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Pseudomonadic Algebras as Algebraic Models of Doxastic Modal Logic

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Abstract. We generalize the notion of a monadic algebra to that of a pseudomonadic algebra. In the same way as monadic algebras serve as algebraic models of epistemic modal system S5, pseudomonadic algebras serve as algebraic models of doxastic modal system KD45. The main results of the paper are: (1) Characterization of subdirectly irreducible and simple pseudomonadic algebras, as well as Tokarz's proper filter algebras; (2) Order-topological representation of pseudomonadic algebras; (3) Complete description of the lattice of subvarieties of the variety of pseudomonadic algebras.

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1 Introduction

Investigation of belief modalities leads to doxastic modal systems in which the principle $\Box p \to p$ is false. Indeed, from the fact that someone believes that p is true does not follow that p is really true. Moreover, if we suppose that an agent's belief is conscious, then we have to accept principles $\Box p \to \Box\Box p$ and $\neg\Box p \to \Box\neg\Box p$, because the agent whose belief is conscious must always believe that what he believes is true and what he does not believe is false. Formalization of the conscious belief of an ideal agent results in doxastic modal system KD45. This system was investigated by many authors. One of the earliest references is Hintikka [8]; Segerberg [16] studied it as the modal system DE4; Nagle [13] and Nagle and Thomason [14] investigated it as a normal extension of the modal system K5; Halpern and Moses [6, 7] discussed the completeness and complexity issues for KD45 and its poly-modal analogues as well as for the systems with mixed S5- and KD45-modalities (see also van der Hoek [9], Meyer and van der Hoek [12], and Meyer [11]).

It was shown in Segerberg [16] that KD45 is locally tabular, hence hereditarily finitely approximable, that every extension of KD45 is in fact its normal extension, that the lattice of extensions of KD45 is countable, and that all extensions of KD45 are finitely axiomatizable. An adequate tableau calculus for KD45 was constructed in Gore [4]. The first algebraic semantics for KD45 – the semantics of proper filter algebras – was described in Tokarz [18].

[1)]I would like to thank Leo Esakia for introducing me to the subject, and my brother Guram for guiding me through the proofs.

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To this we add a systematic algebraic study of KD45. We introduce pseudomonadic algebras as natural generalizations of HALMOS' monadic algebras, and show that they serve as algebraic models of KD45. We characterize TOKARZ's proper filter algebras by showing that they are exactly the well-connected pseudomonadic algebras, and give much simpler proof of the main completeness theorem of TOKARZ. We also characterize simple, subdirectly irreducible and well-connected pseudomonadic algebras and compare them with simple and subdirectly irreducible monadic algebras. In addition, we give a very simple algebraic proof that KD45 is locally tabular – a fact first observed by SEGERBERG [16] using the Kripke semantics. Moreover, we prove a representation theorem for pseudomonadic algebras, and give the order-topological characterization of simple, subdirectly irreducible and well-connected pseudomonadic algebras. The description of the lattice of extensions of KD45 can be found in SEGERBERG [16]. We give an alternative algebraic proof of this result. Finally, we construct a rather simple countable subdirectly irreducible pseudomonadic algebra, and show that KD45 is complete with respect to it.

2 Pseudomonadic algebras

We start by recalling the axiomatization of KD45. Axioms of KD45 are

1. all axiom schemes of classical propositional calculus;
2. $\Box(p \to q) \to (\Box p \to \Box q)$;
3. $\Box p \to \Box\Box p$;
4. $\neg\Box p \to \Box\neg\Box p$;
5. $\Box\neg p \to \neg\Box p$.

The rules of inference of KD45 are modus ponens $(\varphi, \varphi \to \psi/\psi)$ and substitution.

Remark 2.1. Note that we did not list the rule of necessitation $(\varphi/\Box\varphi)$ as one of the rules of inference of KD45 since it follows from SEGERBERG [16] that it is derivable in KD45. Also note that if we use $\Diamond$ as the standard abbreviation of $\neg\Box\neg$, then Axioms (4) and (5) would be rewritten as $\Diamond\Box p \to \Box p$ and $\Box p \to \Diamond p$, respectively.

Now we introduce the main concept of the paper. Let BA denote the variety of Boolean algebras.

Definition 2.1. An algebra $(B, \exists)$ is said to be a *pseudomonadic algebra* if $B \in$ BA, and $\exists$ is a unary operator on B satisfying the following identities for every a, $b \in B$:

1. $\exists 0 = 0$;
2. $\exists(a \vee b) = \exists a \vee \exists b$;
3. $\exists(\exists a \wedge b) = \exists a \wedge \exists b$;
4. $\neg\exists a \leq \exists\neg a$.

It is obvious that the class of all pseudomonadic algebras forms a variety, which we denote by PMA. Note that PMA is a proper extension of HALMOS' variety MA of *monadic algebras* (see [5]). In fact, MA = PMA + $(a \leq \exists a)$ and PMA inherits some properties of MA, which is the reason why we call algebras from PMA pseudomonadic algebras and denote their unary operator by $\exists$ as in monadic case. For the same reason we use $\forall$ as abbreviation of the operator $-\exists-$.

Lemma 2.1. *The following identities hold in every pseudomonadic algebra:*

(1) $\forall a \leq \exists a$;	(6) $\forall\forall a = \forall a$;	(11) $\forall(\forall a \vee b) = \forall a \vee \forall b$;
(2) $\forall 1 = 1$;	(7) $\forall\exists a = \exists a$;	(12) $\forall(a \wedge b) = \forall a \wedge \forall b$;
(3) $\exists 1 = 1$;	(8) $\exists\forall a = \forall a$;	(13) $\forall(-a \vee b) \leq -\forall a \vee \forall b$;
(4) $\forall 0 = 0$;	(9) $\exists - \exists a = -\exists a$;	(14) $\exists(-\exists a \vee a) = 1$;
(5) $\exists\exists a = \exists a$;	(10) $\forall - \forall a = -\forall a$;	(15) $\forall(-\forall a \vee a) = 1$.

Proof.

(1) is equivalent to identity 4 of Definition 2.1.

(2) is equivalent to identity (1) of Definition 2.1.

(3) Since $1 = \forall 1 \leq \exists 1$, we have $\exists 1 = 1$.

(4) is equivalent to (3).

(5) $\exists\exists a = \exists(\exists a \wedge 1) = \exists a \wedge \exists 1 = \exists\, a \wedge 1 = \exists a$.

(6) is equivalent to (5).

(7) Since $\exists a \wedge -\exists a = 0$, we have $\exists(\exists a \wedge -\exists a) = 0$. Thus, $\exists a \wedge \exists - \exists a = 0$, implying $\exists a \leq -\exists - \exists a = \forall\exists a$. Conversely, $\forall a \leq \exists a$ implies $\forall\exists a \leq \exists\exists a$. Therefore, $\forall\exists a \leq \exists a$.

(8), (9), (10), (11) are equivalent to each other and to (7).

(12) is equivalent to identity (2) of Definition 2.1.

(13) is an easy consequence of (12).

(14) $\exists(-\exists a \vee a) = \exists - \exists a \vee \exists a = -\exists a \vee \exists a = 1$.

(15) $\forall(-\forall a \vee a) = \forall(\forall - \forall a \vee a) = \forall - \forall a \vee \forall a = -\forall a \vee \forall a = 1$. $\square$

We also can prove that identity (3) of Definition 2.1 is equivalent to any of the formulas (7) – (11), and that identity (4) is equivalent to any of the formulas (1), (3), (4). Therefore, an equivalent axiomatization of PMA can be obtained by replacing identity (3) by any of the formulas (7) – (11), and identity (4) by any of the formulas (1), (3), (4).

Let us show that pseudomonadic algebras are algebraic models of KD45. By interpreting propositional variables as elements of a pseudomonadic algebra $(B, \exists)$, and connectives $\wedge$, $\vee$, $\to$, $\neg$ and modal operators $\Box$ and $\Diamond$ as the corresponding operators of $(B, \exists)$, we can think of a formula of a modal (propositional) language as a polynomial in $(B, \exists)$. Now for a given valuation V, a formula φ is *true in* $(B, \exists)$ if the corresponding polynomial $V(\varphi)$ is equal to 1. The formula φ is *valid in* $(B, \exists)$ if φ is true in $(B, \exists)$ for every valuation V. Let us associate with a doxastic logic L over KD45 the class $\mathcal{V}_L$ of pseudomonadic algebras in which all the theorems of L are valid. It is easy to check that $\mathcal{V}_L$ always creates a variety[3)] (in particular, $\mathcal{V}_{\mathsf{KD45}} = \mathsf{PMA}$). Conversely, by replacing "=" in an equation by "$\leftrightarrow$" we can associate with every equation a formula in a modal (propositional) language. In this way, with every variety $\mathcal{V} \subseteq \mathsf{PMA}$ is associated a doxastic logic over KD45, denoted here by $L_{\mathcal{V}}$. Now using a Lindenbaum-Tarski type construction we can easily show that

[3)] The crucial point here is the fact that every extension of KD45 is normal, that is the rule of necessitation is derived in it, which follows from Segerberg [16].

the described correspondence is one-to-one. Moreover, as for any L_1, $L_2 \supseteq \mathsf{KD45}$, $L_1 \subseteq L_2$ iff $\mathcal{V}_{L_2} \subseteq \mathcal{V}_{L_1}$, we get the following statement:

Theorem 2.1. *The lattice* Ext$\mathsf{KD45}$ *of all doxastic logics over* $\mathsf{KD45}$ *is dual to the lattice* $\Lambda(\mathsf{PMA})$ *of all varieties of pseudomonadic algebras.*

Hence, all theorems about pseudomonadic algebras can be thought of as theorems about doxastic logics over $\mathsf{KD45}$ and vice versa. We devote the rest of this section to thorough investigation of PMA.

Suppose a pseudomonadic algebra $(B, \exists)$ is given. We call a filter F of B a $\forall$-*filter* if $a \in F$ implies $\forall a \in F$ for each $a \in B$. Since PMA is a (proper) subvariety of the variety of modal algebras, it follows from Blok [2] that there exists a one-to-one correspondence between congruence relations and $\forall$-filters of $(B, \exists)$. As a consequence, PMA is *congruence-distributive* and has the *congruence extension property.*

Recall that an algebra $\mathcal{A}$ is said to be *subdirectly irreducible* if there exists a least nontrivial congruence relation of $\mathcal{A}$. An algebra $\mathcal{A}$ is said to be *simple* if the trivial congruence relation is the only proper congruence relation of $\mathcal{A}$. It is easy to see that an algebra $(B, \exists) \in \mathsf{PMA}$ is subdirectly irreducible iff there exists a least nonunit $\forall$-filter in $(B, \exists)$, and that $(B, \exists)$ is simple iff the unit filter is the only proper $\forall$-filter in $(B, \exists)$. Let us denote the classes of simple and subdirectly irreducible pseudomonadic algebras by $\mathsf{PMA}_{\mathrm{S}}$ and $\mathsf{PMA}_{\mathrm{SI}}$, respectively. Also let MA_{S} and $\mathsf{MA}_{\mathrm{SI}}$ denote the classes of simple and subdirectly irreducible monadic algebras, respectively. From Halmos [5] it is known that $\mathsf{MA}_{\mathrm{SI}} = \mathsf{MA}_{\mathrm{S}}$, which means that MA is a *semi-simple* variety.

Theorem 2.2. $\mathsf{PMA}_{\mathrm{S}} = \mathsf{MA}_{\mathrm{S}}$.

Proof. It is obvious that $\mathsf{MA}_{\mathrm{S}} \subseteq \mathsf{PMA}_{\mathrm{S}}$. For the converse, let $(B, \exists) \in \mathsf{PMA}_{\mathrm{S}}$ and $a \in B$. Since $(B, \exists)$ is simple, there are no proper $\forall$-filters in B. On the other hand, the filter $[\forall a) = \{b \in B : \forall a \leq b\}$ generated by the element $\forall a$ is always an $\forall$-filter. Hence, $a \neq 1$ implies $\forall a = 0$. Thus, for each $a \in B$ we have

$$\forall a = \begin{cases} 1, & \text{if } a = 1, \\ 0, & \text{otherwise,} \end{cases}$$

implying that $\forall a \leq a$ for each $a \in B$. Equivalently, $a \leq \exists a$ for each $a \in B$. Therefore, $(B, \exists) \in \mathsf{MA}_{\mathrm{S}}$. □

However, in contrast to MA, there are subdirectly irreducible algebras in PMA which are not simple. An example would be the algebra shown in Figure 1, where $\forall a = \forall 1 = 1$ and $\forall - a = \forall 0 = 0$ (note that $\{a, 1\}$ is a proper nonunit $\forall$-filter). So, in contrast to MA, we have that PMA is not semi-simple.

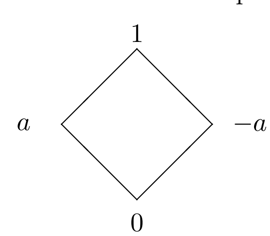


Figure 1

For $(B, \exists) \in \mathsf{PMA}$, let $B_0 = \{\exists a : a \in B\}$. It is routine to prove that

$$B_0 = \{a \in B : \exists a = a\} = \{a \in B : \forall a = a\} = \{\forall a : a \in B\},$$

and that B_0 is a Boolean subalgebra of B.

Definition 2.2. A pseudomonadic algebra $(B, \exists)$ is said to be *well-connected* if from $\exists a \wedge \exists b = 0$ it follows that $\exists a = 0$ or $\exists b = 0$.

Equivalently, $(B, \exists)$ is well-connected iff from $\forall a \vee \forall b = 1$ it follows that $\forall a = 1$ or $\forall b = 1$.

Theorem 2.3. *A pseudomonadic algebra* $(B, \exists)$ *is well-connected iff* $B_0 = \{0, 1\}$.

Proof. It is obvious that if $B_0 = \{0, 1\}$, then $(B, \exists)$ is a well-connected algebra. Conversely, suppose $(B, \exists)$ is well-connected and $a \in B$. Then from $\exists a \wedge -\exists a = 0$ and $-\exists a = \exists - \exists a$ it follows that either $\exists a = 0$ or $\exists a = 1$. Now since every element of B_0 has the form $\exists a$ for some $a \in B$, we obtain that $B_0 = \{0, 1\}$. □

Let us denote the class of well-connected pseudomonadic algebras by $\mathsf{PMA}_{\mathrm{WC}}$. From Theorem 2.3 it follows that $\mathsf{PMA}_{\mathrm{WC}} = \{(B, \exists) \in \mathsf{PMA} : B_0 = \{0, 1\}\}$. Now we are in a position to show that $\mathsf{PMA}_{\mathrm{WC}}$ coincides (up to isomorphism) with the class of proper filter algebras introduced in Tokarz [18]. Recall that a *proper filter algebra* is a pair (B, F) where B is a Boolean algebra and F is a proper filter of B. For our convenience, we denote pseudomonadic algebras by $(B, \forall)$. With $(B, \forall) \in \mathsf{PMA}_{\mathrm{WC}}$ we associate a proper filter algebra $(B, F_\forall)$ by putting $F_\forall = \{a \in B : \forall a = 1\}$; conversely, with a proper filter algebra (B, F) we associate $(B, \forall_F) \in \mathsf{PMA}_{\mathrm{WC}}$ by putting $\forall_F a = 1$ if $a \in F$ and $\forall_F a = 0$ if $a \notin F$. It is rather easy to check that this correspondence is one-to-one. Hence, we arrive at the following characterization of proper filter algebras in terms of pseudomonadic algebras:

Theorem 2.4. *The class of proper filter algebras coincides* (*up to isomorphism*) *with the class of well-connected pseudomonadic algebras.*

Suppose $(B, \forall) \in \mathsf{PMA}_{\mathrm{SI}}$ and $a \neq 1$ is an element of B. If $\forall a \neq 0$, then $[\forall a)$ and $[\neg\forall a)$ would be two nonunit proper $\forall$-filters with the intersection being the unit filter, which contradicts that $(B, \forall)$ is subdirectly irreducible. Therefore, $a \neq 1$ implies $\forall a = 0$, and so $\mathsf{PMA}_{\mathrm{SI}} \subseteq \mathsf{PMA}_{\mathrm{WC}}$. The algebra $(B^*, \forall^*)$ shown in Figure 2, where $\forall^* 1 = \forall^* a = \forall^* b = \forall^* c = 1$ and $\forall^* 0 = \forall^* - a = \forall^* - b = \forall^* - c = 0$, confirms that this inclusion is proper.

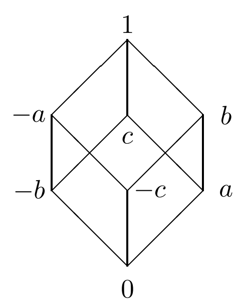


Figure 2

Indeed, it is obvious that $B^*_0 = \{0, 1\}$. On the other hand, $F = \{1, c\}$ and $F' = \{1, b\}$ are nonunit proper $\forall$-filters of B^* such that $F \cap F' = \{1\}$. Therefore, $(B^*, \forall^*)$ is not subdirectly irreducible. As a result, we get $\mathsf{PMA}_{\mathrm{S}} \subset \mathsf{PMA}_{\mathrm{SI}} \subset \mathsf{PMA}_{\mathrm{WC}}$.

This inclusion allows us to simplify considerably TOKARZ's proof of completeness of $\mathsf{KD45}$ with respect to the class of proper filter algebras (cf. [18, Section 6]). To see this, it is a consequence of BIRKHOFF's subdirect representation theorem that PMA is generated by $\mathsf{PMA}_{\mathrm{SI}}$. Since $\mathsf{PMA}_{\mathrm{SI}} \subset \mathsf{PMA}_{\mathrm{WC}}$, it is obvious that PMA is also generated by $\mathsf{PMA}_{\mathrm{WC}}$. Therefore, $\mathsf{KD45}$ is complete with respect to the class $\mathsf{PMA}_{\mathrm{WC}}$. Since $\mathsf{PMA}_{\mathrm{WC}}$ coincides with the class of proper filter algebras, we obtain that $\mathsf{KD45}$ is also complete with respect to the class of proper filter algebras.

Note in passing that the main difference between MA and PMA is the following: in pseudomonadic algebras, as opposed to monadic algebras, it is impossible to reconstruct $\forall$-filters by their restrictions to B_0.

Our next task is to characterize simple and subdirectly irreducible pseudomonadic algebras in terms of proper filter algebras. It is easy to prove that simple algebras correspond to those proper filter algebras (B, F) in which F is the unit filter. To characterize subdirectly irreducible algebras recall that an element $a \neq 1$ of a Boolean algebra B is said to be a *coatom* if $a \leq b$ implies $a = b$ or $b = 1$ for each $b \in B$. Call a filter of a Boolean algebra *minimal* if it is a nonunit filter and no nonunit filter is properly contained in it. It is easy to prove that a filter of a Boolean algebra is minimal iff it is generated by a coatom.

Theorem 2.5. *An algebra $(B, \forall) \in \mathsf{PMA}_{\mathrm{WC}}$ is subdirectly irreducible iff in the corresponding proper filter algebra $(B, F_\forall)$ we have that $F_\forall$ is either the unit filter or a minimal one.*

Proof. The implication from right to left is obvious. For the other implication recall that $F_\forall = \{a \in B : \forall a = 1\}$. Suppose $(B, \forall)$ is a subdirectly irreducible pseudomonadic algebra. If $(B, \forall)$ is simple, then $F_\forall = \{1\}$ by Theorem 2.2. So, assume that $(B, \forall)$ is not simple. Then there exists a least nonunit proper $\forall$-filter F in $(B, \forall)$. Since $B_0 = \{0, 1\}$, every filter contained in F is also a $\forall$-filter, implying that F must be a minimal filter. Further if ∇ is a $\forall$-filter different from F, then $F \subset \nabla$. Since F is a minimal filter, $F = [a)$ for some coatom $a \in B$. Consider the set $\nabla' = \{-a \vee b : b \in \nabla\}$. It is easy to prove that ∇' is a filter, that $\nabla' \subseteq \nabla$, and that $a \notin \nabla'$. Since $\nabla' \subseteq \nabla$, we have that ∇' is also a $\forall$-filter. Since $a \notin \nabla'$, we also have that $F \not\subseteq \nabla'$. Therefore, $\nabla' = \{1\}$. On the other hand, $F \subset \nabla$ implies that there exists $c \in \nabla$ such that $c < a$. Hence, $c \vee -a < 1$, and from $c \vee -a \in \nabla'$ it follows that $\{1\} \subset \nabla'$. The obtained contradiction proves that $F = \nabla$, which means that $F_\forall = F$. Thus, $F_\forall$ is a minimal filter. □

Corollary 2.1. *A pseudomonadic algebra $(B, \forall)$ is subdirectly irreducible iff either it is simple (in which case $\{1\}$ is the only proper $\forall$-filter of $(B, \forall)$), or $\{1\}$ and F are the only proper $\forall$-filters of $(B, \forall)$, where F is a minimal filter of B.*

Recall that a variety $\mathcal{V}$ is said to be *locally finite* if every finitely generated $\mathcal{V}$-algebra is finite. Also recall that a variety $\mathcal{V}$ is *finitely approximable* if it is generated by its finite members. As was mentioned in our introduction, it is known that PMA is locally finite. In the next theorem we give a much simpler proof of this fact.

Theorem 2.6. *PMA is locally finite.*

Proof. For a variety $\mathcal{V}$ of universal algebras, call the class $\mathcal{V}_{\mathrm{SI}}$ *uniformly locally finite* if for each $n \in \omega$ there exists $m(n) \in \omega$ bounding the cardinality of every n-generated algebra $\mathcal{A} \in \mathcal{V}_{\mathrm{SI}}$. It is proved in G. BEZHANISHVILI [1] that a variety $\mathcal{V}$ of finite signature is locally finite iff the class $\mathcal{V}_{\mathrm{SI}}$ is uniformly locally finite. Thus, all we need to show is that $\mathsf{PMA}_{\mathrm{SI}}$ is uniformly locally finite. Suppose $(B[g_1,\dots,g_n],\exists) \in \mathsf{PMA}_{\mathrm{SI}}$ is n-generated, where $g_1,\dots,g_n$ denote the generators of $(B[g_1,\dots,g_n],\exists)$. For each element $a \in B[g_1,\dots,g_n]$ we have that $a = P(g_1,\dots,g_n)$, where P is a polynomial including Boolean operations as well as the operation $\exists$. Since $(B[g_1,\dots,g_n])_0 = \{0,1\}$, every subpolynomial of P which begins with $\exists$ can be replaced by 0 or 1. Then we obtain that $a = P'(g_1,\dots,g_n)$, where P' is a new polynomial containing only Boolean operations. Hence, $B[g_1,\dots,g_n]$ is generated by $g_1,\dots,g_n$ as a Boolean algebra, and since the variety of Boolean algebras is locally finite (cf., e. g., Sikorski [17]), there exists $m(n)$ such that $|B[g_1,\dots,g_n]| \le m(n)$. Therefore, $\mathsf{PMA}_{\mathrm{SI}}$ is uniformly locally finite. □

As a direct consequence we obtain the following

Corollary 2.2.

(1) PMA *and all its subvarieties are finitely approximable.*

(2) *Every extension of* KD45 *has the finite model property.*[4)]

3 Duality theory for pseudomonadic algebras

Suppose X is a nonempty set and R is a binary relation on X. A relation $R \subseteq X^2$ is said to be *serial* if for every $x \in X$ there exists $y \in X$ such that xRy. A relation R is said to be *Euclidean* if xRy and xRz imply yRz for every $x, y, z \in X$ (cf., e. g., Segerberg [16]). Below we will deal only with those pairs (X,R) where X is nonempty and R is a transitive, serial, and Euclidean relation. Note that if in addition R is reflexive, then R becomes an equivalence relation. To see this, suppose xRy. Since R is reflexive, we have that xRx, which implies yRx because R is Euclidean. Therefore, R is symmetric, hence an equivalence relation.

Definition 3.1. We call a relation R a *pseudoequivalence* relation if R is transitive, serial, and Euclidean.

Suppose R is a relation on X. Call $x \in X$ a *quasi-maximal* point, and write $x \in \operatorname{qmax} X$, if xRy implies yRx for every $y \in X$; call x a *minimal* point, and write $x \in \min X$, if there is no $y \in X$ such that yRx. It is obvious that $x\not{R}x$ for each $x \in \min X$, and that $\operatorname{qmax} X \cap \min X = \emptyset$ if R is serial. Moreover, if R is transitive and Euclidean, then $X = \operatorname{qmax} X \cup \min X$. To see this, first note that yRx implies $x \in \operatorname{qmax} X$ for every $x, y \in X$. Therefore, if $x \notin \min X$, then there exists $y \in X$ such that yRx, implying that $x \in \operatorname{qmax} X$. Thus, $X = \operatorname{qmax} X \cup \min X$.

For every $x \in X$ and $A \subseteq X$ let $R(x) = \{y \in X : xRy\}$, $R^{-1}(x) = \{y \in X : yRx\}$, $R(A) = \bigcup_{x\in A} R(x)$, and $R^{-1}(A) = \bigcup_{x\in A} R^{-1}(x)$. A set $A \subseteq X$ is called an *upper cone* if $x \in A$ and xRy imply $y \in A$. *Lower cones* are defined dually. It is easy to prove that A is an upper cone iff $A = R(A)$, and that A is a lower cone iff $A = R^{-1}(A)$.

Consider a topological space (X,τ) and a relation R on X. Following Halmos, call simultaneously closed and open subsets of X *clopens*, and denote the set of clopens of X by $\mathrm{Cp}(X)$. Recall from Esakia [3] and Sambin and Vaccaro [15] that R is said to be a *perfect* relation if $R(x)$ is a closed set for each $x \in X$, and $A \in \mathrm{Cp}(X)$ implies $R^{-1}(A) \in \mathrm{Cp}(X)$.

4) See Segerberg [16], Nagle [13], and Nagle and Thomason [14] for different proofs.

D e f i n i t i o n 3.2. Call a triple (X, τ, R) a *pseudomonadic space* if (X, τ) is a Stone space (i. e. 0-dimensional, compact, and Hausdorff), and R is a perfect pseudoequivalence relation on X.

It is easy to check that for every pseudomonadic space (X, τ, R), the algebra $(\mathrm{Cp}(X), R^{-1})$ is a pseudomonadic algebra. Moreover, every pseudomonadic algebra can be represented this way.

T h e o r e m 3.1 (The Representation Theorem). *For every pseudomonadic algebra $(B, \exists)$ there exists a pseudomonadic space (X, τ, R) such that $(B, \exists)$ is isomorphic to $(\mathrm{Cp}(X), R^{-1})$.*

P r o o f . For a pseudomonadic algebra $(B, \exists)$, let X denote the set of ultrafilters of B. Also for each $a \in B$, let $\varphi(a) = \{x \in X : a \in x\}$, and let τ be the topology on X generated by $F(X) = \{\varphi(a) : a \in B\}$. It is well known from the theory of Boolean algebras that (X, τ) is a Stone space, and that $F(X) = \mathrm{Cp}(X)$. Define a relation R on X by putting xRy iff $a \in y$ implies $\exists a \in x$ for every $a \in B$. (It is easy to check that xRy iff $\forall a \in x$ implies $a \in y$ for every $a \in B$.) Let us show that R is a pseudoequivalence relation. Suppose xRy, yRz and $a \in z$. Then $\exists a \in y$, $\exists\exists a = \exists a \in x$, and so xRz, implying that R is transitive. Further, for $x \in X$, consider the set $I = \{a : \exists a \notin x\}$. It is easy to prove that I is a proper ideal. Hence, by Stone's Theorem there exists an ultrafilter y such that $y \cap I = \emptyset$. But then xRy and R is a serial relation. Finally, if xRy and xRz, then for each $a \in B$ we have $a \in z$ implies $\exists a \in x$. Since $\forall\exists a = \exists a$, we obtain $\forall\exists a \in x$. Therefore, $\exists a \in y$, and so yRz. Thus, R is Euclidean, which together with the above imply that R is a pseudoequivalence relation. To prove that (X, τ, R) is a pseudomonadic space, it remains to show that R is a perfect relation, which follows from Sambin and Vaccaro [15] since PMA is a subvariety of the variety of modal algebras. It also follows from [15] that $\varphi(\exists a) = R^{-1}\varphi(a)$ for every $a \in B$. Thus, (X, τ, R) is a pseudomonadic space, and φ is an isomorphism between $(B, \exists)$ and $(\mathrm{Cp}(X), R^{-1})$. □

Now we extend this theorem to the equivalence of appropriate categories. Suppose (X_1, τ_1, R_1) and (X_2, τ_2, R_2) are pseudomonadic spaces. Call a function $f : (X_1, \tau_1, R_1) \longrightarrow (X_2, \tau_2, R_2)$ a *pseudomonadic morphism* if f is continuous and $fR_1(x) = R_2 f(x)$ for every $x \in X_1$. Denote the category of all pseudomonadic spaces and pseudomonadic morphisms by PMS. Consider PMA as the category of all pseudomonadic algebras and pseudomonadic homomorphisms.

T h e o r e m 3.2. PMA *is dually equivalent to* PMS.

P r o o f . Define (contravariant) functors $\varphi : \mathsf{PMA} \longrightarrow \mathsf{PMS}$, $\psi : \mathsf{PMS} \longrightarrow \mathsf{PMA}$ by putting $\varphi(B, \exists) = (X, \tau, R)$, where (X, τ, R) is the dual space of $(B, \exists)$ constructed in Theorem 3.1, and $\varphi(h) = h^{-1}$, for every $(B, \exists) \in \mathrm{ob}(\mathsf{PMA})$ and $h \in \mathrm{mor}(\mathsf{PMA})$; and $\psi(X, \tau, R) = (\mathrm{Cp}(X), R^{-1})$ and $\psi(f) = f^{-1}$, for every $(X, \tau, R) \in \mathrm{ob}(\mathsf{PMS})$ and $f \in \mathrm{mor}(\mathsf{PMS})$. It follows from Theorem 3.1 that $\varphi(B, \exists) \in \mathrm{ob}(\mathsf{PMS})$. It also follows from [15] that $\varphi(h) \in \mathrm{mor}(\mathsf{PMS})$. Hence, the functor φ is well defined. Further, it is obvious that $\psi(X, \tau, R) \in \mathrm{ob}(\mathsf{PMA})$, and that $\psi(f) \in \mathrm{mor}(\mathsf{PMA})$. Therefore, $\psi : \mathsf{PMS} \longrightarrow \mathsf{PMA}$ is also well defined. Furthermore, it is a consequence of Theorem 3.1 that $\psi \circ \varphi \simeq \mathrm{id}_{\mathsf{PMA}}$. Finally, using [15] once again we obtain that $\varphi \circ \psi \simeq \mathrm{id}_{\mathsf{PMS}}$. Thus, PMA is dually equivalent to PMS. □

As an immediate corollary of Theorem 3.2, we obtain that the category FinPMA of finite pseudomonadic algebras is dually equivalent to the category FinPMS of finite pseudomonadic spaces. Note that every finite pseudomonadic space is discrete, hence the objects of FinPMS are finite couples (X, R) where R is a pseudoequivalence relation on X. Now in a standard way we can obtain the dual characterizations of subalgebras and homomorphic images of a given pseudomonadic algebra. Since the proofs are similar to those in [15], we will omit them.

First observe that there is a one-to-one correspondence between $\forall$-filters of a pseudomonadic algebra and closed upper cones of its dual pseudomonadic space. Hence, there exists a one-to-one correspondence between homomorphic images of a pseudomonadic algebra and closed upper cones of its dual pseudomonadic space. In particular, if $(B, \exists)$ is a finite pseudomonadic algebra, there exists a one-to-one correspondence between $\forall$-filters of $(B, \exists)$ and upper cones of its dual pseudomonadic space. Hence, if $(B, \exists)$ is finite, there exists a one-to-one correspondence between homomorphic images of $(B, \exists)$ and upper cones of its dual pseudomonadic space.

An equivalence relation E on a pseudomonadic space (X, τ, R) is called a *correct partition* of X if $RE(x) \subseteq ER(x)$ for every $x \in X$, and $x \not{E} y$ implies that there exists $A \in \mathrm{Cp}(X)$ such that $E(A) = A$, $x \in A$ and $y \notin A$. (Note that if (X, τ, R) is finite, then this second condition is redundant.) We have that there exists a one-to-one correspondence between subalgebras of a pseudomonadic algebra and correct partitions of its dual pseudomonadic space. In particular, if $(B, \exists)$ is a finite pseudomonadic algebra, then there exists a one-to-one correspondence between subalgebras of $(B, \exists)$ and those partitions of its dual space (X, R) for which $RE(x) \subseteq ER(x)$ for every $x \in X$.

Now we are in a position to give the dual characterization of proper filter algebras (that is, well-connected pseudomonadic algebras), as well as subdirectly irreducible and simple pseudomonadic algebras.

Suppose R is a relation on X. For every $x, y \in X$, say that there exists a *path* from x to y if there exists a finite sequence $\{x_1, \ldots, x_n\} \subseteq X$ such that $x = x_1$, $y = x_n$ and $x_i R x_j$ or $x_j R x_i$ for any $i \neq j$. Call X a *component* if for every $x, y \in X$ there exists a path from x to y.

Theorem 3.3. *Suppose* $(B, \exists) \in \mathsf{PMA}$ *and* (X, τ, R) *is its dual space. Then* $(B, \exists) \in \mathsf{PMA}_{\mathrm{WC}}$ *iff* (X, R) *is a component.*

Proof. Let $(B, \exists) \in \mathsf{PMA}_{\mathrm{WC}}$ and $x, y \in X$. Then since $X = \mathrm{qmax}\, X \cup \min X$, we have $x \in \mathrm{qmax}\, X$ or $x \in \min X$. In both cases there exists $z \in \mathrm{qmax}\, X$ such that xRz (if $x \in \mathrm{qmax}\, X$, we can put $z = x$). Therefore, for each $a \in B$ we have that $a \in z$ implies $\exists a \in x$. Since $B_0 = \{0, 1\}$ and x is a proper filter, $\exists a \in x$ implies that $\exists a = 1$. Thus, $\exists a \in y$, and so yRz. Hence, $\{x, z, y\}$ is a path from x to y, implying that (X, R) is a component. Conversely, suppose (X, R) is a component and $A \subseteq X$. If $A \cap \mathrm{qmax}\, X \neq \emptyset$, then $R^{-1}(A) = X$; and if $\mathrm{qmax}\, X \cap A = \emptyset$, then $R^{-1}(A) = \emptyset$. Therefore, $B_0 = \{0, 1\}$, and so $(B, \exists) \in \mathsf{PMA}_{\mathrm{WC}}$. $\Box$

Corollary 3.3. $(B, \exists) \in \mathsf{PMA}$ *is a proper filter algebra iff its dual space is a component.*

We can think of the dual space of $(B, \exists) \in \mathsf{PMA}_{\mathrm{WC}}$ as it is shown in Figure 3. Note that if the number of minimal elements of X is finite, then every minimal element of X is an isolated point, and hence $\operatorname{qmax} X$ is a clopen. Therefore, if $|\min X| < \infty$, we also have that $A \in \mathrm{Cp}(X)$ implies $R(A) \in \mathrm{Cp}(X)$.

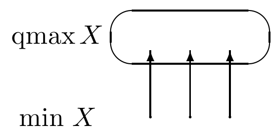


Figure 3

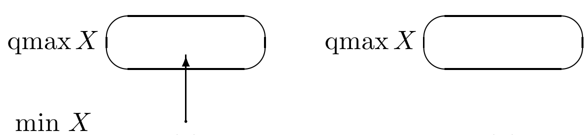
T h e o r e m 3.4. $(B, \exists) \in \mathsf{PMA}$ *is subdirectly irreducible iff its dual space* (X, τ, R) *is a component and* $\min X = \{x\}$ *or* $\min X = \emptyset$.

P r o o f. Suppose $(B, \exists)$ is a pseudomonadic algebra and (X, τ, R) is its dual space. If $(B, \exists)$ is subdirectly irreducible, then $(B, \exists)$ is well-connected, and so (X, R) is a component. Moreover, if there are at least two distinct elements x and y in $\min X$, then we can find two closed upper cones A_1 and A_2 such that $x \notin A_1$, $y \notin A_2$ and $A_1 \cup A_2 = X$. Therefore, there is no greatest proper closed upper cone in (X, R), which means that $(B, \exists)$ is not subdirectly irreducible. Conversely, if (X, R) is a component and $\min X = \{x\}$ or $\min X = \emptyset$, then $\operatorname{qmax} X$ or $\emptyset$ is a greatest proper closed upper cone, respectively, implying that $(B, \exists)$ is subdirectly irreducible. $\square$

Note that if $\min X = \{x\}$, then x is an isolated point.

C o r o l l a r y 3.4. $(B, \exists) \in \mathsf{PMA}$ *is simple iff its dual space* (X, τ, R) *is a component and* $\min X = \emptyset$. *In other words,* $(B, \exists) \in \mathsf{PMA}$ *is simple iff* $X = \operatorname{qmax} X$.

We can think of the dual spaces of subdirectly irreducible and simple pseudomonadic algebras as it is shown in Figures 4(a) and 4(b).



(a) Figure 4 (b)

We conclude this section by showing that if $(B, \exists)$ is a subdirectly irreducible pseudomonadic algebra, then every homomorphic image of $(B, \exists)$ is isomorphic to a subalgebra of $(B, \exists)$. For a class $\mathcal{K}$ of algebras, denote by $\mathbf{H}(\mathcal{K})$, $\mathbf{S}(\mathcal{K})$, and $\mathbf{I}(\mathcal{K})$ the classes of homomorphic images, subalgebras, and isomorphic copies of algebras from $\mathcal{K}$, respectively. We want to prove that if $(B, \exists) \in \mathsf{PMA}_{\mathrm{SI}}$, then $\mathbf{H}(B, \exists) \subseteq \mathbf{IS}(B, \exists)$. Suppose (X, τ, R) is the dual space of $(B, \exists)$. Then either $X = \operatorname{qmax} X$ or $X = \operatorname{qmax} X \cup \{x\}$. In the former case, X has no nontrivial proper homomorphic images, and in the latter case, the only nontrivial proper homomorphic image of $(B, \exists)$ corresponds to the upper closed cone $\operatorname{qmax} X$. Pick a point y from $\operatorname{qmax} X$ and define a partition E of X by putting $E(x) = \{x, y\}$ and $E(z) = \{z\}$ if z is different from x and y. Since $\operatorname{qmax} X$ is a clopen, it is routine to check that E is a correct partition of X, and that X/E is isomorphic to $\operatorname{qmax} X$. Therefore, the algebra of all clopens of $\operatorname{qmax} X$ is isomorphic to a subalgebra of $(B, \exists)$, and so $\mathbf{H}(B, \exists) \subseteq \mathbf{IS}(B, \exists)$.

Note that a similar argument yields that if $(B, \exists) \in \mathsf{PMA}_{\mathrm{WC}}$ and the set $\min X$ of minimal elements of its dual space (X, τ, R) is finite, then $\mathbf{H}(B, \exists) \subseteq \mathbf{IS}(B, \exists)$. Therefore, we have that $\mathbf{H}(B, \exists) \subseteq \mathbf{IS}(B, \exists)$ for every finite well-connected algebra. However, this result can not be generalized to all well-connected algebras, let alone all pseudomonadic algebras.

4 The lattice $\mathbf{\Lambda}(\mathsf{PMA})$

Consider the set $\mathcal{X}$ of all finite nonisomorphic subdirectly irreducible pseudomonadic algebras and define a partial order $\leq$ on $\mathcal{X}$ by putting

$$(B, \exists) \leq (B', \exists') \text{ iff } (B, \exists) \in \mathbf{IS}(B', \exists').$$

Remark 4.1. It follows from the previous section that

$$(B, \exists) \in \mathbf{IS}(B', \exists') \text{ iff } (B, \exists) \in \mathbf{HS}(B', \exists').$$

Denote by $(\mathbb{N}, \leq)$ the set of natural numbers with its usual ordering and consider the set $\mathbb{N} \sqcup \mathbb{N} = (\mathbb{N} \times \{0\}) \cup (\mathbb{N} \times \{1\})$ consisting of two disjoint copies of the set of natural numbers. Define an order R on $\mathbb{N} \sqcup \mathbb{N}$ by putting $(n, i)R(m, j)$ iff $n \leq m$ and $i \leq j$ for n, $m \in \mathbb{N}$ and i, $j \in \{0, 1\}$ (see Figure 5).

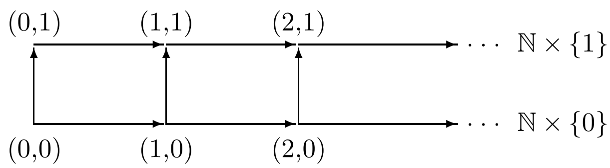


Figure 5

From the dual characterization of subdirectly irreducible pseudomonadic algebras it directly follows that $(\mathcal{X}, \leq)$ is isomorphic to $(\mathbb{N} \sqcup \mathbb{N}, R)$. Recall that a set $A \subseteq \mathbb{N} \sqcup \mathbb{N}$ is a lower cone of $\mathbb{N} \sqcup \mathbb{N}$ if $(n, i) \in A$ and $(m, j)R(n, i)$ imply $(m, j) \in A$, where n, $m \in \mathbb{N}$ and i, $j \in \{0, 1\}$. Denote by $\mathcal{CON}_{\downarrow}(\mathbb{N} \sqcup \mathbb{N})$ the set of all lower cones of $\mathbb{N} \sqcup \mathbb{N}$. Also for $(n, i) \in \mathbb{N} \sqcup \mathbb{N}$ let $\downarrow(n, i) = \{(m, j) \in \mathbb{N} \sqcup \mathbb{N} : (m, j)R(n, i)\}$ denote the least lower cone containing (n, i). Put

$$\begin{aligned}
\mathcal{M}_1 &= \{\downarrow(n, 0) : n \in \mathbb{N}\} \cup \{\downarrow(n, 1) : n \in \mathbb{N}\},\\
\mathcal{M}_2 &= \{\downarrow(n, 1) \cup \downarrow(n + k + 1, 0) : n,\ k \in \mathbb{N}\},\\
\mathcal{M}_3 &= \{\mathbb{N} \times \{0\} \cup \downarrow(n, 1) : n \in \mathbb{N}\}.
\end{aligned}$$

Then one can easily count that

$$\mathcal{CON}_{\downarrow}(\mathbb{N} \sqcup \mathbb{N}) = \{\emptyset, \mathbb{N} \times \{0\}, \mathbb{N} \sqcup \mathbb{N}\} \cup \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3.$$

Therefore, we can think of $\mathcal{CON}_{\downarrow}(\mathbb{N} \sqcup \mathbb{N})$ as it is shown in Figure 6. Here a^n_{n+k+1} denotes $\downarrow(n, 1) \cup \downarrow(n + k + 1, 0)$.

Theorem 4.1 (The Representation Theorem for $\Lambda(\mathsf{PMA})$)). *$\Lambda(\mathsf{PMA})$ is isomorphic to $\mathcal{CON}_{\downarrow}(\mathbb{N} \sqcup \mathbb{N})$.*

Proof. We first show that $\Lambda(\mathsf{PMA})$ is isomorphic to $\mathcal{CON}_{\downarrow}\mathcal{X}$. We define $\varphi : \Lambda(\mathsf{PMA}) \longrightarrow \mathcal{CON}_{\downarrow}(\mathcal{X})$ by putting $\varphi(\mathcal{V}) = \mathcal{X}_{\mathcal{V}}$ for every $\mathcal{V} \in \Lambda(\mathsf{PMA})$, where $\mathcal{X}_{\mathcal{V}}$ denotes the set of all finite nonisomorphic subdirectly irreducible algebras from $\mathcal{V}$. Since $(B', \exists') \leq (B, \exists)$ iff $(B', \exists')$ is isomorphic to a subalgebra of $(B, \exists)$, we have that

$(B, \exists) \in \mathcal{X}_{\mathcal{V}}$ and $(B', \exists') \leq (B, \exists)$ imply $(B', \exists') \in \mathcal{X}_{\mathcal{V}}$. Therefore, φ is well defined. Moreover, since PMA is congruence-distributive, $\varphi(\mathcal{V}_1 \vee \mathcal{V}_2) = \mathcal{X}_{\mathcal{V}_1 \vee \mathcal{V}_2} = \mathcal{X}_{\mathcal{V}_1} \cup \mathcal{X}_{\mathcal{V}_2} = \varphi(\mathcal{V}_1) \cup \varphi(\mathcal{V}_2)$. It is also easy to check that $\varphi(\mathcal{V}_1 \wedge \mathcal{V}_2) = \varphi(\mathcal{V}_1) \cap \varphi(\mathcal{V}_2)$. Thus, φ is a lattice homomorphism. Furthermore, since PMA is locally finite, $\mathcal{V}_1 \neq \mathcal{V}_2$ implies $\mathcal{X}_{\mathcal{V}_1} \neq \mathcal{X}_{\mathcal{V}_2}$ for every $\mathcal{V}_1, \mathcal{V}_2 \in \Lambda(\mathsf{PMA})$. Hence, φ is injective. It is left to be shown that φ is surjective. For this we need to see that for every $A \in \mathcal{CON}_{\downarrow}(\mathcal{X})$ there exists a variety $\mathcal{V} \in \Lambda(\mathsf{PMA})$ such that $\varphi(\mathcal{V}) = A$. Suppose $A \in \mathcal{CON}_{\downarrow}(\mathcal{X})$. Consider the variety $\mathcal{V}$ generated by $\{(B, \exists) : (B, \exists) \in A\}$. It is obvious that $A \subseteq \varphi(\mathcal{V})$. The other inclusion follows from the fact that every finite subdirectly irreducible pseudomonadic algebra is a splitting algebra, which is a consequence of PMA being a subvariety of the variety of K4-algebras and the standard splitting technique in modal logic (for details consult, e. g., Kracht [10, Chapter 7]). Therefore, $\Lambda(\mathsf{PMA})$ is isomorphic to $\mathcal{CON}_{\downarrow}\mathcal{X}$. Now since $(\mathcal{X}, \leq)$ is isomorphic to $(\mathbb{N} \sqcup \mathbb{N}, R)$, we obtain that $\Lambda(\mathsf{PMA})$ is isomorphic to $\mathcal{CON}_{\downarrow}(\mathbb{N} \sqcup \mathbb{N})$. □

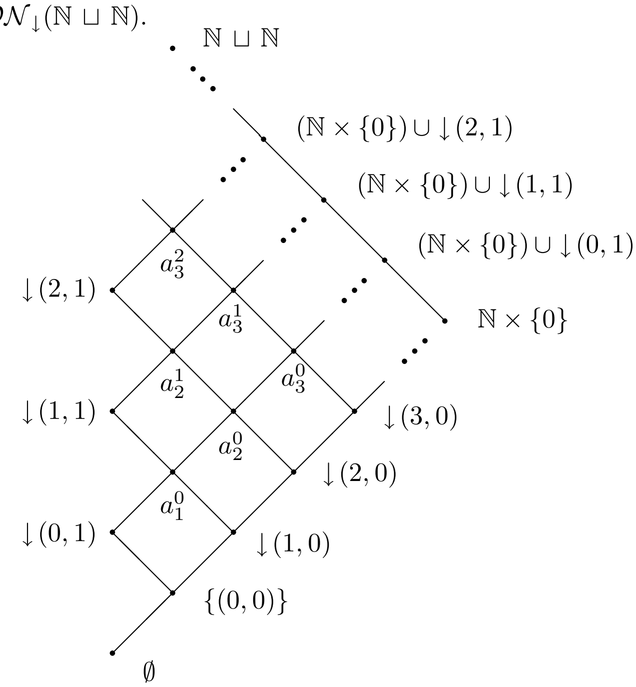


Figure 6

Corollary 4.1 (Segerberg [16]). ExtKD45 *is isomorphic to* $\mathcal{CON}(\mathbb{N} \sqcup \mathbb{N})$.

Proof. This follows from Theorem 4.1 since ExtKD45 is dual to $\Lambda(\mathsf{PMA})$ and $\mathcal{CON}(\mathbb{N} \sqcup \mathbb{N})$ is dual to $\mathcal{CON}_{\downarrow}(\mathbb{N} \sqcup \mathbb{N})$. □

We will close the paper by constructing a countable subdirectly irreducible pseudomonadic algebra generating PMA, and a countable simple monadic algebra generating MA.

Let $\mathbb{N}$ be the set of all natural numbers with the discrete topology. Consider the one point compactification $\alpha\,\mathbb{N}$ of $\mathbb{N}$. It is well known that $\alpha\,\mathbb{N}$ is a Stone space (cf., e. g., Sikorski [17]). Let $\{x\}$ be a singleton set (with the discrete topology).

Consider the topological sum $X = \alpha\,\mathbb{N} \bigoplus \{x\}$ and define a binary relation R on X by putting nRm iff either both $n, m \in \alpha\,\mathbb{N}$ or $n = x$ and $m \in \alpha\,\mathbb{N}$. It is easy to check that (X, τ, R) is a pseudomonadic space, and that $(\mathrm{Cp}(X), R^{-1})$ is a subdirectly irreducible pseudomonadic algebra. Moreover, every finite subdirectly irreducible pseudomonadic algebra is isomorphic to a subalgebra of $(\mathrm{Cp}(X), R^{-1})$. Therefore, PMA is generated by $(\mathrm{Cp}(X), R^{-1})$. In the case of MA we consider $\alpha\,\mathbb{N}$ and define an equivalence relation R on $\alpha\,\mathbb{N}$ by putting nRm for every $n, m \in \alpha\,\mathbb{N}$. Then $(\alpha\,\mathbb{N}, R)$ is a Halmos space and $(\mathrm{Cp}(\alpha N), R^{-1})$ is a simple monadic algebra generating MA. As an immediate consequence of these observations we obtain the following

Proposition 4.1.

(1) $\mathsf{KD45}$ *is complete with respect to* $(\mathrm{Cp}(X), R^{-1})$.

(2) $\mathsf{S5}$ *is complete withe respect to* $(\mathrm{Cp}(\alpha\mathbb{N}), R^{-1})$.

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