-1}(\lambda_{\mathbb{X}}(a_1, \ldots, a_n)) && \text{(Follows from (8.2))} \\
&= \lambda_B(\pi^{-1}(a_1), \ldots, \pi^{-1}(a_n)) && \text{(Naturality of } \lambda) \\
&\subseteq \lambda_B\big((\pi')^{-1} \circ \pi'[\pi^{-1}(a_1)], \ldots, (\pi')^{-1} \circ \pi'[\pi^{-1}(a_n)]\big) && \text{(Monotonicity of } \lambda) \\
&= \lambda_B\big((\pi')^{-1}(B[a_1]), \ldots, (\pi')^{-1}(B[a_n])\big) && (B[a] = \pi' \circ \pi^{-1}(a)) \\
&\subseteq \lambda_B((\pi')^{-1}(a_1'), \ldots, (\pi')^{-1}(a_n')) && \text{(Monotonicity of } \lambda) \\
&= (\mathsf{T}\pi')^{-1}(\lambda_{\mathbb{X}'}(a_1', \ldots, a_n')). && \text{(Naturality of } \lambda)
\end{aligned}$$

Therefore

$$\gamma'(x') = (\mathsf{T}\pi')(\beta(x, x')) \in \lambda_{\mathbb{X}'}(a_1', \ldots, a_n'),$$

as desired. □

The collection of Λ-bisimulations between two models enjoys the following interesting property.

Proposition 8.6. *Let* Λ *be a geometric modal signature of a functor* $\mathsf{T} : \mathbf{Top} \to \mathbf{Top}$ *and let* $\mathfrak{X} = (\mathbb{X}, \gamma, V)$ *and* $\mathfrak{X}' = (\mathbb{X}', \gamma', V')$ *be two geometric* T*-models. The collection of* Λ*-bisimulations between* $\mathfrak{X}$ *and* $\mathfrak{X}'$ *forms a complete lattice.*

Proof. It is obvious that the collection of Λ-bisimulations is a poset. We will show that this collection is closed under taking arbitrary unions; the result then follows from the fact that any complete semilattice is also a complete lattice, see e.g. [8, Theorem 4.2].

Let J be some index set and for all $j \in J$ let B_j be Λ-bisimulations between $\mathfrak{X}$ and $\mathfrak{X}'$ and set $B = \bigcup_{j \in J} B_j$. We claim that B is a Λ-bisimulation.

Let (a_i, a_i') be B-coherent pairs of opens. Suppose xBx' and $\gamma(x) \in \lambda_{\mathbb{X}}(a_1, \ldots, a_n)$. Then there is $j \in J$ with xB_jx' hence $x \in V(p)$ iff $x' \in V'(p)$. As $B_j[a_i] \subseteq B[a] \subseteq a_i'$ and $B_j^{-1}[a'] \subseteq B^{-1}[a'] \subseteq a_i$, all B-coherent pairs (a_i, a_i') are also B_j-coherent. Since B_j is a Λ-bisimulation we get $\gamma'(x') \in \lambda_{\mathbb{X}'}(a_1', \ldots, a_n')$. The converse direction is proven symmetrically. □

We know by now that Λ-bisimilarity implies modal equivalence. Furthermore, we have seen in Theorem 7.15 that modal equivalence coincides with behavioural equivalence whenever T is an endofunctor on **Top** which preserves sobriety and Λ is a Scott-continuous characteristic geometric modal signature. In order to prove a converse, i.e. that behavioural equivalence implies Λ-bisimilarity, we need to assume that the geometric modal signature is strong.

Recall that two elements x, x' in two models are behaviourally equivalent in $\mathbf{Mod}(\mathsf{T})$, notation: $\simeq_{\mathbf{Mod}(\mathsf{T})}$, if there exist morphisms f, f' in $\mathbf{Mod}(\mathsf{T})$ such that $f(x) = f'(x')$.

Proposition 8.7. *Let T be an endofunctor on $\mathbf{C}$ and Λ a strongly monotone geometric modal signature for T. Let $\mathfrak{X} = (\mathbb{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathbb{X}', \gamma', V')$ be two geometric T-models. Then $\simeq_{\mathbf{Mod}(\mathsf{T})} \subseteq \leftrightarroweq_{\Lambda}$.*

Proof. Suppose x and x' are behaviourally equivalent. Then there are some geometric T-model $\mathfrak{Y} = (\mathbb{Y}, \nu, V_{\mathbb{Y}})$ and T-model morphisms $f : \mathfrak{X} \to \mathfrak{Y}$ and $f' : \mathfrak{X}' \to \mathfrak{Y}$ such that $f(x) = f'(x')$. We will show that

$$B = \{(u, u') \in X \times X' \mid f(u) = f'(u')\}. \tag{8.3}$$

is a Λ-bisimulation B linking x and x'.

Clearly xBx'. It follows from Proposition 3.7 that u and u' satisfy precisely the same formulas whenever $(u, u') \in B$. Suppose $\lambda \in \Lambda$ is n-ary and for $1 \leq i \leq n$ let (a_i, a_i') be a B-coherent pair of opens. Suppose uBu' and $\gamma(u) \in \lambda_{\mathbb{X}}(a_1, \ldots, a_n)$. We will show that $\gamma'(u') \in \lambda_{\mathbb{X}'}(a_1', \ldots, a_n')$. The converse direction is similar. By monotonicity and naturality of λ we obtain

$$\gamma(u) \in \lambda_{\mathbb{X}}(a_1, \ldots, a_n) \subseteq \lambda_{\mathbb{X}}(f^{-1}(f[a_1]), \ldots, f^{-1}(f[a_n])) = (\mathsf{T}f)^{-1}(\lambda_{\mathbb{Y}}(f[a_1], \ldots, f[a_n])),$$

so $(\mathsf{T}f)(\gamma(u)) \in \lambda_{\mathbb{Y}}(f[a_1], \ldots, f[a_n])$. (The $f[a_i]$ need not be open in $\mathbb{Y}$, but since λ is strong, $\lambda_{\mathbb{Y}}(f[a_1], \ldots, f[a_n])$ is defined.) Because f and f' are coalgebra morphisms and $f(u) = f'(u')$ we have $(\mathsf{T}f)(\gamma(u)) = \nu(f(u)) = \nu(f'(u')) = (\mathsf{T}f')(\gamma'(u'))$. Finally, we get

$$\begin{aligned}
\gamma'(u') &\in (\mathsf{T}f')^{-1}(\lambda_{\mathbb{Y}}(f[a_1], \ldots, f[a_n])) \\
&= \lambda_{\mathbb{X}'}((f')^{-1}(f[a_1]), \ldots, (f')^{-1}(f[a_n])) && \text{(Naturality of } \lambda) \\
&= \lambda_{\mathbb{X}'}(B[a_1], \ldots, B[a_n]) && \text{(Strong monotonicity of } \lambda) \\
&\subseteq \lambda_{\mathbb{X}'}(a_1', \ldots, a_n'). && \text{(Coherence of } (a_i, a_i'))
\end{aligned}$$

This proves the proposition. □

Remark 8.8. If $\mathbf{C} = \mathbf{KHaus}$ in the proposition above, then Proposition 3.15 allows us to drop the assumption that Λ be strong.

Let T be an endofunctor on **Top** and let Λ be a geometric modal signature for T. The following diagram summarises the results from Propositions 8.4 and 8.7 and Theorem 7.15.

The arrows indicate that one form of equivalence implies the other. Here (1) holds if T preserves weak pullbacks, (2) is true when Λ is Scott-continuous and characteristic and T preserves sobriety (cf. Theorem 7.15), and (3) holds when Λ is strongly monotone. Note that the converse of (2) always holds, because morphisms preserve truth (Proposition 3.7).

$$\begin{tikzcd}
\mathrel{\underline{\leftrightarrow}} \arrow[r] & \mathrel{\underline{\leftrightarrow}}_\Lambda \arrow[r] & \equiv_\Lambda \arrow[r, shift left, "(2)"] & \simeq_{\mathbf{Mod}(\mathsf{T})} \arrow[l, shift left] \arrow[lll, bend right, "(1)"'] \arrow[ll, bend left, "(3)"]
\end{tikzcd} \tag{8.4}$$

As stated in the introduction we are not only interested in endofunctors on **Top**, but also in endofunctors on full subcategories of **Top**, in particular **KHaus**.

The implications in the diagram hold for endofunctors on **Sob** as well (use Remark 7.16). Moreover, with some extra effort it can be made to work for endofunctors on **KSob** as well. In order to achieve this, we have to redo the proof for the bi-implication between modal equivalence and behavioural equivalence. This is the content of the following theorem.

Theorem 8.9. *Let T be an endofunctor on* **KSob***, Λ a Scott-continuous characteristic geometric modal signature for T and $\mathfrak{X} = (\mathbb{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathbb{X}', \gamma', V')$ two geometric T-models. Then $\equiv_\Lambda \, = \, \simeq_{\mathbf{Mod}(\mathsf{T})}$.*

Proof. If x and x' are behaviourally equivalent then they are modally equivalent by Proposition 3.7. The converse direction can be proved using similar reasoning as in Section 7. The major difference is the following: We define an equivalence relation $\equiv_2$ on $\mathrm{GML}(\Phi, \Lambda)$ by $\varphi \equiv_2 \psi$ iff $[\![\varphi]\!]^{\mathfrak{X}} = [\![\psi]\!]^{\mathfrak{X}}$ and $[\![\varphi]\!]^{\mathfrak{X}'} = [\![\psi]\!]^{\mathfrak{X}'}$. (Note that $\mathfrak{X}$ and $\mathfrak{X}'$ are now fixed.) That is, $\varphi \equiv_2 \psi$ iff φ and ψ are satisfied by precisely the same states in $\mathfrak{X}$ and $\mathfrak{X}'$ (compare Definition 7.2). The frame $\mathbf{E}_2 := \mathrm{GML}(\Phi, \Lambda)/{\equiv_2}$ can then be shown to be a compact frame and hence $\mathbb{Z}_2 := \mathsf{pt}\mathbf{E}_2$ is a compact sober space. The remainder of the proof is analogous to the proof of Theorem 7.15. A detailed proof can be found in [16, Theorem 3.34]. $\square$

We summarise the results for **Top** and two of its full subcategories:

Theorem 8.10. *Let T be an endofunctor on* **Top**, **Sob** *or* **KSob** *and Λ a Scott-continuous characteristic strongly monotone geometric modal signature for T. If x and x' are two states in two geometric T-models, then*

$$x \mathrel{\underline{\leftrightarrow}}_\Lambda x' \quad \textit{iff} \quad x \equiv_\Lambda x' \quad \textit{iff} \quad x \simeq_{\mathbf{Mod}(\mathsf{T})} x'.$$

9. Conclusion

We have started building a framework for coalgebraic *geometric* logic and we have investigated examples of concrete functors. There are still many unanswered and interesting questions. We outline possible directions for further research.

Modal equivalence versus behavioural equivalence: From Theorem 8.10 we know that modal equivalence and behavioural equivalence coincide in $\mathbf{Mod}(\mathsf{T})$ if T is an endofunctor on **KSob**, **Sob** or an endofunctor on **Top** which preserves sobriety. A natural question is whether the same holds when T is an endofunctor on **KHaus**.

When does a lifted functor restrict to KHaus?: We know of two examples, namely the powerset functor with the box and diamond lifting, and the monotone functor on **Set** with the box and diamond lifting, where the lifted functor on **Top** restricts to **KHaus**. It would be interesting to investigate whether there are explicit conditions guaranteeing that the lift of a functor restricts to **KHaus**. These conditions could be either for the **Set**-functor one starts with, or the collection of predicate liftings for this functor, or both.

Bisimulations: In [3] the authors define Λ-bisimulations (which are inspired by [15]) between **Set**-coalgebas. In this paper we define Λ-bisimulations between **C**-coalgebras. A similar definition yields a notion of Λ-bisimulation between **Stone**-coalgebras, where the interpretants of the proposition letters are clopen sets, see [16, Definition 2.19]. This raises the question whether a more uniform treatment of Λ-bisimulations is possible, which encompasses all these cases.

Modalities and finite observations: Geometric logic is generally introduced as the logic of finite observations, and this explains the choice of connectives ($\wedge$, $\bigvee$ and, in the first-order version, $\exists$). We would like to understand to which degree *modalities* can safely be added to the base language, without violating the (semantic) intuition of finite observability. Clearly there is a connection with the requirement of *Scott-continuity* (preservation of directed joins), and we would like to make this connection precise, specifically in the topological setting.

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