Bounded Proofs and Step Frames

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Abstract. The longstanding research line investigating free algebra constructions in modal logic from an algebraic and coalgebraic point of view recently lead to the notion of a one-step frame [14], [8]. A one-step frame is a two-sorted structure which admits interpretations of modal formulae without nested modal operators. In this paper, we exploit the potential of one-step frames for investigating proof-theoretic aspects. This includes developing a method which detects when a specific rule-based calculus Ax axiomatizing a given logic L has the so-called bounded proof property. This property is a kind of an analytic subformula property limiting the proof search space. We define conservative one-step frames and prove that every finite conservative one-step frame for Ax is a p-morphic image of a finite Kripke frame for L iff Ax has the bounded proof property and L has the finite model property. This result, combined with a 'one-step version' of the classical correspondence theory, turns out to be quite powerful in applications. For simple logics such as **K**, **T**, **K4**, **S4**, etc, establishing basic metatheoretical properties becomes a completely automatic task (the related proof obligations can be instantaneously discharged by current first-order provers). For more complicated logics, some ingenuity is needed, however we successfully applied our uniform method to Avron's cut-free system for **GL** and to Goré's cut-free system for **S4.3**.

1 Introduction

The method of describing free algebras of modal logics by approximating them with finite partial algebras is longstanding. The key points of this method are that every free algebra is approximated by partial algebras of formulas of modal complexity n, for $n \in \omega$, and that dual spaces of these approximants can be described explicitly [1], [16]. The basic idea of this construction can be traced back to [15]. In recent years there has been a renewed interest in this method e.g., [6], [8], [9], [14], [17]. In this paper we apply the ideas originating from this line of research to investigate proof-theoretic aspects of modal logics. In particular, we will concentrate on the *bounded proof property*. An axiomatic system Ax has the bounded proof property (the bpp, for short) if every formula ϕ of modal complexity at most n derived in Ax from some set Γ containing only formulae of modal complexity at most n, can be derived from Γ in Ax by only

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using formulae of modal complexity at most n. The bounded proof property is a kind of an analytic subformula property limiting the proof search space. This property holds for proof systems enjoying the subformula property (the latter is a property that usually follows from cut elimination). The bounded proof property depends on an axiomatization of a logical system. That is, one axiomatization of a logic may have the bpp and the other not. Examples of such axiomatizations will be given in Section 5 of the paper.

The main tools of our method are the one-step frames introduced in [14] and [8]. A one-step frame is a two-sorted structure which admits interpretations of modal formulae without nested modal operators. We show that an axiomatic system Ax axiomatizing a logic L has the bpp and the finite model property (the fmp) iff every one step-frame validating Ax is a p-morphic image of a finite standard (aka Kripke) frame for L. This gives a purely semantic characterization of the bpp. The main advantage of this criterion is that it is relatively easy to verify. In the next subsection we give an example explaining the details of our machinery step-by-step. Here we just list the main ingredients. Given an axiom of a modal logic, we rewrite it into a one-step rule, that is, a rule of modal complexity 1. One-step rules can be interpreted on one-step frames. We use an analogue of the classical correspondence theory, to obtain a first-order condition (or a condition of first-order logic enriched with fixed-point operators) for a one-step frame corresponding to the one-step rule. Finally, we need to find a standard frame p-morphically mapped onto any finite one-step frame satisfying this first-order condition. This part is not automatic, but we have some standard templates. For example, we define a procedure modifying the relation of a one-step frame so that the obtained frame is standard. In easy cases, e.g., for modal logics such as **K**, **T**, **K4**, **S4**, this frame is a frame of the logic and is p-morphically mapped onto the one-step frame. The bpp and fmp for these logics follow by our criterion. For more complicated systems such as **S4.3** and **GL**, we show using our method that Avron's cut-free system for **GL** [2] and Goré's cut-free system for **S4.3** [19] provide axiomatic systems with the bpp.

A worked out example. In order to explain the basic idea of our technique, we proceed by giving a rather simple (but still significant) example. Consider the modal logic obtained by adding to the basic normal modal system **K** the 'density' axiom:

$$\Box\Box x \to \Box x. \tag{1}$$

First Step: we replace (1) by equivalent derived rules having modal complexity 1. The obvious solution is to replace the modalized subformulae occurring inside the modal operator by an extra propositional variable. Thus the first candidate is the rule $y \leftrightarrow \Box x / \Box y \to \Box x$. A better solution (suggested by the proof of Proposition 1) is to take advantage of the monotonicity and to use instead the rule

$$\frac{y \to \Box x}{\Box y \to \Box x} \tag{2}$$

Often, the method suggested by the proof of Proposition 1 gives 'good' rules, but for more complicated logics one needs some ingenuity to find the right system of derived rules replacing the axioms (this is substantially the kind of ingenuity needed to find rules leading to cut eliminating systems).

Second Step: this step may or may not succeed, but it is entirely algorithmic. It relies on a light modification of the well-known modal correspondence machinery. We first observe that inference rules having modal complexity 1 can be interpreted in the so-called *one-step frames*. A one-step frame is a quadruple $\mathcal{S} = (W_1, W_0, f, R)$, where W_0, W_1 are sets, $f : W_1 \to W_0$ is a map and $R \subseteq W_1 \times W_0$ is a relation between W_1 and W_0. In the applications, we need two further requirements (called *conservativity* requirements) on such a one-step frame $\mathcal{S}$: for the purpose of the present discussion, we may ignore the second requirement and keep only the first one, which is just the surjectivity of f. Formulae of modal complexity 1 (i.e., without nested modal operators) can be interpreted in one-step frames as follows: propositional variables are interpreted as subsets of W_0; when we apply modal operators to subsets of W_0, we produce subsets of W_1 using the modal operator $\Box_R$ canonically induced by R. In particular, for $y \subseteq W_0$ the operator $\Box_R$ is defined as $\Box_R y = \{w \in W_1 \mid R(w) \subseteq y\}$, where $R(w) = \{v \in W_0 \mid (w, v) \in R\}$. Whenever we need to compare, say y and $\Box_R x$, we apply the inverse image f (denoted by f^*) to y in order to obtain a subset of W_1. Thus, a one-step frame $\mathcal{S} = (W_1, W_0, f, R)$ validates (2) iff we have

$$\forall x, y \subseteq W_0 \ (f^*(y) \subseteq \Box_R x \Rightarrow \Box_R y \subseteq \Box_R x).$$

The standard correspondence machinery for Sahlqvist formulae shows that in the two-sorted language of one-step frames this condition has the following first-order equivalent:

$$\forall w \forall v \ (wRv \Rightarrow \exists k \ (wRf(k) \ \& \ kRv)). \tag{3}$$

In relational composition notation this becomes $R \subseteq R \circ f^o \circ R$, where f^o is the binary relation such that $wf^o v$ iff $f(w) = v$. We may call (3) the *step-density* condition. In fact, notice that for standard frames, where we have $W_1 = W_0$ and $f = id$, step-density condition becomes the customary density condition, see (6) below.

Third Step: our main result states that both the finite model property and bounded proof property (for the global consequence relation) are guaranteed provided we are able to show that *any finite conservative one-step frame validating our inference rules is a p-morphic image of a standard finite frame for our original logic*. The formal definition of a p-morphic image for one-step frames will be given in Definition 5. Here we content ourselves to observing that, in our case, in order to apply the above result and obtain the fmp and bpp, we need to prove that, given a conservative finite step-dense frame $\mathcal{S} = (W_1, W_0, f, R)$, there are a finite dense frame $\mathfrak{F} = (V, S)$ and a surjective map $\mu : V \longrightarrow W_1$ such that $R \circ \mu = f \circ \mu \circ S$. In concrete examples, the idea is to take $V := W_1$ and $\mu := id_{W_1}$. So the whole task reduces to that of finding $S \subseteq W_1 \times W_1$ such that $R = f \circ S$. That is, S should satisfy

$$\forall w \forall v \ (wRv \ \Leftrightarrow \ \exists w' \ (wSw' \ \& \ f(w') = v)). \tag{4}$$

Some ingenuity is needed in the general case to find the appropriate S (indeed our problem looks quite similar to the problem of finding appropriate filtrations case-by-case). As in the case of filtrations, there are standard templates that often work for the cases of arbitrary relations, transitive relations, etc. The basic template for the case of an arbitrary relation is that of taking S to be $f^o \circ R$, namely

$$\forall w \forall w' \ (wSw' \Leftrightarrow \exists v \ (wRv \ \& \ f(w') = v)). \tag{5}$$

Notice that what we need to prove in the end is that, assuming (3), the so-defined S satisfies (4) and

$$\forall w \forall v \ (wSv \Rightarrow \exists k \ (kSv \ \& \ wSk)). \tag{6}$$

Thus, taking into consideration that f is also surjective, i.e.,

$$\forall v \ \exists w \ f(w) = v, \tag{7}$$

(because $\mathcal{S}$ is conservative), we need the validity of the implication

$$(7) \ \& \ (3) \ \& \ (5) \ \Rightarrow \ (4) \ \& \ (6).$$

The latter is a deduction problem in first-order logic that can be solved affirmatively along the lines indicated in Section 5. The problem can be efficiently discharged by provers like SPASS, E, Vampire.[3]

In summary, the above is a purely algorithmic procedure, that may or may not succeed (in case it does not succeed, one may try to invent better solutions for the derived rules of Step 1 and/or for defining the relation S in Step 3). In case the procedure succeeds, we really obtain quite a lot of information about our logic, because we get altogether: (i) completeness via the finite model property; (ii) decidability; (iii) the bounded proof property; (iv) first-order definability; (v) canonicity (as a consequence of (i)+(iv), via known results in modal logic). Further applications concern the step-by-step descriptions of finitely generated free algebras (following the lines of [14] and [8]). But we will not deal with free algebras in this paper.

The large amount of information that one can obtain from successful runs of the method might suggest that the event of success is quite rare. This is true in essence, but we shall see in the paper that (besides simple systems such as **K**, **T**, **K4**, **S4**) the procedure can be successfully applied to more interesting case studies such as the linear system **S4.3** and the Gödel-Löb system **GL**. In the latter case we have definability not in first-order logic, but in first-order logic enriched with fixed-point operators. However, for finite one-step frames (as for finite standard **GL**-frames), this condition boils down to a first-order condition.

The paper is organized as follows. In Section 2 we recall the basic definitions of logics and decision problems. In Section 3 we introduce one-step frames and state our main results. In Section 4 we discuss the correspondence theory for one-step frames. In Section 5 we supply illustrative examples and case studies.

[3] SPASS `http://www.spass-prover.org/` (in the default configuration) took less than half a second to solve the above problem with a 47-line proof.

Section 6 provides concluding remarks and discusses future work. For space reasons, we can only limit ourselves to giving the definitions, main results and important examples. For all the other details (especially the proofs), the reader is referred to the accompanying online Technical Report [7].

2 Logics and Decision Problems

Modal formulae are built from propositional variables $x, y, \ldots$ by using the Booleans $(\neg, \wedge, \vee, 0, 1)$ and a modal operator $\Diamond$ (further connectives such as $\rightarrow, \Box$ are defined in the standard way). Underlined letters stand for tuples of unspecified length formed by distinct elements. Thus, we may use $\underline{x}$ for a tuple $x_1, \ldots, x_n$. When we write $\phi(\underline{x})$ we want to stress that ϕ contains at most the variables $\underline{x}$. The same convention applies to sets of formulae: if Γ is a set of formulae and we write $\Gamma(\underline{x})$, we mean that all formulae in Γ are of the kind $\phi(\underline{x})$. The modal complexity of a formula ϕ counts the maximum number of nested modal operators in ϕ (the precise definition is by an obvious induction). The polarity (positive/negative) of an occurrence of a subformula in a formula ϕ is defined inductively: ϕ is positive in ϕ, the polarity is preserved through all connectives, except $\neg$ that reverses it. When we say that a propositional variable is positive (negative) in ϕ we mean that all its occurrences are such.

A *logic* is a set of modal formulae containing tautologies, Aristotle's principle (namely $\Box(x \rightarrow y) \rightarrow (\Box x \rightarrow \Box y)$) and closed under uniform substitution, modus ponens and necessitation (namely $\phi/\Box\phi$).

We are interested in the *global consequence relation decision problem* for modal logics [20, Ch. 3.1]. This can be formulated as follows: given a logic L, a finite set $\Gamma = \{\phi_1, \ldots, \phi_n\}$ of formulae and a formula ψ, decide whether $\Gamma \vdash_L \psi$. Here the notation $\Gamma \vdash_L \psi$ means that there is a proof of ψ using tautologies, Aristotle's principle and the formulae in Γ, as well as necessitation, modus ponens and substitution instances of formulae from a set of axioms for L (notice that uniform substitutions cannot be applied to formulae in Γ).

In proof theory, logics are specified via axiomatic systems consisting of inference rules (axioms are viewed as 0-premises rules). Formally, an *inference rule* is an $n+1$-tuple of formulae, written in the form

$$\frac{\phi_1(\underline{x}), \ldots, \phi_n(\underline{x})}{\psi(\underline{x}).} \tag{8}$$

An *axiomatic system* Ax is a set of inference rules. We write $\vdash_{\mathrm{Ax}} \phi$ to mean that ϕ has a proof using tautologies and Aristotle's principle as well as modus ponens, necessitation and inferences from Ax. When we say that a proof uses an inference rule such as (8), we mean that the proof can introduce at any step i a formula of the kind $\psi\sigma$ provided it already introduced in the previous steps $j_1, \ldots, j_n < i$ the formulae $\phi_1\sigma, \ldots, \phi_n\sigma$, respectively. Here σ is a substitution and notation $\psi\sigma$ denotes the application of the substitution σ to ψ.

Given a finite set $\Gamma = \{\phi_1, \ldots, \phi_n\}$, an inference system Ax and a formula ψ, we write $\Gamma \vdash_{\mathrm{Ax}} \psi$ to mean that ψ has a proof using tautologies, Aristotle's principle and elements from Γ as well as modus ponens, necessitation and inferences from Ax (again notice that uniform substitution cannot be applied to members of Γ). We need some care when replacing a logic L with an inference system Ax, because we want global consequence relation to be preserved, in the sense of Proposition 1(ii) below. To this aim, we need to use derivable rules: the rule (8) is *derivable* in a logic L iff $\{\phi_1, \ldots, \phi_n\} \vdash_L \psi$. We say that the inference rule (8) is *reduced* iff (i) the formulae $\phi_1, \ldots, \phi_n, \psi$ have modal complexity at most 1; (ii) every propositional variable occuring in (8) occurs within a modal operator[4] An axiomatic system is *reduced* iff all inference rules in it are reduced.

Definition 1 *An axiomatic system* Ax *is* adequate for a logic L (*or* Ax *is an axiomatic system* for L) *iff* (*i*) *it is reduced;* (*ii*) *all rules in* Ax *are derivable in* L*;* (*iii*) $\vdash_{\mathrm{Ax}} \phi$ *for all* $\phi \in L$*.*

Proposition 1 (*i*) *For any modal logic* L*, there always exists an axiomatic system* Ax*, which is adequate for* L*.*

(*ii*) *If* Ax *is an axiomatic system for* L*, then* $\Gamma \vdash_{\mathrm{Ax}} \psi$ *iff* $\Gamma \vdash_L \psi$ *for all* Γ, ψ*.*

Proof. We just indicate how to prove (i) by sketching an algorithm replacing every rule (8) by one or more reduced rules. Applying exhaustively this algorithm to the formulae in L (or just to a set of axioms for L) viewed as zero-premises rules, we obtain the desired axiomatic system for L. Notice that the algorithm has a large degree of non determinism, so proof-theoretic properties of the outcome may be influenced by the way we run it.

Take a formula α having modal complexity at least one and take an occurrence of it located inside a modal operator in (8). We can obtain an equivalent rule by replacing this occurrence by a new propositional variable y and by adding as a further premise $\alpha \to y$ (resp. $y \to \alpha$) if the occurrence of α is positive within ψ or negative within one of the ϕ_i's (resp. if the occurrence of α is negative within ψ or positive within one of the ϕ_i's). Continuing in this way, in the end, only formulae of modal complexity at most 1 will occur in the rule. If a variable x does does not occur inside a modal operator in (8), one can add $\Box(x \vee \neg x)$ as a further premise (alternatively, one can show that x is eliminable from (8)). ⊣

Thus by Proposition 1(ii), the global consequence relation $\Gamma \vdash_{\mathrm{Ax}} \phi$ does not depend on an axiomatic system Ax chosen for a given logic L. However, deciding $\Gamma \vdash_{\mathrm{Ax}} \phi$ is easier for 'nicer' axiomatic systems. In particular, the bounded proof property below may hold only for 'nice' axiomatic systems for a logic L.

When we write $\Gamma \vdash^n_{\mathrm{Ax}} \phi$ we mean that ϕ can be proved from Ax, Γ (in the above sense) by using a proof in which only formulae of modal complexity at most n occur.

Definition 2 *We say that* Ax *has the* bounded proof property (*the bpp, for short*) *iff for every formula* ϕ *of modal complexity at most* n *and for every* Γ *containing only formulae of modal complexity at most* n*, we have*

$$\Gamma \vdash_{\mathrm{Ax}} \phi \quad \Rightarrow \quad \Gamma \vdash^n_{\mathrm{Ax}} \phi.$$

[4] Requirement (ii) is just to avoid possible misunderstanding in Definition 4.

It should be clear that the bpp for a finite axiom system Ax which is adequate for L implies the decidability of the global consequence relation problem for L. This is because we have a bounded search space for possible proofs: in fact, there are only finitely many non-provably equivalent formulae containing a given finite set of variables and having the modal complexity bounded by a given n. Notice that in a proof witnessing $\Gamma(\underline{x}) \vdash^n_{\mathrm{Ax}} \phi(\underline{x})$ we can freely suppose that only the variables $\underline{x}$ occur, because extra variables can be uniformly replaced by, say, a tautology.

3 Step frames

The aim of this section is to supply a semantic framework for investigating proofs and formulae of modal complexity at most 1. We first recall the definition of one-step frames from [14] and [8], and define conservative one-step frames.

Definition 3 *A* one-step frame *is a quadruple* $\mathcal{S} = (W_1, W_0, f, R)$*, where* W_0, W_1 *are sets,* $f : W_1 \to W_0$ *is a map and* $R \subseteq W_1 \times W_0$ *is a relation between* W_1 *and* W_0*. We say that* $\mathcal{S}$ *is* conservative *iff* f *is surjective and the following condition is satisfied for all* $w_1, w_2 \in W_1$:

$$f(w_1) = f(w_2) \ \&\ R(w_1) = R(w_2) \quad \Rightarrow \quad w_1 = w_2. \tag{9}$$

We shall use the notation $\mathcal{S}^*$ to indicate the so-called complex algebra (one-step modal algebra in the terminology of [7, 8]) formed by the 4-tuple

$$\mathcal{S}^* = (\wp(W_0), \wp(W_1), f^*, \Diamond_R),$$

where f^* is the Boolean algebra homomorphism given by inverse image along f and $\Diamond_R$ is the semilattice morphism associated with R. The latter is defined as follows: for $A \subseteq W_0$, we have $\Diamond_R(A) = \{w \in W_1 \mid R(w) \cap A \neq \emptyset\}$.

Notice that a one-step frame $\mathcal{S} = (W_1, W_0, f, R)$, where $W_0 = W_1$ and $f = id$ is just an ordinary Kripke frame. For clarity, we shall sometimes call Kripke frames *standard* frames.

We spell out what it means for a one-step frame to validate a reduced axiomatic system Ax. Notice that only formulae of modal complexity at most 1 are involved.

An $\mathcal{S}$-*valuation* $\mathtt{v}$ on a one-step frame $\mathcal{S} = (W_1, W_0, f, R)$ is a map associating with each variable x an element $\mathtt{v}(x) \in \wp(W_0)$. For every formula ϕ of complexity 0, we define $\phi^{\mathtt{v}0} \in \wp(W_0)$ inductively as follows:

$$x^{\mathtt{v}0} = \mathtt{v}(x) \ \ \text{(for every variable } x);$$
$$(\phi \wedge \psi)^{\mathtt{v}0} = \phi^{\mathtt{v}0} \cap \psi^{\mathtt{v}0}; \qquad (\phi \vee \psi)^{\mathtt{v}0} = \phi^{\mathtt{v}0} \cup \psi^{\mathtt{v}0}; \qquad (\neg\phi)^{\mathtt{v}0} = W_0 \setminus (\phi^{\mathtt{v}0}).$$

For each formula ϕ of complexity 0, we define $\phi^{\mathtt{v}1} \in \wp(W_1)$ as $f^*(\phi^{\mathtt{v}0})$. For ϕ of complexity 1, $\phi^{\mathtt{v}1} \in \wp(W_1)$ is defined inductively as follows:

$$(\Diamond\phi)^{\mathtt{v}1} = \Diamond_R(\phi^{\mathtt{v}0}); \ (\phi\wedge\psi)^{\mathtt{v}1} = \phi^{\mathtt{v}1}\cap\psi^{\mathtt{v}1}; \ (\phi\vee\psi)^{\mathtt{v}1} = \phi^{\mathtt{v}1}\cup\psi^{\mathtt{v}1}; \ (\neg\phi)^{\mathtt{v}1} = W_1\setminus(\phi^{\mathtt{v}1}).$$

Definition 4 *We say that $\mathcal{S}$* validates the inference rule (8) *iff for every $\mathcal{S}$-valuation* $\mathtt{v}$,*we have that*

$$\phi_1^{\mathtt{v}1} = W_1, \ldots, \phi_n^{\mathtt{v}1} = W_1, \ \textit{imply} \ \psi^{\mathtt{v}1} = W_1.$$

We say that $\mathcal{S}$ validates an axiomatic system Ax (*written* $\mathcal{S} \models \mathrm{Ax}$) *iff $\mathcal{S}$ validates all inferences from* Ax.

Notice that it might well be that $\mathrm{Ax}_1, \mathrm{Ax}_2$ are both adequate for the same logic L, but that only one of them is validated by a given $\mathcal{S}$ (see Section 5).

We can specialize the notion of a valuation to standard frames $\mathfrak{F} = (W, R)$ and obtain well-known definitions from the literature. In particular, given a valuation $\mathtt{v}$, for any formula ϕ (of any modal complexity) we can define $\phi^{\mathtt{v}}$ by

$$\begin{aligned} &x^{\mathtt{v}} = \mathtt{v}(x) \quad \text{(for every variable } x); \\ &(\Diamond\phi)^{\mathtt{v}} = \Diamond_R(\phi^{\mathtt{v}}); \quad (\phi \wedge \psi)^{\mathtt{v}} = \phi^{\mathtt{v}} \cap \psi^{\mathtt{v}}; \\ &(\phi \vee \psi)^{\mathtt{v}} = \phi^{\mathtt{v}} \cup \psi^{\mathtt{v}}; \quad (\neg\phi)^{\mathtt{v}} = W \setminus (\phi^{\mathtt{v}}). \end{aligned}$$

We say that $\mathfrak{F}$ *is a frame for L* [10, 20] iff $\phi^{\mathtt{v}} = W$ for all $\mathtt{v}$ and all $\phi \in L$.

We now introduce morphisms of one-step frames. In the definition below, we use $\circ$ to denote relational composition: for $R_1 \subseteq X \times Y$ and $R_2 \subseteq Y \times Z$, we have $R_2 \circ R_1 := \{(x, z) \in X \times Z \mid \exists y \in Y \ ((x, y) \in R_1 \ \& \ (y, z) \in R_2)\}$. Notice that the relational composition applies also when one or both of R_1, R_2 are functions.

Definition 5 *A* p-morphism *between step frames $\mathcal{F}' = (W_1', W_0', f', R')$ and $\mathcal{F} = (W_1, W_0, f, R)$ is a pair of surjective maps $\mu : W_1' \longrightarrow W_1, \quad \nu : W_0' \longrightarrow W_0$ such that*

$$f \circ \mu = \nu \circ f' \qquad \text{and} \qquad R \circ \mu = \nu \circ R'. \tag{10}$$

Notice that, when $\mathcal{F}'$ is standard (i.e., $W_1' = W_0'$ and $f' = id$), ν must be $f \circ \mu$ and (10) reduces to

$$R \circ \mu = f \circ \mu \circ R'. \tag{11}$$

In the next section we formulate a semantic criterion for an axiomatic system to enjoy the bounded proof property in terms of one-step frames. For this we need to recall extensions of one step-frames [8].

Definition 6 *A* one-step extension *of a one-step frame $\mathcal{S}_0 = (W_1, W_0, f_0, R_0)$ is a one-step frame $\mathcal{S}_1 = (W_2, W_1, f_1, R_1)$ satisfying $R_0 \circ f_1 = f_0 \circ R_1$. A class* K *of one-step frames has the* extension property *iff every conservative one-step frame $\mathcal{S}_0 = (W_1, W_0, f_0, R_0)$ in K has an extension $\mathcal{S}_1 = (W_2, W_1, f_1, R_1)$ such that f_1 is surjective and $\mathcal{S}_1$ is also in K.*

Theorem 1 *An axiomatic system* Ax *has the bpp iff the class of finite one-step frames validating* Ax *has the extension property.*

We point out again that the proofs of this and other results of this paper can be found in [7]. The characterization of the bpp obtained in Theorem 1 may not be easy to handle, because in concrete examples one would like to avoid managing one-step extensions and would prefer to work with standard frames instead. This is possible, if we combine the bpp with the finite model property.

Definition 7 *A logic L has the* (*global*) finite model property*, the* fmp *for short, if for every finite set of formulae Γ and for every formula ϕ we have $\Gamma \not\vdash_L \phi$ iff there exist a finite frame $\mathfrak{F} = (W, R)$ for L and a valuation v on $\mathfrak{F}$ such that $(\bigwedge \Gamma)^{\mathrm{v}} = W$ and $\phi^{\mathrm{v}} \neq W$.*

We are ready to state our main result.

Theorem 2 *Let L be a logic and* Ax *an axiomatic system adequate for L. The following two conditions are equivalent:*

(i) Ax *has the bpp and L has the fmp;*
(ii) *Every finite conservative one-step frame validating* Ax *is a p-morphic image of some finite frame for L.*

4 One-Step Correspondence

In this section we develop the correspondence theory for one-step frames based on the classical correspondence theory for standard frames.

We will start by reformulating Definition 4. Notice that a one-step frame $\mathcal{S} = (W_1, W_0, f, R)$ is a two-sorted structure for the language $\mathcal{L}_f$ having a unary function and a binary relation symbol. The complex algebra $\mathcal{S}^* = (\wp(W_0), \wp(W_1), f^*, \Diamond_R)$ is also a two-sorted structure for the first-order language $\mathcal{L}_a$ having two sorts, Boolean operations for each of them, and two-sorted unary function symbols that we call i and $\Diamond$ (they are interpreted in $\mathcal{S}^*$ as f^* and $\Diamond_R$, respectively). As a first step, we reformulate the validity of inference in terms of truth of a formula in the language $\mathcal{L}_a$. We need to turn modal formulae ϕ of complexity at most 1 into $\mathcal{L}_a$-terms. This is easily done as follows: just replace every occurrence of a variable x *which is not located inside a modal connective* in ϕ by $i(x)$. Let us call $\tilde{\phi}$ the result of such replacement. The following fact is then clear.

Proposition 2 *A step frame $\mathcal{S}$ validates a reduced inference rule of the kind* (8) *iff considering $\mathcal{S}^*$ as a two-sorted $\mathcal{L}_a$-structure, we have*

$$\mathcal{S}^* \models \forall \underline{x}\ (\tilde{\phi}_1 = 1\ \& \cdots \&\ \tilde{\phi}_n = 1 \to \tilde{\psi} = 1). \tag{12}$$

If we rewrite (12) in terms of the $\mathcal{L}_f$-structure $\mathcal{S}$, we realize that this is a truth relation regarding a *second order* formula, because the quantifiers $\forall \underline{x}$ range over tuples of subsets. The idea (borrowed from correspondence theory) is to perform symbolic manipulations on (12) and to convert it into a first-order $\mathcal{L}_f$-condition. This procedure works for many concrete examples, although there are cases where it fails. We follow a long line of research, e.g., [3–5, 12, 13, 18, 22] (see also [10, 20]). Similarly to these papers, our basic method is to perform symbolic manipulations on the algebraic language $\mathcal{L}_a$.

We start by enriching $\mathcal{L}_a$. The enrichment comes from the following observations. Let $\mathcal{F} = (W_0, W_1, f, R)$ be a one-step frame. First of all, the morphism $i := f^* : \wp(W_0) \to \wp(W_1)$ has a left i^* and a right adjoint $i_!$. In fact i^* is the direct image along f and $i_!$ is $\neg i^* \neg$. The operator $\Diamond : \wp(W_0) \to \wp(W_1)$ (we skip the index R) also has a right adjoint, which is the Box operator $\blacksquare$ induced by the converse relation R^o of R. We shall make use also of the related Diamond $\blacklozenge$ defined as $\neg\blacksquare\neg$. Thus we enrich $\mathcal{L}_a$ with extra unary function symbols $i^*, i_!, \blacksquare, \blacklozenge$ of appropriate sorts. In addition, we shall make use of the letters w_i^0, w_i^1 to denote *nominals*, namely quantified variables ranging over atoms (i.e., singleton subsets) of $\wp(W_0), \wp(W_1)$, respectively. For simplicity and for readability, we shall avoid the superscript $(-)^1, (-)^0$ indicating the sort of nominals. However, we shall adopt the convention of using preferably the variables $w, w_0, w', \dots$ for nominals of sort 1, the variables $v, v_0, v', \dots$ for nominals of sort 0 and the letters $u, u_0, u', \dots$ for nominals of unspecified sort (i.e., for nominals that might be of both sorts, which are useful in preventing, e.g., rule duplications). We call $\mathcal{L}_a^+$ the enriched language.

The idea is the following. We want to analyze validity of the inference rule (8) in a one-step frame $\mathcal{F}$. We initialize our procedure to:

$$\forall \underline{x}\ (1 \leq \tilde{\phi}_1\ \&\cdots\&\ 1 \leq \tilde{\phi}_n \to 1 \leq \tilde{\psi}). \tag{13}$$

Here and below, we use abbreviations such as $\alpha \leq \beta$ to mean $\alpha \to \beta = 1$. Usually, we omit external quantifiers $\forall \underline{x}$ and use sequent notation, so that (13) is written as

$$1 \leq \tilde{\phi}_1,\ \dots,\ 1 \leq \tilde{\phi}_n \Rightarrow 1 \leq \tilde{\psi}. \tag{14}$$

We then try to find a sequence of applications of the rules below ending with a formula where *only quantifiers for nominals* occur (that is, the variables $\underline{x}$ have been eliminated). If we succeed, the standard translation can easily and automatically *convert the final formula into a first-order formula in the language* $\mathcal{L}_f$. It is possible to characterize syntactic classes (e.g., Sahlqvist-like classes and beyond) where the procedure succeeds, but for the purposes of this paper we are not interested in the details of such characterizations. They can be obtained in a straightforward way by extending the well-known characterizations, see e.g., [3, 13, 18]). The rules we use are divided into three groups:

(a) Any set of invertible rules in classical first-order sequent calculus. We refer the reader to proof-theory textbooks such as [21] for more details on this;
(b) Rules for managing nominal quantifiers (see Table 1);
(c) Adjunction rules (see Table 2);
(d) Ackermann rules (see Table 3).

Rules (a)-(b)-(c) are local, in the sense that they can be applied simply by replacing the formula above the line by the formula below the line (or vice versa).

Rules (d) to the contrary require checking global monotonicity conditions at the whole sequent level. Ackermann rules eliminate the quantified variables $\underline{x}$ one by one in successful runs.

When we start from a logic L, we first need to convert the axioms into reduced inference rules. The method indicated in the proof of Proposition 1 has the big advantage of *introducing new quantified variables that can be easily eliminated via the adjunction and the Ackermann rules*, as is shown in the example below.

$\dfrac{\tilde{\phi} \leq \tilde{\psi}}{\forall u\ (u \leq \tilde{\phi}\ \rightarrow\ u \leq \tilde{\psi})}$	$\dfrac{u \leq \tilde{\psi}_1 \wedge \tilde{\psi}_2}{u \leq \tilde{\psi}_1\ \&\ u \leq \tilde{\psi}_2}$	$\dfrac{u \leq \tilde{\psi}_1 \vee \tilde{\psi}_2}{u \leq \tilde{\psi}_1\ \text{or}\ u \leq \tilde{\psi}_2}$
$\dfrac{u \leq \neg\tilde{\psi}}{u \nleq \tilde{\psi}}\quad \dfrac{u \nleq \tilde{\psi}}{\tilde{\psi} \leq \neg u}$	$\dfrac{w \leq \Diamond\tilde{\psi}}{\exists v\ (w \leq \Diamond v\ \&\ v \leq \tilde{\psi})}$	$\dfrac{v \leq \blacklozenge\tilde{\psi}}{\exists w\ (v \leq \blacklozenge w\ \&\ w \leq \tilde{\psi})}$
$\dfrac{u \leq 1}{\top}$	$\dfrac{u \leq 0}{\bot}$	$\dfrac{v \leq i^*(\tilde{\psi})}{\exists w\ (v \leq i^*(w)\ \&\ w \leq \tilde{\psi})}$

Table 1. Nominals Rules

$\dfrac{\tilde{\phi} \leq \Box\tilde{\psi}}{\blacklozenge\tilde{\phi} \leq \tilde{\psi}}$	$\dfrac{\tilde{\phi} \leq \blacksquare\tilde{\psi}}{\Diamond\tilde{\phi} \leq \tilde{\psi}}$
$\dfrac{\tilde{\phi} \leq i(\tilde{\psi})}{i^*(\tilde{\phi}) \leq \tilde{\psi}}$	$\dfrac{\tilde{\phi} \leq i_!(\tilde{\psi})}{i(\tilde{\phi}) \leq \tilde{\psi}}$

Table 2. Adjunction Rules

$\dfrac{\Gamma, x \leq \tilde{\phi} \Rightarrow \Delta}{\Gamma(\tilde{\phi}/x) \Rightarrow \Delta(\tilde{\phi}/x)}$	(x is not in ϕ, is positive in all Γ, negative in all Δ)
$\dfrac{\Gamma, \tilde{\phi} \leq x \Rightarrow \Delta}{\Gamma(\tilde{\phi}/x) \Rightarrow \Delta(\tilde{\phi}/x)}$	(x is not in ϕ, is negative in all Γ, positive in all Δ)

Table 3. Ackermann Rules

Example 1 Let us consider the system **K4** that is axiomatized by the axiom $\Box x \to \Box\Box x$. Since this axiom does not have modal complexity 1, we turn it into the inference rule

$$\frac{\Box x \leq y}{\Box x \leq \Box y} \tag{15}$$

following the algorithm in the proof of Proposition 1. We then initialize our procedure to $\Box x \leq i(y) \Rightarrow \Box x \leq \Box y$. By adjunction rules, we obtain

$$i^*(\Box x) \leq y \Rightarrow \Box x \leq \Box y.$$

We can immediately eliminate y via the Ackermann rules and get $\Box x \leq \Box i^*(\Box x)$. We now use nominals rules together with rules (a) (i.e., invertible rules in classical sequent calculus) and get $w \leq \Box x \Rightarrow w \leq \Box i^*(\Box x)$ (notice that the nominal variable w is implicitly universally quantified here). By adjointness we obtain a sequent $\blacklozenge w \leq x \Rightarrow w \leq \Box i^*(\Box x)$ to which the Ackermann rules apply yielding:

$$w \leq \Box i^*(\Box \blacklozenge w).$$

This is a condition involving only (one) quantified variable for nominals. Thus, in the language $\mathcal{L}_f$ for one-step frames it is first-order definable (to do the unfolding, it is sufficient to notice that the nominal w stands in fact for the set $\{w' \in W_1 \mid w' = w\}$). After appropriate simplifications, we obtain

$$\forall w\, \forall v\ (R(w,v) \to \exists w_1\ (f(w_1) = v\ \&\ R(w_1) \subseteq R(w))). \tag{16}$$

5 Examples and Case Studies

In this section we show how to apply Theorem 2 first to basic, and later to more elaborate examples. The methodology is the following. We have three steps, as pointed out in Section 1:

– starting from a logic L, we produce an equivalent axiomatic system Ax_L with reduced rules (there is a default procedure for that, see the proof of Proposition 1);
– we apply the correspondence machinery of Section 4 and try to obtain a first-order formula α_L in the two-sorted language $\mathcal{L}_f$ of one-step frames characterizing the one-step frames validating Ax_L;
– we apply Theorem 2 and try to prove that conservative finite one-step frames satisfying α_L are p-mophic images of standard frames for L.

If we succeed, we obtain both the fmp and bpp for L. In examples, given a finite conservative one-step frame $\mathcal{F} = (X, Y, f, R)$ satisfying α_L, the finite frame required by Theorem 2 is often based on X and the p-morphism is the identity. Thus one must simply define a relation S on X in such a way that (11) holds (with $R' = S$). Condition (11), taking into consideration that μ is the identity, reduces to (4). There are standard templates for S. We give some examples below where the procedure succeeds, supplying also the relevant hints for the definition of the right S.

- $\boxed{L = \mathbf{K}}$: this is the basic normal modal logic. To obtain the appropriate S, we take $S := f^o \circ R$, i.e., we put wSw' iff $f(w') \in R(w)$.
- $\boxed{L = \mathbf{T}}$: this is the logic axiomatized by $\Box x \to x$. The one-step correspondence gives $f \subseteq R$ as the semantic condition equivalent to being a one-step frame for L. To obtain the appropriate S we again take $S := f^o \circ R$.
- $\boxed{L = \mathbf{K4}}$: this is the logic axiomatized by $\Box x \to \Box\Box x$. As we know, this axiom can be turned into the equivalent rule (15) and the one-step correspondence gives (16) as the semantic condition equivalent to being a one-step frame for **K4**. We take S to be $(f^o \circ R) \cap \geq_R$ (where $w \geq_R w'$ is defined as $R(w) \supseteq R(w')$); this is the same as saying that wSw' holds iff $R(w) \supseteq \{f(w')\} \cup R(w')$.
- $\boxed{L = \mathbf{S4}}$ Here one can combine the previous two cases. However, the definition of S as $(f^o \circ R) \cap \geq_R$ simplifies to $\geq_R$ by reflexivity.

The details required to justify the claims are straightforward but sometimes a bit involved (they considerably simplify by using a relational formalism), see [7, Sec. 8] for the details. All the claims we need are easy for current provers (the SPASS prover for instance solves each of the above problems in less than half a second on a common laptop).

Remark 1 Notice that the definition of a conservative finite one-step frame (Definition 3) has two conditions. However, only the first one (namely surjectivity of f) is used in the computations above. In fact, it is not clear whether Theorem 2 holds if we drop the second condition (9) in the definition of a one-step conservative frame.

A case study: S4.3. As a more elaborated example we take **S4.3**, which is **S4** plus the axiom

$$\Box(\Box x \to y) \vee \Box(\Box y \to x).$$

Applying the algorithm from the proof of Proposition 1, we obtain a rule which is 'bad' (the bpp fails for the related axiomatic system, see [7, Sec. 9] for details).

Instead of a rule obtained by the procedure of Proposition 1, we axiomatize **S4.3** by using the reflexivity axiom for **T** and the following infinitely many rules proposed by R. Goré [19]:

$$\frac{\cdots \Box y \to x_j \vee \bigvee_{j \neq i} \Box x_i \cdots}{\Box y \to \bigvee_{i=1}^{n} \Box x_i} \tag{17}$$

The rules are indexed by n and the n-th rule has n premises, according to the values $j = 1, \ldots, n$. If we collectively do the correspondence theory on these rules, we obtain the following condition on *finite* one-step frames:

$$\forall w\, \forall S \subseteq R(w)\, \exists v \in S\, \exists w'\, (f(w') = v \;\&\; S \subseteq R(w') \subseteq R(w)). \tag{18}$$

This condition is sufficient to prove that a finite one-step frame (X, Y, f, R) satisfying (18) can be extended to a finite frame (X', R') which is a frame for **S4.3**. In the proof, we do not take X' to be X, but we define X' via a specific construction (see [7, Thm. 3]). Summing everything up, we obtain:

Theorem 3 **S4.3** *axiomatized by Goré's rules* (17) *has the bpp and fmp.*

A case study: GL. The Gödel-Löb modal logic **GL** can be axiomatized by the axiom $\Box(\Box x \to x) \to \Box x$. This system is known to have the fmp and to be complete with respect to the class of finite irreflexive transitive frames. From the proof-theoretic side, the following rule

$$\frac{x \wedge \Box x \wedge \Box y \to y}{\Box x \to \Box y} \tag{19}$$

has been proposed by Avron in [2], and shown to lead to a cut-eliminating system. We now analyze the axiomatic system for **GL** consisting of the only rule (19). If we analyze the validity of rule (19) in a *finite* one-step frame, we obtain the following condition

$$\forall w \ (R(w) \subseteq \{f(w') \mid R(w') \subset R(w)\}). \tag{20}$$

Notice that condition (20) implies the one-step transitivity condition (16). Using Theorem 2 and a specific construction of the p-morphic extension (see [7, Thm. 4] for details), we obtain:

Theorem 4 **GL** *axiomatized by Avron's rule* (19) *has the bpp and fmp.*

We conclude by mentioning that it is possible to apply our results for showing that the axiomatic system for **GL** obtained by adding to the transitivity rule (15) the well-known Löb rule $\Box x \to x / x$ is indeed unsatisfactory from a proof-theoretic point of view, because the bpp fails for it [7, Ex. 3].

6 Conclusions and Future Work

We have developed a uniform semantic method for analyzing proof systems of modal logics. The method relies on p-morphic extensions of finite one-step frames. In simple cases, by a one-step version of the classical correspondence theory, the application of our methodology is completely algorithmic. This is a concrete step towards *mechanizing the metatheory* of propositional modal logic.

We also analyzed our approach in two nontrivial cases, namely for the cut-free axiomatizations of **S4.3** and **GL** known from the literature. We succeeded in both cases in proving the fmp and bpp by our methods. The proofs are not entirely mechanical, but from the details given in [7] it emerges that they are still based on a common feature: an induction on the cardinality of accessible worlds in finite one-step frames.

For future, it will be important to see whether this method can fruitfully apply to complicated logics arising in computer science applications (such as dynamic logic, linear or branching time temporal logics, the modal μ-calculus, etc.). Another important series of questions concerns the clarification of the relationship between our techniques and standard techniques employed in filtrations and analytic tableaux. Finally, a comparison with the algebraic approach (in a non-distributive context) to cut elimination via MacNeille completions for the full Lambek calculus **FL** developed in [11] might bring further fruitful consequences.

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