Universal models for the positive fragment of intuitionistic logic

Nick Bezhanishvili, Dick de Jongh, Apostolos Tzimoulis and Zhiguang Zhao

Abstract. We describe the n-universal model $\mathcal{U}^\star(n)$ of the positive fragment of the intuitionistic propositional calculus IPC. We show that $\mathcal{U}^\star(n)$ is isomorphic to a generated submodel of $\mathcal{U}(n)$ – the n-universal model of IPC. Using $\mathcal{U}^\star(n)$, we give an alternative proof of Jankov's theorem stating that the intermediate logic KC, the logic of the weak law of excluded middle, is the greatest intermediate logic extending IPC that proves exactly the same positive formulas as IPC.

1 Introduction

In this paper, we use the tools of universal models to study the positive fragment of intuitionistic propositional calculus IPC, i.e., formulas containing only propositional variables, $\wedge$, $\vee$ and $\rightarrow$. Fragments of intuitionistic logic have been thoroughly investigated in the literature. For a detailed historic account we refer to [20]. Among these fragments, the locally finite ones, i.e., the fragments where for each $n \in \omega$ there are only finitely many non-equivalent formulas in n variables, attracted more attention. For example, [13] proved a classic result that the $[\wedge, \rightarrow]$-fragment of IPC is locally finite. The positive fragment is not locally finite, and as a result it has not received much attention in the literature. The major interest for the study of this fragment comes from minimal logic [19], the sublogic of intuitionistic logic obtained by dropping the axiom $\bot \rightarrow \varphi$.

Universal models of intuitionistic logic can be seen as duals to free Heyting algebras. The basic idea underlying the construction of universal models can be traced back to [9]. Universal models for the full IPC were described in [1,16,21,22]; for a detailed exposition see also [6, Section 8.7], [4, Section 3.2] and [11, Section 3]. We refer to [15, Section 3.2.1] for an overview of the history of universal models. The universal model for the $[\wedge, \rightarrow]$-fragment of IPC is characterized in [14], [23] and [5]. Universal models for other locally finite fragments of IPC are discussed[1] in [10, 17]. In this paper we focus on the $[\wedge, \vee, \rightarrow]$-fragment of IPC.

The contribution of the present paper can be listed as follows:

– We describe the n-universal model $\mathcal{U}^\star(n)$ of the positive fragment of IPC and show that it is isomorphic to a generated submodel of the n-universal model $\mathcal{U}(n)$ of IPC and at the same time is a (positive morphism) quotient of $\mathcal{U}(n)$. We study the properties of $\mathcal{U}^\star(n)$ as well as its connection with the n-Henkin model $\mathcal{H}^\star(n)$ for the positive fragment of IPC.

[1] We note that [10, 17] do not discuss universal (*exact* in their terminology) models of non-locally finite fragments of IPC.

– Using $\mathcal{U}^\star(n)$, we give an alternative proof of Jankov's theorem that the intermediate logic KC, the logic of the weak law of excluded middle, is the greatest intermediate logic extending IPC that proves exactly the same positive formulas as IPC.

The paper is organized as follows: In Section 2, we recall all the basic notions and results used consequently in the paper. We also discuss the top-model property and its relationship with the positive fragment of IPC. In Section 3, we define the universal models for the positive fragment of IPC. We also recall the definition of positive morphisms and show that every finite Kripke model can be mapped via a positive morphism into the universal model. We also define positive Jankov-de Jongh formulas and prove an analogue of the Jankov-de Jongh theorem for these formulas. In Section 4, we discuss the relationship between the n-Henkin models and the n-universal models of the positive fragment and in Section 5 we give an alternative proof of Jankov-de Jongh and Jankov's theorems. In Section 6 we summarize obtained results and discuss some future research directions.

2 Preliminaries

2.1 Basic notations

In this section, we briefly recall the relational semantics for the intuitionistic propositional calculus IPC. For a detailed study of IPC we refer to [6].

Definition 1 (Kripke frames and models) *A* Kripke frame *is a pair* $\mathfrak{F} = (W, R)$ *where* W *is a set and* R *is a partial order on it. A* Kripke model *is a triple* $\mathfrak{M} = (W, R, V)$ *where* (W, R) *is a Kripke frame and* V *is a partial map* $V : \mathrm{Prop} \to \mathscr{P}(W)$ *(where* Prop *is the set of propositional variables and* $\mathscr{P}(W)$ *is the powerset of* W*) such that for any* $w, w' \in W$ *we have that* $w \in V(p)$ *and* wRw' *imply* $w' \in V(p)$*.*

The valuation can be extended to all formulas in a standard way. We call the upward closed subsets of W (with respect to R) *upsets*. The set of all upsets of W is denoted by $\mathrm{Up}(W)$. As usual $w \in V(\varphi)$ will be denoted as $w \models \varphi$.

Definition 2 (General frames)

1. *A* general frame *is a triple* $\mathfrak{F} = (W, R, \mathcal{P})$*, where* (W, R) *is a Kripke frame and* $\mathcal{P}$ *is a family of upsets containing* $\emptyset$ *and closed under* $\cap, \cup$ *and the following operation* $\Rightarrow$*: for every* $X, Y \subseteq W$*,*

$$X \Rightarrow Y = \{x \in W : \forall y \in W(xRy \wedge y \in X \to y \in Y)\}.^2$$

Elements of the set $\mathcal{P}$ *are called* admissible sets*.*

2. *A general frame* $\mathfrak{F} = (W, R, \mathcal{P})$ *is called* refined *if for any* $x, y \in W$*,*

[2] In fact, $\Rightarrow$ is just the Heyting implication of the Heyting algebra of all upsets of W.

$$\forall X \in \mathcal{P}(x \in X \to y \in X) \Rightarrow xRy.$$

3. *$\mathfrak{F}$ is called* compact*, if for any families $\mathcal{X} \subseteq \mathcal{P}$ and $\mathcal{Y} \subseteq \{W \backslash X : X \in \mathcal{P}\}$, for which $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property (i.e., finite intersections of the elements of $\mathcal{X} \cup \mathcal{Y}$ are non-empty), we have $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$.*
4. *A general frame $\mathfrak{F}$ is called a* descriptive frame *if it is refined and compact.*[3]

By an *n-formula* we mean a formula built from $p_1, \ldots, p_n$. An *n-model* is a model where Prop $= \{p_1, \ldots, p_n\}$. Next we recall some frame and model constructions that will be used consequently.

Definition 3 (Generated subframe and generated submodel)

1. *For any Kripke frame $\mathfrak{F} = (W, R)$ and $X \subseteq W$, the* subframe of $\mathfrak{F}$ generated by X *is $\mathfrak{F}_X = (R(X), R')$, where $R(X) = \{w' \in W : wRw'$ for some $w \in X\}$ and R' is the restriction of R to $R(X)$. If $X = \{w\}$, then we denote $\mathfrak{F}_X$ by $\mathfrak{F}_w$ and $R(X)$ by $R(w)$.*
2. *For any Kripke frame $\mathfrak{F} = (W, R)$, any valuation V on $\mathfrak{F}$ and $X \subseteq W$, the* submodel of $\mathfrak{M} = (\mathfrak{F}, V)$ generated by X *is $\mathfrak{M}_X = (\mathfrak{F}_X, V')$, where V' is the restriction of V to $R(X)$. If X is a singleton $\{w\}$, then we denote $\mathfrak{M}_X$ by $\mathfrak{M}_w$.*
3. *For any general frame $\mathfrak{F} = (W, R, \mathcal{P})$ and any $X \subseteq W$, the* (general) subframe of $\mathfrak{F}$ generated by X *is $\mathfrak{F}_X = (R(X), R', \mathcal{Q})$, where $(R(X), R')$ is the subframe of (W, R) generated by X, and $\mathcal{Q} = \{U \cap R(X) : U \in \mathcal{P}\}$.*

Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a descriptive frame and let $W' \in \mathcal{P}$. Let $\mathfrak{G} = (W', R', \mathcal{Q})$ denote a general frame such that R' is the restriction of R to W' and $\mathcal{Q} = \{U \cap W' : U \in \mathcal{P}\}$. For a proof of the next lemma we refer to, e.g., [23]. In terms of Esakia spaces this lemma states that a restriction of the order and topology of an Esakia space to a clopen upset in it yields again an Esakia space.

Lemma 4 *Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a descriptive frame and let $W' \in \mathcal{P}$. Then $\mathfrak{G} = (W', R', \mathcal{Q})$ is a descriptive frame.*

Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a descriptive frame. A *descriptive* (or an *admissible*) *valuation on* $\mathfrak{F}$ is a map $V :$ Prop $\to \mathcal{P}$. A pair $(\mathfrak{F}, V)$ is a *descriptive model* is $\mathfrak{F}$ is a descriptive frame and V a descriptive valuation on $\mathfrak{F}$. The truth and validity of formulas in Kripke and descriptive frames and models are defined in a standard way. Next we recall the definition of p-morphisms.

[3] Descriptive general frames are essentially the same as Esakia spaces (see e.g., [4, Section 2.3]). This topological perspective explains why compact general frames are called "compact" (the corresponding topology is compact). This also explains why $\mathcal{Q}$ in Definition 3(3) is defined this way.

Definition 5 (p-morphism)

1. *Let $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$ be Kripke frames. A map $f : W \to W'$ is called a* p-morphism *from $\mathfrak{F}$ to $\mathfrak{F}'$ if*
   - *wRw' implies $f(w)R'f(w')$ for any $w, w' \in W$;*
   - *$f(w)R'v'$ implies $\exists v \in W(wRv \wedge f(v) = v')$.*
2. *Let $\mathfrak{F} = (W, R, \mathcal{P})$ and $\mathfrak{G} = (V, S, \mathcal{Q})$ be general frames. We call a Kripke frame p-morphism f of (W, R) to (V, S) a* (general frame)p-morphism *of $\mathfrak{F}$ to $\mathfrak{G}$, if*
$$\forall X \in \mathcal{Q}, f^{-1}(X) \in \mathcal{P}.$$
3. *A* p-morphism *f from $\mathfrak{M} = (W, R, V)$ to $\mathfrak{M}' = (W', R', V')$ is a p-morphism from (W, R) to (W', R') such that $w \in V(p) \Leftrightarrow f(w) \in V'(p)$ for every $p \in$* Prop*. For models based on general frames, we also require the condition for p-morphisms between general frames. For n-models, the definition is similar.*

The extra condition on p-morphisms in Definition 5.2 is again best explained by viewing descriptive frames as Esakia spaces. This condition is then just equivalent to continuity.

Next we recall the definition of n-Henkin model, which is the canonical model for the n-variable fragment of IPC.

Definition 6 (n-Henkin model)

1. *An* n-theory *is a set of n-formulas closed under deduction in* IPC*.*
2. *A set of formulas Γ has the* disjunction property*, if for all n-formulas φ, ψ, we have that $\varphi \vee \psi \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$.*
3. *The* n-canonical model *or* n-Henkin model *$\mathcal{H}(n) = (W_n, R_n, V_n)$ is a model where W_n consists of all consistent n-theories with the disjunction property, R_n is the subset relation, and $\Gamma \in V_n(p)$ iff $p \in \Gamma$.*

2.2 The n-universal model for the full language of IPC

In this section we recall the definition of the n-universal model for the full language of IPC, state its main properties, recall the definition of the de Jongh formulas and the statement of the Jankov-de Jongh theorem. Proofs of all the results stated here can be found in [4, Chapter 3], [6, Section 8.6] and [11, Section 3].

In the following, we use the terminology *color* to denote the valuation at a world in an n-model. In general, an n-color (n can be omitted if it is clear from the context) is a sequence $c_1 \ldots c_n$ of 0's and 1's. The set of all n-colors is denoted by C^n. We define the order on colors as follows:

$$c_1 \ldots c_n \leq c'_1 \ldots c'_n \text{ iff } c_i \leq c'_i, \text{ for } 1 \leq i \leq n.$$

We write $c_1 \dots c_n < c'_1 \dots c'_n$ if $c_1 \dots c_n \leq c'_1 \dots c'_n$ but $c_1 \dots c_n \neq c'_1 \dots c'_n$.

A *coloring* on $\mathfrak{F} = (W, R)$ is a map $col : W \to C^n$ satisfying $uRv \Rightarrow col(u) \leq col(v)$. It is easy to see that colorings and valuations are in 1-1 correspondence. Given $\mathfrak{M} = (W, R, V)$, we can reconstruct the valuation by the coloring $col_V : W \to C^n$, where $col_V(w) = c_1 \dots c_n$, and for each $1 \leq i \leq n$ we have $c_i = 1$ if $w \in V(p_i)$, and 0 otherwise. We call $col_V(w)$ the *color of w under V*.

In any frame $\mathfrak{F} = (W, R)$, we say that $X \subseteq W$ *totally covers* w (notation: $w \prec X$), if X is the set of all immediate successors of w. When $X = \{v\}$, we write $w \prec v$. A set $X \subseteq W$ is called an *anti-chain* if $|X| > 1$ and for every $w, v \in X$, $w \neq v$ implies that $\neg(wRv)$ and $\neg(vRw)$. If uRv we say that *u is under v*.

We can now inductively define the n-universal model $\mathcal{U}(n)$ by cumulative layers $\mathcal{U}(n)^k$ for $k \in \omega$, where each layer contains all the points w such that the longest chain starting from w has length k, omitting n if it is clear from the context.

Definition 7 (n-universal model)

- *The first layer $\mathcal{U}(n)^1$ consists of 2^n nodes with the 2^n different n-colors under the discrete ordering.*
- *For $k \geq 1$, under each element w in $\mathcal{U}(n)^k$, for each color $s < col(w)$, we put a new node v in $\mathcal{U}(n)^{k+1}$ such that $v \prec w$ with $col(v) = s$, and we take the reflexive transitive closure of the ordering.*
- *For $k \geq 1$, under any finite anti-chain X with at least one element in $\mathcal{U}(n)^k$ and any color s with $s \leq col(w)$ for all $w \in X$, we put a new element v in $\mathcal{U}(n)^{k+1}$ such that $col(v) = s$ and $v \prec X$ and we take the reflexive transitive closure of the ordering.*

The whole model $\mathcal{U}(n)$ is the union of its layers.

It is easy to see from the construction that every $\mathcal{U}(n)^k$ is finite. As a consequence, the generated submodel $\mathcal{U}(n)_w$ is finite for any node w in $\mathcal{U}(n)$.

We now state some properties of the n-universal model. For a proof of the next lemma, we refer to, e.g., [6, Section 8.6], [23, Theorem 3.2.3] and [11, Lemma 11].

Lemma 8 *Let $\mathfrak{M}$ be a finite rooted Kripke n-model. Then there exist a unique $w \in \mathcal{U}(n)$ and a unique p-morphism f mapping $\mathfrak{M}$ onto $\mathcal{U}(n)_w$.*

The next theorem shows that $\mathcal{U}(n)$ is a counter-model to every n-formula not provable in IPC. This justifies the name "universal model" for $\mathcal{U}(n)$. For a proof, we refer to, e.g., [23, Theorem 3.2.4] and [11, Theorem 13].

Theorem 9

1. *For any n-formula φ we have $\mathcal{U}(n) \models \varphi$ iff $\vdash_{\mathsf{IPC}} \varphi$.*
2. *For any n-formulas φ and ψ, and for all $w \in \mathcal{U}(n)$ we have*

$$(\mathcal{U}(n), w \models \varphi \Rightarrow \mathcal{U}(n), w \models \psi) \text{ iff } \varphi \vdash_{\mathsf{IPC}} \psi.$$

In the following, we recall the definition of de Jongh formulas for the full language of IPC and the fact that these formulas define point-generated submodels of universal models.

For any node w in an n-model $\mathfrak{M}$, if $w \prec \{w_1, \ldots, w_m\}$, then we let

$$\begin{aligned}
\text{prop}(w) &= \{p_i | w \models p_i, 1 \leq i \leq n\}, \\
\text{notprop}(w) &= \{q_i | w \not\models q_i, 1 \leq i \leq n\}, \\
\text{newprop}(w) &= \{r_j | w \not\models r_j \text{ and } w_i \models r_j \text{ for each } 1 \leq i \leq m, \text{ for } 1 \leq j \leq n\}.
\end{aligned}$$

Here newprop(w) denotes the set of atoms which are "about to be true in w", i.e., the atoms that are false in w but are true in its all proper successors. For the definition of a depth of a point in a frame we refer to [6, p. 43] or [4, 3.1.9]. Roughly speaking, a point w of a universal model has *depth* k if belongs to the k-th layer of $\mathcal{U}(n)$. The depth of a point w will be denoted by $d(w)$.

Definition 10 *Let w be a point in $\mathcal{U}(n)$. We inductively define the corresponding* de Jongh formulas *φ_w and ψ_w:*

If $d(w) = 1$, then let

$$\varphi_w = \bigwedge \text{prop}(w) \wedge \bigwedge \{\neg p_k | p_k \in \text{notprop}(w), 1 \leq k \leq n\},$$

and

$$\psi_w = \neg \varphi_w.$$

If $d(w) > 1$, and $\{w_1, \ldots, w_m\}$ is the set of all immediate successors of w, then define

$$\varphi_w = \bigwedge \text{prop}(w) \wedge (\bigvee \text{newprop}(w) \vee \bigvee_{i=1}^{m} \psi_{w_i} \to \bigvee_{i=1}^{m} \varphi_{w_i}),$$

and

$$\psi_w = \varphi_w \to \bigvee_{i=1}^{m} \varphi_{w_i}.$$

The most important properties of the de Jongh formulas are recalled in the following proposition. For a proof, we refer to [4, Theorem 3.3.2].

Proposition 11 *For every $w \in \mathcal{U}(n)$, we have:*

- $V(\varphi_w) = R(w)$,
- $V(\psi_w) = \mathcal{U}(n) \backslash R^{-1}(w)$, *where* $R^{-1}(w) = \{w' \in \mathcal{U}(n) : w' R w\}$.

Now we state more properties of the universal model and de Jongh formulas. For a proof of the next proposition we refer to [11, Corollary 19]. We let

$$\begin{aligned}
Cn_n(\varphi) &= \{\psi : \psi \text{ is an } n\text{-formula such that } \vdash_{\mathsf{IPC}} \varphi \to \psi\}, \\
Th_n(\mathfrak{M}, w) &= \{\varphi : \varphi \text{ is an } n\text{-formula such that } \mathfrak{M}, w \models \varphi\},
\end{aligned}$$

We will omit n if it is clear from the context.

Proposition 12 *For any point w in $\mathcal{U}(n)$, $Th_n(\mathcal{U}(n), w) = Cn_n(\varphi_w)$.*

The next lemma states that $\mathcal{U}(n)_w$ is isomorphic to the submodel of $\mathcal{H}(n)$ generated by the theory axiomatized by the de Jongh formula of w. For a proof, we refer to [11, Lemma 20].

Lemma 13 *For any $w \in \mathcal{U}(n)$, let φ_w be the de Jongh formula of w, then we have that $\mathcal{H}(n)_{Cn(\varphi_w)}$ is isomorphic to $\mathcal{U}(n)_w$.*

Let Upper($\mathfrak{M}$) denote the submodel $\mathfrak{M}_{\{w \in W | d(w) < \omega\}}$ generated by all the points of finite depth. Intuitively, Upper($\mathfrak{M}$) is the "upper" part of $\mathfrak{M}$. It can be shown that the n-universal model is isomorphic to the upper part of the n-Henkin model, i.e., to Upper($\mathcal{H}(n)$). For a proof, we refer to, e.g., [4, Theorem 3.2.9] and [11, Theorem 39].

Theorem 14 Upper($\mathcal{H}(n)$) *is isomorphic to $\mathcal{U}(n)$.*

The following result follows from Proposition 11 and Lemma 13. For a proof see [11, Corollary 21].

Proposition 15 *Let $\mathfrak{M}$ be any model and w be a point in $\mathcal{U}(n) = (W, R, V)$. For any point x in $\mathfrak{M}$, if $\mathfrak{M}, x \models \varphi_w$, then there exists a unique point v satisfying $\mathfrak{M}, x \models \varphi_v, \mathfrak{M}, x \not\models \varphi_{v_1}, \ldots, \mathfrak{M}, x \not\models \varphi_{v_m}$, where $v \prec \{v_1, \ldots, v_m\}$, and wRv.*

In the following we state the Jankov-de Jongh theorem for the full language of IPC. For a proof we refer to [4, Theorem 3.3.3] and [11, Theorem 26].

Theorem 16 (Jankov-de Jongh theorem for IPC) *Let $\mathfrak{G}$ be a descriptive frame and $w \in U(n)$ for some $n \in \omega$. Then $\mathfrak{G} \not\models \psi_w$ iff there is an n-valuation V on $\mathfrak{G}$ such that $\mathcal{U}(n)_w$ is a p-morphic image of a generated submodel of $(\mathfrak{G}, V)$.*

Notice that for each finite rooted frame $\mathfrak{F}$ there is a valuation V such that $(\mathfrak{F}, V)$ is isomorphic to $\mathcal{U}(n)_w$ for some $n \in \omega$ and $w \in U(n)$. (For this it is sufficient to introduce a new propositional variable p_s for each s in $\mathfrak{F}$ and let $V(p_s) = R(s)$.) So the above theorem applies to any finite rooted $\mathfrak{F}$.

2.3 The top-model property

The *positive fragment* of IPC consists of the formulas constructed only by $\wedge, \vee, \to$. We denote this language by $\mathcal{L}_{\wedge,\vee,\to}$. Formulas in this fragment will be called *positive formulas*[4]. For the other fragments of IPC the notation is similar.

By replacing every occurrence of $\bot$ by $\neg(p \to p)$, every formula is IPC-equivalent to a $\bot$-free formula. For simplicity of discussion, we restrict our attention to $\bot$-free formulas (i.e. formulas in $\mathcal{L}_{\wedge,\vee,\to,\neg}$) only.

Definition 17 (Top model) *A Kripke model $\mathfrak{M} = (W, R, V)$ is called a* top model, *if it has a node $t \in W$ such that:*

[4] Notice that some authors, e.g., [6] call such formulas *negative-free*.

– *t is a successor of all nodes, i.e., we have wRt for each $w \in W$;*
– *all propositional variables are true in t.*

The node t is called the top point *of $\mathfrak{M}$.*

Definition 18 (Top-model property) *We say that a formula φ has the* top-model property*, if for all Kripke models $\mathfrak{M} = (W, R, V)$, all $w \in W$ we have $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M}^+, w \models \varphi$, where $\mathfrak{M}^+ = (W^+, R^+, V^+)$ is obtained by adding a top point t to $\mathfrak{M}$.*

The next proposition states that there is a procedure which with any intuitionistic formula φ associates a positive formula φ^* or $\bot$ equivalent to φ over top models. The algorithm describing how to compute φ^* given φ is provided in [12, Theorem 5].

Proposition 19 *There is an algorithm which transforms any formula φ in $\mathcal{L}_{\wedge,\vee,\to,\neg}$ into a formula φ^* in $\mathcal{L}_{\wedge,\vee,\to} \cup \{\bot\}$ such that for any top model $\mathfrak{M}$ and any node w in $\mathfrak{M}$, we have $\mathfrak{M}, w \models \varphi \leftrightarrow \varphi^*$. Furthermore if $\varphi^* \neq \bot$ and $\psi^* \neq \bot$ then $(\varphi \to \psi)^* = \varphi^* \to \psi^*$, $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$ and $(\varphi \vee \psi)^* = \varphi^* \vee \psi^*$.*

3 The universal models for the positive fragment of IPC

3.1 The universal model

We will now proceed by defining the n-universal model, $\mathcal{U}^\star(n)$, for the positive fragment of IPC. This model closely resembles the n-universal model for IPC: it is a generated submodel of it, and as we shall see below, is also a positive morphism quotient (see Definition 22 and Lemma 27). We now define $\mathcal{U}^\star(n) = (U^\star(n), R^\star, V^\star)$ inductively in a similar way as we defined $\mathcal{U}(n)$.

Definition 20

– *The first layer $\mathcal{U}^\star(n)^1$ consists of $2^n - 1$ nodes with all the different n-colors* — excluding the color $1 \ldots 1$ — *under the discrete ordering.*
– *For $k \geq 1$, under each element w in $\mathcal{U}^\star(n)^k$, for each color $s < col(w)$, we put a new node v in $\mathcal{U}^\star(n)^{k+1}$ such that $v \prec w$ with $col(v) = s$, and we take the reflexive transitive closure of the ordering.*
– *For $k \geq 1$, under any finite anti-chain X with at least one element in $\mathcal{U}^\star(n)^k$ and any color s with $s \leq col(w)$ for all $w \in X$, we put a new element v in $\mathcal{U}^\star(n)^{k+1}$ such that $col(v) = s$ and $v \prec X$, and we take the reflexive transitive closure of the ordering.*

The whole model $\mathcal{U}^\star(n)$ is the union of its layers.

Notice that $\mathcal{U}^\star(1)$ is very different from the Rieger-Nishimura ladder $\mathcal{U}(1)$. It is well known that $\mathcal{U}(1)$ is infinite while $\mathcal{U}^\star(1)$ consists of a single point that does not satisfy p. The only formulas satisfied at this point are the classical

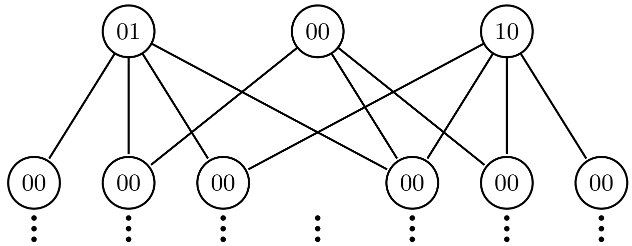

Fig. 1. The first two layers of $\mathcal{U}^\star(2)$

tautologies. For $n > 1$ we have that $\mathcal{U}^\star(n)$ is infinite. Below we present the first two layers of $\mathcal{U}^\star(2)$. The third layer consists of 72 points.

It is known (see e.g., [4, Theorem 3.2.19]) that $\mathcal{U}(n)$, where $n \geq 2$, has uncountably many upsets, and therefore, as our language is countable, not all of them are definable. A similar result holds for $\mathcal{U}^\star(n)$.

Lemma 21 *There are uncountably many upsets in $\mathcal{U}^\star(n)$ for $n \geq 2$. Thus, not all upsets of $\mathcal{U}^\star(n)$ are definable.*

Proof. We prove the result for $n = 2$. The case $n > 2$ follows since the underlying frame of $\mathcal{U}^\star(2)$ is a generated subframe of the underlying frame of $\mathcal{U}^\star(n)$ for $n \geq 2$. We show that $\mathcal{U}^\star(2)$ has a countable anti-chain. As every subset of this anti-chain generates a unique upset, the latter implies that there are $2^{\aleph_0}$ upsets of $\mathcal{U}^\star(2)$. This, in turn, means that not all upsets of $\mathcal{U}^\star(2)$ are definable.

The proof proceeds similarly to the proof that $\mathcal{U}(2)$ has a countable anti-chain (see e.g., [4, Theorem 3.2.19(2)]). The points of the first layer of $\mathcal{U}^\star(2)$ are of the colors 01, 00 and 10. It is easy to see that we can embed the Rieger-Nishimura ladder into the submodel of $\mathcal{U}^\star(2)$ that contains the points that are not below the 10-point. Hence there exists a countable chain of points below the 00-point, let us call them a_n (where $a_n R^\star a_m$ for $n \geq m$), such that the upsets they generate do not contain the 10-point. This means that the 10-point and a_n form an anti-chain for each $n \in \omega$. Therefore for each $n \in \omega$ there exists a point b_n totally covered by a_n and the 10-point. We will now argue that $\{b_n : n \in \omega\}$ is a countable anti-chain. Indeed, let $m \neq n$. We have that $\neg(a_n R^\star b_m)$, which implies that $\neg(b_n R^\star b_m)$ by the definition of b_n. By a symmetric argument, we also have $\neg(a_m R^\star b_n)$ and thus $\neg(b_m R^\star b_n)$. This concludes the proof of the lemma.

□

3.2 Positive morphisms

We will now recall the definition of positive morphisms between descriptive frames and models. In [6, Section 9.1] these morphisms are called *dense subreductions*. Positive morphisms are closely related to *strong partial Esakia morphisms* of [2]. However, strong Esakia morphisms satisfy additional conditions which guarantee a duality between these morphisms and $(\wedge, \vee, \to)$-homomorphisms between Heyting algebras.

Given two intuitionistic models (W, R, V) and (W', R', V') and a partial map $f : W \to W'$, for each $X \subseteq W'$ we let $f^*(X) = W \setminus R^{-1}(f^{-1}[W' \setminus X])$.

Definition 22 *Let (W, R, V) and (W', R', V') be models. A* positive morphism *is a partial map $f : (W, R, V) \to (W', R', V')$ such that:*

1. *If $w, v \in \mathrm{dom}(f)$ and wRv then $f(w)R'f(v)$* (forth condition)*;*
2. *If $w \in \mathrm{dom}(f)$ and $f(w)R'v$, then there exists some $u \in \mathrm{dom}(f)$ such that $f(u) = v$ and wRu* (back condition)*;*
3. *If $w \in \mathrm{dom}(f)$ and vRw, then $v \in \mathrm{dom}(f)$;*
4. *For every $p \in \mathrm{Prop}$ we have $V(p) = f^*(V'(p))$.*

The last condition of Definition 22 guarantees that $f^*(V'(p))$ is an admissible upset. For descriptive frames $(W, R, \mathcal{P})$ and $(W', R', \mathcal{P}'')$, the corresponding condition is: $U \in \mathcal{P}'$ implies $f^*(U) \in \mathcal{P}$, which ensures that f^* is a well-defined map between $\mathcal{P}'$ and $\mathcal{P}$. In fact, conditions (1)-(2) ensure that it preserves $\wedge$ and $\to$, and condition (3) yields that it also preserves $\vee$, so f^* is a $(\wedge, \vee, \to)$-homomorphism.

Lemma 23 *Let $f : W \to W'$ be a positive morphism. If $X \subseteq W'$ is an upset of W', then $f^*(X) = f^{-1}[X] \cup (W \setminus \mathrm{dom}(f))$.*

Proof. Let X be an upset of W'. Then $W' \setminus X$ is a downset of W'. By Definition 22(3), $w \in \mathrm{dom}(f)$ and $u \in R^{-1}(w)$ imply $u \in \mathrm{dom}(f)$ and Definition 22(1) yields $f(u)Rf(w)$. Since $W' \setminus X$ is a downset, $w \in f^{-1}[W' \setminus X]$ implies $u \in f^{-1}[W' \setminus X]$. Thus, $R^{-1}(f^{-1}[W' \setminus X]) = f^{-1}[W' \setminus X]$.

Therefore, we have that $f^*(X) = W \setminus R^{-1}(f^{-1}[W' \setminus X]) = W \setminus f^{-1}[W' \setminus X]$. But $W \setminus f^{-1}[W' \setminus X] = f^{-1}[X] \cup (W \setminus \mathrm{dom}(f))$, which finishes the proof of the lemma. □

We will now give a more convenient characterization of positive morphisms.

Lemma 24 *A partial function $f : (W, R, V) \to (W', R', V')$ is a positive morphism iff the following conditions hold:*

1^*. *If $w, v \in \mathrm{dom}(f)$ and wRv then $f(w)R'f(v)$;*
2^*. *If $w \in \mathrm{dom}(f)$ and $f(w)R'v$ then there exists some $u \in \mathrm{dom}(f)$ such that $f(u) = v$ and wRu;*
3^*. *If $w \in \mathrm{dom}(f)$ and vRw, then $v \in \mathrm{dom}(f)$;*
4^*. *For every $p \in \mathrm{Prop}$ and $w \in \mathrm{dom}(f)$ we have $w \in V(p) \iff f(w) \in V'(p)$;*
5^*. $\mathrm{dom}(f) \supseteq \{w \in W : \exists p \in \mathrm{Prop}\ w \notin V(p)\}$.

Proof. We need to prove that under the assumptions (1)-(3) of the definition of positive morphisms, (4) is equivalent to (4^*) and (5^*).

Let us assume (4^*) and (5^*). By Lemma 23 we have that $f^*(V'(p)) = f^{-1}[V'(p)] \cup W \setminus \mathrm{dom}(f)$. By ($4^*$) we have that $f^{-1}[V'(p)] = V(p) \cap \mathrm{dom}(f)$.

We also have that (5*) implies $W \setminus \mathrm{dom}(f) \subseteq V(p)$ since every element outside the domain of f satisfies all propositional variables. Therefore $V(p) = (V(p) \cap \mathrm{dom}(f)) \cup W \setminus \mathrm{dom}(f)$ and thus $f^*(V'(p)) = V(p)$.

For the other direction assume (4). Then $w \in \mathrm{dom}(f)$ and $f^{-1}[V'(p)] \cup W \setminus \mathrm{dom}(f) = V(p)$ yield that $w \in V(p)$ iff $f(w) \in V'(p)$. So (4*) holds. Also, for any $p \in \mathrm{Prop}$ we have that $W \setminus \mathrm{dom}(f) \subseteq V(p)$ and hence all elements outside the domain of f satisfy all propositional variables. So (5*) holds. □

From now on, we will use this alternative characterization of positive morphisms. Obviously, every p-morphism is a positive morphism. Moreover, notice that if for all $w \in W$, there is some propositional variable p such that p is not satisfied in w, then the positive morphisms are p-morphisms. Finally, it is easy to check that the composition of two positive morphisms is a positive morphism.

The essential difference between p-morphisms and positive morphisms is that the latter are partial maps – domains of such maps may not contain worlds that satisfy all propositional variables. The reason why we can ignore these worlds when dealing with the positive fragment of IPC lies in a simple fact (which can be easily checked by induction) that in such worlds all positive formulas are true. Next we show that positive morphisms preserve positive formulas.

Proposition 25 *Let $f : (W, R, V) \to (W', R', V')$ be a positive morphism. Then for every positive formula φ and $w \in \mathrm{dom}(f)$ we have*

$$(W, R, V), w \models \varphi \quad \textit{iff} \quad (W', R', V'), f(w) \models \varphi.$$

Proof. We proceed by induction on the complexity of φ. The base case, i.e. when φ is a propositional variable, follows directly from Lemma 24(4). Now suppose that f preserves the positive formulas φ and ψ. That f also preserves $\varphi \vee \psi$ and $\varphi \wedge \psi$ trivially follows from the semantic definitions of the connectives and the induction hypothesis.

Let us now assume that $(W, R, V), w \models \varphi \to \psi$. Let $f(w)R'v$ and assume that $(W', R', V'), v \models \varphi$. Then by the definition of the positive morphisms, there is some $u \in \mathrm{dom}(f)$ such that $f(u) = v$ and wRu. By the induction hypothesis, we have $(W, R, V), u \models \varphi$. Hence $(W, R, V), u \models \psi$, which by the induction hypothesis gives us that $(W', R', V'), f(u) \models \psi$. So $(W', R', V'), f(w) \models \varphi \to \psi$.

For the converse direction, let us assume that $(W', R', V'), f(w) \models \varphi \to \psi$ and for some u such that wRu we have $(W, R, V), u \models \varphi$. If $u \in \mathrm{dom}(f)$, then the induction hypothesis readily implies that $(W, R, V), u \models \psi$. If $u \notin \mathrm{dom}(f)$, then by Lemma 24(5) for every propositional variable p we have that $u \in V(p)$, which implies that $(W, R, V), u \models \psi$, since all positive formulas are true in such worlds. □

The next corollary is a consequence of the proposition above.

Corollary 26 *Every formula in $\mathcal{L}_{\wedge,\vee,\to} \cup \{\bot\}$ has the top-model property.*

Proof. Let $\mathfrak{M} = (W, R, V)$ be an arbitrary Kripke model. We define a partial map $f : \mathfrak{M}^+ \to \mathfrak{M}$ such that it is the identity on all the elements of W and it is undefined in the top node. It is easy to see that f is a positive morphism. The result now follows directly from Proposition 25 for positive formulas. Finally, notice that for $\perp$ the result is trivially true. $\square$

By the construction of the two universal models, we can see that $\mathcal{U}^\star(n)$ contains all the points in $\mathcal{U}(n)$ which are not below the node where all propositional variables are true. Therefore, it follows that $\mathcal{U}^\star(n)$ is isomorphic to $\mathcal{N} = (N, R, V)$ with $N = \{w \in \mathcal{U}(n) : w^\star \notin R(w)\}$, a generated submodel of $\mathcal{U}(n)$, where $w^\star$ is the greatest node of $\mathcal{U}(n)$ such that $\mathrm{col}(w^\star)_i = 1$ for every $i \leq n$. By Corollary 26, $(\mathcal{U}^\star(n))^+$ satisfies the same positive formulas as $\mathcal{U}^\star(n)$. Again, by the construction of the models, it follows that $(\mathcal{U}^\star(n))^+$ is (isomorphic to) a generated submodel of $\mathcal{U}(n)$, whose domain consists of the elements of $U(n)$ whose only successor of depth 1 satisfies all propositional variables. Let us call this submodel $\mathcal{M}(n)$, and let $G : (\mathcal{U}^\star(n))^+ \to \mathcal{M}(n)$ be this isomorphism.

The models $\mathcal{U}^\star(n)$ and $(\mathcal{U}^\star(n))^+$ can be viewed as two different ways of describing the universal models of the positive fragment of IPC. In the first approach, there are IPC-satisfiable positive formulas (for example $p_1 \wedge \ldots \wedge p_n$) that are satisfied nowhere in $\mathcal{U}^\star(n)$ and hence are indistinguishable from $\perp$ in this model. This is not the case in $\mathcal{U}(n)$, where every IPC-satisfiable formula is satisfied in some world. In $(\mathcal{U}^\star(n))^+$ all positive formulas are satisfied at the topmost point, and hence this model can distinguish positive formulas from $\perp$. As we will see below, every finite rooted model can be mapped onto a generated submodel of $\mathcal{U}^\star(n)$ via a positive morphism, which is not the case for $(\mathcal{U}^\star(n))^+$. On the other hand, for every finite rooted model $\mathfrak{M}$, the model $\mathfrak{M}^+$ can be mapped onto a generated submodel of $(\mathcal{U}^\star(n))^+$ via a p-morphism.

Lemma 27 *There exists a surjective positive morphism $F : \mathcal{U}(n) \to \mathcal{U}^\star(n)$ with $\mathrm{dom}(F) = \{w \in \mathcal{U}(n) : \exists p \in \mathrm{Prop}(w \notin V(p))\}$, and for every $w \in \mathrm{dom}(F)$ we have that the restriction of F to $\mathcal{U}(n)_w$ maps $\mathcal{U}(n)_w$ onto $\mathcal{U}^\star(n)_{F(w)}$.*

Proof. We will define F by induction on the depth of the elements of $\mathcal{U}(n)$ in such a way that the color of $F(w)$ is the same as the color of w. If $d(w) = 1$, then $F(w) = w'$, where $d(w') = 1$ and $\mathrm{col}(w) = \mathrm{col}(w')$. Let us now assume that F is defined for the elements of $\mathcal{U}(n)$ of depth m. Let $d(w) = m + 1$ and let us assume that $w \prec \{w_1, \ldots, w_k\}$. Let $A \subseteq F[\{w_1, \ldots, w_k\}]$ be the set of the R-minimal elements of $F[\{w_1, \ldots, w_k\}]$. Then A is finite as it is a subset of a finite set. If A is empty then let $F(w)$ be the element of $\mathcal{U}^\star(n)$ of depth 1 with the same color as w. If $A = \{u\}$ and u has the same color as w, then let $F(w) = u$. Otherwise, by the construction of $\mathcal{U}^\star(n)$, there is a unique $v \prec A$ (by the induction hypothesis for F) with the same color as w and we let $F(w) = v$.

It remains to show that F is a surjective positive morphism, and that for every $w \in \mathrm{dom}(F)$, the restriction of F to $\mathcal{U}(n)_w$ maps $\mathcal{U}(n)_w$ onto $\mathcal{U}^\star(n)_{F(w)}$.

That $w \in V(p)$ if an only if $F(w) \in V^\star(p)$ follows from the construction of F. It is also easy to see by the above construction that if uRw then $F(u)R^\star F(w)$. The surjectivity of F can be shown by viewing $\mathcal{U}^\star(n)$ as the generated submodel $\mathcal{N}$ of $\mathcal{U}(n)$ presented above. Then it is routine to check that F is the identity function on $\mathcal{N}$.

Next we show that the restriction of F to $\mathcal{U}(n)_w$ maps $\mathcal{U}(n)_w$ onto $\mathcal{U}^\star(n)_{F(w)}$. Since all elements of $\mathcal{U}^\star(n)$ have finite depth, it suffices to show that for all $u \in \mathrm{dom}(F)$, all the immediate successors of $F(u)$ are images of successors of u. Indeed, from the definition of F, the immediate successors of $F(w)$ form a subset of $F[\{w_1, \ldots, w_k\}]$, where $w_1, \ldots, w_k$ are the only immediate successors of w. Therefore, by an easy induction on immediate successors we can show that every element in $\mathcal{U}^\star(n)_{F(w)}$ is the image of some element in $\mathcal{U}(n)_w$.

Finally, the back clause that $F(w)R^\star v$ implies the existence of some u with wRu and such that $F(u) = v$ follows from the fact that the restriction of F to $\mathcal{U}(n)_w$ maps $\mathcal{U}(n)_w$ onto $\mathcal{U}^\star(n)_{F(w)}$. □

In the proof of Lemma 27 the map F is defined explicitly. An alternative proof of this lemma can be obtained by describing the same map F indirectly in the following way. Let us fix the injective partial map $i : (\mathcal{U}^\star(n))^+ \to \mathcal{U}^\star(n)$ between the two versions of the universal models to be the identity on $\mathcal{U}^\star(n)$ and undefined in the top node of $(\mathcal{U}^\star(n))^+$ (similarly to the positive morphism defined in the proof of Corollary 26). Moreover, by Proposition 25, for every $w \in U(n)$ we have that w satisfies the same positive formulas in $\mathcal{U}(n)$ and $(\mathcal{U}(n))^+$. Furthermore, by Lemma 8, there exists a unique p-morphism f_w from $((\mathcal{U}(n))^+)_w$ to $\mathcal{U}(n)$, and in particular to $\mathcal{M}(n)$ (see the paragraph after Corollary 26), since $\mathcal{M}(n)$ is a top model. By the uniqueness, we have that $f = \bigcup_{w \in U(n)} f_w$ is a p-morphism from $(\mathcal{U}(n))^+$ onto $\mathcal{M}(n)$. Then we can define $F = i \circ G^{-1} \circ f$ (where G is as in the paragraph after Corollary 26). This function is a positive morphism since it is a composition of positive morphisms. It is onto because f covers $\mathcal{M}(n)$ and G^{-1} and i are onto. Finally, since f was a union of maps from $((\mathcal{U}(n))^+)_w$ onto $\mathcal{U}(n)_{f(w)}$, it follows that the restriction of F to $\mathcal{U}(n)_w$ maps $\mathcal{U}(n)_w$ onto $\mathcal{U}^\star(n)_{F(w)}$.

Lemma 27 gives analogues of Lemma 8 and Theorem 9 for positive morphisms.

Lemma 28 *Let $\mathfrak{M} = (W, R, V)$ be a finite rooted intuitionistic n-model such that there exist $x \in W$ and $p \in \mathrm{Prop}$ with $x \notin V(p)$. Then there exist a unique $w \in U^\star(n)$ and a unique positive morphism f mapping $\mathfrak{M}$ onto $\mathcal{U}^\star(n)_w$.*

Proof. Given any finite rooted intuitionistic n-model $\mathfrak{M}$, Lemma 8 implies that there is a unique $w \in U(n)$ and a p-morphism f from $\mathfrak{M}$ onto $\mathcal{U}(n)_w$. By taking the F from Lemma 27, it follows that $F \circ f$ (with domain $\{x \in W : f(x) \in \mathrm{dom}(F)\}$) is a positive morphism (as a composition of positive morphisms) of $\mathfrak{M}$ onto $\mathcal{U}^\star(n)_{F(w)}$. Finally, since there exist $x \in W$ and $p \in \mathrm{Prop}$ such that $x \notin V(p)$ it follows that $\mathrm{dom}(F \circ f) \neq \varnothing$.

To show the uniqueness, we first observe that given two positive morphisms g_1, g_2 from $\mathfrak{M}$ to $\mathcal{U}^\star(n)$, we have

$$\mathrm{dom}(g_1) = \mathrm{dom}(g_2) = \{x \in W : \exists p \in \mathrm{Prop}(x \notin V(p))\},$$

because there do not exist points of $\mathcal{U}^\star(n)$ that satisfy all $p \in \mathrm{Prop}$. Notice that when restricted to $\mathrm{dom}(g_1)$ both g_1 and g_2 become p-morphisms. Thus, if

$g_1 \neq g_2$, then there exist two different p-morphisms (g_1 and g_2) from dom(g_1) to $\mathcal{U}(n)$ (since $\mathcal{U}^\star(n)$ is a generated subframe of $\mathcal{U}(n)$), contradicting Lemma 8. $\square$

The next theorem shows that $\mathcal{U}^\star(n)$ is indeed a "universal model" for all positive formulas.

Theorem 29 *For every positive n-formula φ, $\mathcal{U}^\star(n) \models \varphi$ iff $\vdash_{\mathsf{IPC}} \varphi$.*

Proof. The right to left direction is trivial. For the converse, let us assume that $\nvdash_{\mathsf{IPC}} \varphi$, i.e. there is a finite rooted model $\mathfrak{M}$ such that $\mathfrak{M}, x \not\models \varphi$, where x is the root of $\mathfrak{M}$. Since φ is positive, we have that x does not satisfy all propositional variables. Then, by Lemma 28, there exists a unique $w \in U^\star(n)$ and a positive morphism f from $\mathfrak{M}$ onto $\mathcal{U}^\star(n)_w$. By Proposition 25, it follows that $\mathcal{U}^\star(n), f(x) \not\models \varphi$. $\square$

3.3 The Jankov-de Jongh formulas

We will now define the de Jongh formulas for the positive fragment of IPC (for the description of the de Jongh formulas for the $[\wedge, \rightarrow]$-fragment of IPC, see [5]). These will be used in the next section for proving Jankov's theorem. We will present two ways of constructing the formulas: one that mirrors the construction of the standard de Jongh formulas, and one that derives the formulas through the procedure cited in Section 2.3. For $w \in U^\star(n)$ let $\mathrm{prop}(w), \mathrm{newprop}(w)$ and $\mathrm{notprop}(w)$ be defined as for the elements of $U(n)$.

Definition 30 *Let $w \in U^\star(n)$. We will define the formulas $\varphi^\star_w$ and $\psi^\star_w$ by induction on the depth of w:*

- *If $d(w) = 1$, then define*

$$\varphi^\star_w = \bigwedge \mathrm{prop}(w) \wedge (\bigvee \mathrm{notprop}(w) \rightarrow \bigwedge \mathrm{notprop}(w))$$

  and

$$\psi^\star_w = \varphi^\star_w \rightarrow \bigwedge_{i \in n} p_i.$$

- *If $d(w) > 1$, then let $w \prec \{w_1, \ldots, w_r\}$ and define*

$$\varphi^\star_w = \bigwedge \mathrm{prop}(w) \wedge (\bigvee \mathrm{newprop}(w) \vee \bigvee_{i \leq r} \psi^\star_{w_i} \rightarrow \bigvee_{i \leq r} \varphi^\star_{w_i})$$

  and

$$\psi^\star_w = \varphi^\star_w \rightarrow \bigvee_{i \leq r} \varphi^\star_{w_i}.$$

The construction is motivated by the following observation: As we noted $(\mathcal{U}^\star(n))^+$ is a generated submodel of $\mathcal{U}^\star(n)$. Using the original de Jongh formula φ_w, for w the greatest element of $(\mathcal{U}^\star(n))^+$, we can define the de Jongh formulas from depth 2, using exactly the same construction as for the standard de Jongh formulas. Only now there is no need to take into consideration the ψ_w formula. This is because every positive formula is satisfied in a world that satisfies all propositional variables, and hence all positive formulas are true in w.

The above leads to the second way of constructing the de Jongh formulas for $\mathcal{U}^\star(n)$.

Definition 31 *For every $w \in \mathcal{U}^\star(n)$, we define $\varphi^\star_w$ and $\psi^\star_w$ as $[\varphi_{G(w)}]^*$ and $[\psi_{G(w)}]^*$ respectively, where $[\cdot]^*$ is the operation discussed in Proposition 19.*

The next proposition shows that the two definitions are in fact equivalent.

Proposition 32 *The formulas defined in Definitions 30 and 31 are equivalent.*

Proof. The proof is by induction on the depth of w. For $d(w) = 1$, we note that $[\varphi_{G(w)}]^*$ is $\bigwedge \mathrm{prop}(w) \wedge (\bigvee \mathrm{notprop}(w) \to \bigwedge \mathrm{prop}(w) \wedge \bigwedge \mathrm{notprop}(w))$, which is clearly equivalent to $\bigwedge \mathrm{prop}(w) \wedge (\bigvee \mathrm{notprop}(w) \to \bigwedge \mathrm{notprop}(w))$. So $\varphi^\star_w$ is equivalent to $[\varphi_{G(w)}]^*$. Next we show that $\psi^\star_w$ is equivalent to $[\psi_{G(w)}]^*$. Since $G(w)$ is of depth 2 and its only successor is the node $w^\star$ where all propositional variables are true, by Proposition 19, $[\psi_{G(w)}]^* = [\varphi_{G(w)} \to \varphi_{w^\star}]^* = [\varphi_{G(w)}]^* \to [\varphi_{w^\star}]^* = [\varphi_{G(w)}]^* \to \bigwedge_{i \in n} p_i$, which is equivalent to $\psi^\star_w = \varphi^\star_w \to \bigwedge_{i\in n} p_i$.

For $d(w) = k + 1$, since φ and ψ formulas are inductively constructed in the same manner (in Definitions 10 and 30), by the induction hypothesis and the preservation of operations mentioned in Proposition 19, the equivalence follows immediately. □

We can now show that these formulas are indeed "positive analogues" of the standard de Jongh formulas (see Proposition 11).

Proposition 33 *For every $w \in \mathcal{U}^\star(n)$, we have:*

- $V^\star(\varphi_w) = R^\star(w)$;
- $V^\star(\psi_w) = \mathcal{U}^\star(n) \setminus (R^\star)^{-1}(w)$.

Proof. By Proposition 19 we have that any formula σ is equivalent to $[\sigma]^*$ in the top models. Hence $\varphi^\star_w$ is satisfied in the same worlds of $\mathcal{M}(n)$ (which is isomorphic to $(\mathcal{U}^\star(n))^+$, see the paragraph after Corollary 26) as φ_w (and likewise for ψ_w). But since $\varphi^\star_w$ are positive formulas, by Corollary 26, they will be satisfied in the same worlds in $(\mathcal{U}^\star(n))$. □

The proposition above implies that two distinct points of $\mathcal{U}^\star(n)$ can be distinguished via a positive formula. Indeed, if $w_1 \neq w_2$ are two worlds in $\mathcal{U}^\star(n)$, then either $\neg(w_1 R w_2)$ or $\neg(w_2 R w_1)$. In the first case $\mathcal{U}^\star(n), w_2 \not\models \varphi_{w_1}$, while in the second case $\mathcal{U}^\star(n), w_1 \not\models \varphi_{w_2}$.

4 n-Henkin models

Let us denote the n-Henkin model for the positive fragment of IPC by $\mathcal{H}^\star(n)$. We write

$$\mathrm{Cn}_n^\star(\varphi) = \{\psi \in \mathcal{L}_{\wedge,\vee,\to} : \psi \text{ is an } n\text{-formula and } \vdash_{\mathsf{IPC}} \varphi \to \psi\}$$

and

$$\mathrm{Th}_n^\star(\mathfrak{M}, w) = \{\varphi \in \mathcal{L}_{\wedge,\vee,\to} : \varphi \text{ is an } n\text{-formula and } \mathfrak{M}, w \models \varphi\}.$$

The following proposition is analogous to Proposition 12.

Proposition 34 *For any point $w \in \mathcal{U}^\star(n)$ we have $\mathrm{Th}_n^\star(\mathcal{U}^\star(n), w) = \mathrm{Cn}_n^\star(\varphi_w^\star)$.*

Proof. It follows from Proposition 33 that the right hand side is a subset of the left hand side. For the other direction, assume $\mathcal{U}^\star(n), w \models \sigma$. Then if $\nvdash_{\mathsf{IPC}} \varphi_w^\star \to \sigma$, there is a finite model $\mathfrak{M}$ whose root, x, satisfies $\varphi_w^\star$ and does not satisy σ. Then, since x does not satisfy all positive formulas, it does not satisfy all propositional variables. Hence there is a positive morphism f with non-empty domain from $\mathfrak{M}$ to $\mathcal{U}^\star(n)$. Since x satisfies φ_w, by Proposition 25 we have that $f(x)$ also satisfies φ_w. By Proposition 33, this implies that $f(x) \in R^\star(w)$. Finally, since $\mathcal{U}^\star(n), w \models \sigma$ we get $\mathcal{U}^\star(n), f(x) \models \sigma$, which contradicts Proposition 25 as $\mathfrak{M}, x \nvDash \sigma$. □

The next lemma will be used in the proof that the universal model is isomorphic to the upper part of the Henkin model (Theorem 37).

Lemma 35 *Let Γ be an n-theory of the positive fragment of IPC. If $\Gamma \supseteq \mathrm{Cn}_n^\star(\varphi_w^\star)$ for some $w \in U^\star(n)$, then either there exists some $v \in R^\star(w)$ such that $\Gamma = \mathrm{Cn}_n^\star(\varphi_v^\star)$, or Γ contains all positive formulas.*

Proof. Let $\Gamma \supseteq \mathrm{Cn}_n^\star(\varphi_w^\star)$ and let v be such that wRv and $\varphi_v^\star \in \Gamma$ while for all immediate successors of v (let $v_1, \dots, v_k$ be all the immediate successors of v) we have that $\Gamma \cap \{\varphi_{v_1}^\star, \dots, \varphi_{v_k}^\star\} = \varnothing$.

If this v is unique we can see that $\Gamma = \mathrm{Cn}_n^\star(\varphi_v^\star)$. The right to left inclusion is trivial. For the converse inclusion we observe that for every $\sigma \in \Gamma$ we have $\sigma \wedge \varphi_v^\star \nvDash \varphi_{v_1}^\star \vee \cdots \vee \varphi_{v_k}^\star$ which implies by Theorem 29 that there is a point of $\mathcal{U}^\star(n)$ that satisfies $\sigma \wedge \varphi_v^\star$ but not $\varphi_{v_1}^\star \vee \cdots \vee \varphi_{v_k}^\star$. By Proposition 33, there is only one such element, v. Hence $\sigma \in \mathrm{Th}_n^\star(\mathcal{U}^\star(n), v)$, which by Proposition 34 means that $\sigma \in \mathrm{Cn}_n^\star(\varphi_v^\star)$.

To complete the proof, we will show that the aforementioned v is unique or has depth 1. If $d(v) > 1$ and there is an element u ($v \neq u$) with the aforementioned property, then Proposition 33 implies that $\neg(vR^\star w)$ and $\neg(wR^\star v)$ and hence $\psi_v^\star \in \mathrm{Th}_n^\star(\mathcal{U}^\star(n), u)$, thus $\psi_v^\star \in \Gamma$. Therefore, since Γ has the disjunction property, there is some immediate successor v_i of v, such that $\varphi_{v_i}^\star \in \Gamma$. This is a contradiction. So if $d(v) > 1$, then v is unique.

Finally, if $\varphi_v^\star, \varphi_u^\star \in \Gamma$, where $v \neq u$ and $d(v) = d(u) = 1$, then we can assume without loss of generality that there is some propositional variable q true in v but not true in u. By the definition of $\varphi_v^\star$ we have that $q \in \Gamma$. By the definition of $\varphi_u^\star$ we have that $q \to \bigwedge_{i \leq n} p_i \in \Gamma$. Hence all propositional variables are in Γ, which implies that Γ contains all positive formulas. □

The next three statements are the positive-fragment analogues of Lemmas 13 and 14 and Proposition 15, respectively. Notice that the n-Henkin model here contains a top point where every positive formula is true.

Lemma 36 *For any $w \in \mathcal{U}^\star(n)$ we have that $\mathcal{H}^\star(n)_{\mathrm{Cn}^\star(\varphi_w^\star)}$ is isomorphic to $(\mathcal{U}^\star(n)_w)^+$.*

Proof. We will show that the function $g : (\mathcal{U}^\star(n)_w)^+ \to \mathcal{H}^\star(n)_{\mathrm{Cn}^\star(\varphi_w^\star)}$, where $g(v) = \mathrm{Cn}_n^\star(\varphi_v^\star)$ and the topmost element is mapped to the set of all positive formulas, is the required isomorphism. Proposition 33 implies that g is injective and Lemma 35 implies that g is surjective. The frame relations are preserved back and forth by the following chain of equivalences:

$$\begin{array}{lll} & uR^\star v & \\ \text{iff} & R^\star(v) \subseteq R^\star(u) & \\ \text{iff} & V^\star(\varphi_v^\star) \subseteq V^\star(\varphi_u^\star) & \text{(Proposition 33)} \\ \text{iff} & \vdash_{\mathsf{IPC}} \varphi_v^\star \to \varphi_u^\star & \text{(Theorem 29)} \\ \text{iff} & \varphi_u^\star \in \mathrm{Cn}_n^\star(\varphi_v^\star) & \\ \text{iff} & \mathrm{Cn}_n^\star(\varphi_u^\star) \subseteq \mathrm{Cn}_n^\star(\varphi_v^\star) & \\ \text{iff} & g(u) \subseteq g(v). & \end{array}$$

□

The next theorem shows that in the same way n-universal models for IPC are the "upper-parts" of the n-Henkin models, the n-universal models for positive IPC are the "upper-parts" of the n-Henkin models of positive IPC.

Theorem 37 $\mathrm{Upper}(\mathcal{H}^\star(n))$ *is isomorphic to* $(\mathcal{U}^\star(n))^+$.

Proof. As above, the isomorphism will be given by the function $g : (\mathcal{U}^\star(n))^+ \to \mathrm{Upper}(\mathcal{H}^\star(n))$, such that $g(v) = \mathrm{Cn}_n^\star(\varphi_v^\star)$ and the topmost element is mapped to the set of all positive formulas. That this map is injective follows from Proposition 34 and the fact that two distinct points of $(\mathcal{U}^\star(n))^+$ are separated by a positive formula (see the paragraph after Proposition 33). That the map preserves the relation follows from the fact that intuitionistic truth is upward preserving. What is left to show is that it is onto. Let $x \in \mathrm{Upper}(\mathcal{H}^\star(n))$, and x does not contain all positive formulas. Then, by Lemma 28, there is a positive morphism, f (which is non-empty by the assumptions for x) from $\mathrm{Upper}(\mathcal{H}^\star(n))_x$ onto some $\mathcal{U}^\star(n)_w$. Then we observe by Proposition 25 that $\mathrm{Th}_n^\star(\mathcal{U}^\star(n), w) = x$, i.e., by Proposition 34, $x = \mathrm{Cn}_n^\star(\varphi_w^\star)$. Therefore, g is surjective. □

Corollary 38 *Let $\mathfrak{M} = (W, R, V)$ be any n-model and let $X \subseteq V(\varphi^\star_w)$ be a non-empty set for some $w \in U^\star(n)$. Then there is a unique positive morphism f from $\mathfrak{M}_X$ to $\mathcal{U}^\star(n)_w$. Furthermore, if $\mathfrak{M}_X$ is rooted and does not satisfy all positive formulas, then there is a unique $v \in U^\star(n)$ with $wR^\star v$ and such that f maps $\mathfrak{M}_X$ onto $\mathcal{U}^\star(n)_v$.*

Proof. Since $X \subseteq V(\varphi^\star_w)$, for each $x \in X$ and $y \in W$ with xRy we have $\mathrm{Th}^\star_n(\mathfrak{M}, y) \supseteq \mathrm{Cn}^\star_n(\varphi^\star_w)$. By Lemma 35 such a theory is equal to some $\mathrm{Cn}^\star_n(\varphi^\star_v)$ or contains all positive formulas. We define a positive morphism f as follows:

$$f(y) = \begin{cases} u, & \text{if } \exists u \text{ such that } \mathrm{Th}^\star_n(\mathfrak{M}, y) = \mathrm{Cn}^\star_n(\varphi^\star_u); \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

If the domain of f is empty then it is vacuously a positive morphism. If the domain is non-empty, by the definition of f the only non-trivial step to show that f is a positive morphism is the back condition. For this we have: if $vR^\star u$ and $f(y) = v$, then by Proposition 33, it is the case that $\mathfrak{M}, y \nvDash \psi^\star_u$. Hence there is some $z \in W$ with yRz such that $\mathfrak{M}, z \models \varphi^\star_u$ and $\mathfrak{M}, z \nvDash \bigvee_{i \le l} \varphi^\star_{u_i}$. This yields that $\mathrm{Th}^\star_n(\mathfrak{M}, z) = \mathrm{Cn}^\star_n(\varphi^\star_u)$, i.e. $f(z) = u$.

Finally, if $\mathfrak{M}_X$ is rooted and does not satisfy all positive formulas, then the root, x, is in the domain of f. Then we let $v = f(x)$. The back condition immediately yields that f is onto. □

Note that the underlying Kripke frame of $\mathcal{U}^\star(n)_w = (U^\star(n)_w, R^\star(n)_w, V^\star(n)_w)$ described in the previous lemma can be viewed as the general frame $(U^\star(n)_w,$ $R^\star(n)_w$, $Up(\mathcal{U}^\star(n)_w))$, which is a descriptive frame since W is finite.

5 Jankov's theorem for KC

In this section, we will first prove an analogue of the Jankov-de Jongh theorem (Theorem 16). This theorem will be used afterwards for an alternative proof of Jankov's theorem for KC.

Theorem 39 (Jankov-de Jongh theorem for positive fragment of IPC) *For every descriptive frame $\mathfrak{G}$ and $w \in U^\star(n)$ we have that $\mathfrak{G} \nvDash \psi^\star_w$ iff there is an n-valuation V on $\mathfrak{G}$ such that $\mathcal{U}^\star(n)_w$ is the image, through a positive morphism, of a generated submodel of $(\mathfrak{G}, V)$.*

Proof. Let $\mathcal{U}^\star(n)_w$ be the image, through a positive morphism f, of a generated submodel $\mathcal{K}$ of $(\mathfrak{G}, V)$. Proposition 33 implies that $\mathcal{U}^\star(n)_w, w \nvDash \psi^\star_w$. Since f is a positive morphism, Proposition 25 yields that $\mathcal{K}, x \nvDash \psi^\star_w$ for every $x \in f^{-1}[\{w\}]$. Now, because $\mathcal{K}$ is a generated submodel of $(\mathfrak{G}, V)$, we have that $(\mathfrak{G}, V), x \nvDash \psi^\star_w$, i.e. $\mathfrak{G} \nvDash \psi^\star_w$.

For the other direction, let us assume that there is some valuation and some x such that $(\mathfrak{G}, V), x \nvDash \psi^\star_w$. This implies that there is some y_0 such that xRy_0 and $(\mathfrak{G}, V), y_0 \models \varphi^\star_w$, while $(\mathfrak{G}, V), y_0 \nvDash \varphi^\star_{w_i}$, for all immediate successors w_i of w.

We take $(\mathfrak{G}, V)_{V(\varphi_w^\star)}$, the submodel of $(\mathfrak{G}, V)$ generated by $V(\varphi_w^\star)$. We note that by the above observation $V(\varphi_w^\star) \neq \varnothing$. Furthermore, we have that $(\mathfrak{G}, V)_{V(\varphi_w^\star)}$ does not satisfy all positive formulas since $y_0 \in V(\varphi_w^\star)$ and $(\mathfrak{G}, V), y_0 \not\models \varphi_{w_i}^\star$, for all immediate successors w_i of w.

Therefore, by Corollary 38, we have that there is a positive morphism f from $(\mathfrak{G}, V)_{V(\varphi_w^\star)}$ to $\mathcal{U}^\star(n)_w$. It is onto because $\mathrm{Th}^\star((\mathfrak{G}, V), y_0) = \mathrm{Cn}_n^\star(\varphi_w^\star)$ and hence $f(y_0) = w$.

Finally, we have that $(\mathfrak{G}, V)_{V(\varphi_w)}$ is a descriptive model, by Lemma 4, since it is based on $V(\varphi_w)$. To show that the positive morphism is also descriptive, we only need to show that $f^{-1}[R^\star(v)] \cup (\mathfrak{G} \setminus \mathrm{dom}(f)) = V(\varphi_v^\star)$, for $v \in \mathcal{U}^\star(n)_w$. For the left to right inclusion we observe that anything outside the domain of f satisfies all positive formulas and f preserves positive formulas. For the right to left assume that $x \in V(\varphi_v^\star)$. Then $x \in V(\varphi_w^\star)$ and by Lemma 35 we get that $f(x) \in R^\star(w)$ or x satisfies all propositional variables and hence it is not in the domain of f. $\square$

We recall that KC is complete with respect to the finite frames with a topmost node. Thus, by reflecting on Corollary 26, one can easily see that KC proves exactly the same positive formulas as IPC. In [18], Jankov proved that KC is maximal with that property (see also [6, Ex. 9.17] for a proof via Zakharyaschev's canonical formulas). In [11] an alternative proof based on the universal model for IPC is given.

Using the universal model for positive formulas we will provide yet another proof of this theorem, more perspicuous than the one in [11]. For this we will need the following auxiliary lemma.

Lemma 40 *Let $\mathfrak{F}$ be a descriptive frame with a topmost element, let $\mathfrak{G}$ be a descriptive frame, V and V' be admissible valuations and $f : (\mathfrak{G}, V) \to (\mathfrak{F}, V')$ a descriptive positive morphism between models. Then f can be extended to a descriptive frame p-morphism.*

Proof. First assume that the map f is total. Then it is a frame p-morphism. Now suppose f is not total. Then we extend f to f' such that for every $y \in \mathfrak{G} \setminus \mathrm{dom}(f)$ we have $f'(y) = x_0$, where x_0 is the topmost element of $\mathfrak{F}$. We claim that f' is the desired frame p-morphism. That the forth condition holds is easy to see, since everything in $\mathfrak{F}$ is below x_0. For the back condition the only possible problem may arise if some $f'(y) R x_0$. In that case, if $y \in \mathrm{dom}(f)$ then $f(y) R x_0$ and by the definition of positive morphisms a witness for the back condition exists. If $y \notin \mathrm{dom}(f)$ then the witness is y. It remains to show that the f'-pre-image of an admissible set is admissible. Let Q be an admissible set in $\mathfrak{F}$. By the construction of f we have that $f'^{-1}[Q] = f^{-1}[Q] \cup (\mathfrak{G} \setminus \mathrm{dom}(f))$, which is admissible since, by Lemma 23, it is equal to $f^*(Q)$ and f is a positive morphism between descriptive frames. $\square$

Finally, we will give our alternative proof of Jankov's theorem stating that KC is the greatest intermediate logic that proves exactly the same positive formulas as IPC.

Theorem 41 (*Jankov*) *For every logic $\mathcal{L} \nsubseteq \mathsf{KC}$ there exists some positive formula σ such that $\mathcal{L} \vdash \sigma$ while $\mathsf{IPC} \nvdash \sigma$.*

Proof. Let us assume that $\mathcal{L} \nsubseteq \mathsf{KC}$. Then $\mathcal{L} \vdash \chi$ and $\mathsf{KC} \nvdash \chi$ for some formula χ. As KC is complete with respect to finite rooted frames with a topmost element (see, e.g., [6, Proposition 2.37 and Theorem 5.33]), there is a a finite rooted frame with a topmost element, $\mathfrak{F} = (W, R)$ with $\mathfrak{F} \nvDash \chi$. We define a valuation, V, on $\mathfrak{F}$ such that each of its elements has a different color and that there is a propositional variable, q, not satisfied at the topmost element. A way to do this is to introduce a propositional variable p_x for each $x \in W$ such that $V(p_x) = R(x)$ and $V(q) = \varnothing$. By Lemma 28, there is some $w \in U(n)$ and a positive morphism from $(\mathfrak{F}, V)$ onto $\mathcal{U}^\star(n)_w$. Since each element of $(\mathfrak{F}, V)$ has a different color, the positive morphism is 1-1 and since in every element of W at least one propositional variable is not satisfied, the positive morphism has W as its domain, hence $(\mathfrak{F}, V)$ is isomorphic to $\mathcal{U}^\star(n)_w$.

We claim that the required positive formula, σ is $\psi^\star_w$. For contradiction, let us assume that $\mathcal{L} \nvdash \psi^\star_w$. Then, as every logic is complete with respect to descriptive frames (e.g., [6, Theorem 8.36]), there exists a descriptive $\mathcal{L}$-frame, $\mathfrak{G}$ such that $\mathfrak{G} \nvDash \psi^\star_w$. By Theorem 39 there is a valuation V' on $\mathfrak{G}$, a generated submodel $\mathcal{K}$ of $(\mathfrak{G}, V')$, and a descriptive positive morphism f, from $\mathcal{K}$ onto $(\mathfrak{F}, V)$. By Lemma 40, f can be extended to a descriptive frame p-morphism f'. Since $\mathfrak{G}$ is an $\mathcal{L}$-frame and $\chi \in \mathcal{L}$, we have that $\mathfrak{G} \models \chi$. As f' is a descriptive frame p-morphism, $\mathfrak{G} \models \chi$ implies that $\mathfrak{F} \models \chi$, contradicting the assumption that $\mathfrak{F} \nvDash \chi$. □

6 Conclusions and future directions

In this paper we described the universal models for the positive fragment of IPC, and using these models gave an alternative proof of Jankov's theorem which states that the logic KC of the weak law of excluded middle is the greatest logic that proves the same positive formulas as IPC. The main technical ingredients of our proofs are positive morphisms and Jankov-de Jongh formulas.

We also briefly underline some future research directions. In this paper we do not discuss algebraic aspects of universal models for positive IPC. It would be interesting to describe in all detail the algebraic counterparts of these universal models together with a full duality theory for the corresponding algebras. We refer to recent work [3] and [7] for topological dualities for similar algebraic structures. Here we only give a small hint towards algebraic analogues of the two different n-universal models $\mathcal{U}^\star(n)$ and $(\mathcal{U}^\star(n))^+$ for positive IPC discussed in Section 3.

From an algebraic point of view, the two universal models correspond to the Lindenbaum-Tarski algebras for the languages $\mathcal{L}_{\wedge,\vee,\to}$ and $\mathcal{L}_{\wedge,\vee,\to} \cup \{\bot\}$, respectively. In fact, one can show that the definable upsets of $\mathcal{U}^\star(n)$ form an algebra isomorphic to the Lindenbaum-Tarski algebra of the positive IPC. On the other hand, the definable upsets of $(\mathcal{U}^\star(n))^+$ form an algebra which is isomorphic to the Lindenbaum-Tarski algebra of the positive IPC with an additional bottom element $\bot$.

Finally, we point out a connection with minimal logic. Minimal logic can be seen as arising from positive intuitionistic logic by interpreting one propositional variable as the falsum without giving it any special properties and defining negation in the standard manner. The n-universal model for minimal logic is therefore directly available as the $n+1$-universal model of positive intuitionistic logic developed above. Recently, minimal logic with negation as a primitive and its sublogics have been studied in [8]. Colacito extended this work in [7] with proof-theoretic and algebraic results using top frames. We believe that the universal models for positive intuitionistic logic described in this paper will find fruitful application in this area as will the construction of the accompanying Jankov-de Jongh formulas.

Acknowledgement We would like to thank the referees of this paper for careful reading and many useful comments.

References

1. F. Bellissima. Finitely generated free Heyting algebras. *Journal of Symbolic Logic*, 51:152–165, 1986.
2. G. Bezhanishvili and N. Bezhanishvili. An algebraic approach to canonical formulas: Intuitionistic case. *Review of Symbolic Logic*, 2(3):517–549, 2009.
3. G. Bezhanishvili, T. Moraschini, and J. Raftery. Epimorphisms in varieties of residuated structures. 2016. Submitted.
4. N. Bezhanishvili. *Lattices of Intermediate and Cylindric Modal Logics*. PhD thesis, University of Amsterdam, 2006.
5. N. Bezhanishvili, D. Coumans, S. J. van Gool, and D. de Jongh. Duality and universal models for the meet-implication fragment of IPC. In *Proceedings of the 10th International Tbilisi Conference on Logic, Language, and Computation*, TbiLLC'13, pages 97–116. Springer-Verlag, 2013.
6. A. Chagrov and M. Zakharyaschev. *Modal Logic*. Oxford Logic Guides. Clarendon Press, 1997.
7. A. Colacito. Minimal and subminimal logic of negation. *Master's Thesis, University of Amsterdam*, 2016.
8. A. Colacito, D. de Jongh, and A. L. Vargas Sandoval. Subminimal negation. In *Proceedings of "Coherence and Truth", Conference in Memoriam Franco Montagna*. 2015. To be published in Soft Computing.
9. D. de Jongh. *Investigations on the Intuitionistic Propositional Calculus*. PhD thesis, University of Wisconsin, 1968.
10. D. de Jongh, L. Hendriks, and G. R. de Lavalette. Computations in fragments of intuitionistic propositional logic. *Journal of Automated Reasoning*, 7(4):537–561, 1991.
11. D. de Jongh and F. Yang. Jankov's theorems for intermediate logics in the setting of universal models. In *Proceedings of the 8th International Tbilisi Conference on Logic, Language, and Computation*, TbiLLC'09, pages 53–76. Springer-Verlag, 2011.
12. D. de Jongh and Z. Zhao. Positive formulas in intuitionistic and minimal logic. In *Proceedings of the 10th International Tbilisi Conference on Logic, Language, and Computation*, TbiLLC'13, pages 175–189. Springer-Verlag, 2013.

13. A. Diego. *Sur les Algèbres de Hilbert*, volume 21. Gauthier-Villars, Louvain, E. Nauwelaerts, 1966.
14. S. Ghilardi. Irreducible models and definable embeddings. In D. Gabbay, L. Csirmaz, and M. de Rijke, editors, *Logic Colloquium 92 (Proceedings)*, Studies in Logic, Language and Information, pages 95–213. CSLI Publications, 1995.
15. J. Goudsmit. *Intuitionistic Rules: Admissible Rules of Intermediate Logics*. PhD thesis, Utrecht University, 2015.
16. R. Grigolia. *Free Algebras of Non-Classical Logics*. "Metsniereba", Tbilisi, 1987. (Russian).
17. L. Hendriks. *Computations in propositional logic*. PhD thesis, University of Amsterdam, 1996.
18. V. A. Jankov. Calculus of the weak law of the excluded middle (In Russian). *Izv. Akad. Nauk SSSR*, Ser. Mat. 32(5):1044–1051, 1968.
19. I. Johansson. Der Minimalkalkül, Ein Reduzierter Intuitionistischer Formalismus. *Compositio Mathematica*, 4:119–136, 1937.
20. G. Renardel de Lavalette, A. Hendriks, and D. de Jongh. Intuitionistic Implication without Disjunction. *Journal of Logic and Computation*, 22(3):375–404, 2012.
21. V. Rybakov. Rules of inference with parameters for intuitionistic logic. *Journal of Symbolic Logic*, 57:33–52, 1992.
22. V. Shehtman. Rieger-Nishimura lattices. *Soviet Mathematics Doklady*, 19:1014–1018, 1978.
23. F. Yang. Intuitionistic subframe formulas, NNIL-formulas and n-universal models. *Master's Thesis, University of Amsterdam*, 2008.