Transfer results for hybrid logic
Part I: the case without satisfaction operators

Nick Bezhanishvili and Balder ten Cate

September 6, 2005

Abstract

We define for every Kripke complete modal logic L its hybrid companion L_H and investigate which properties transfer from L to L_H. For a specific class of logics, we present a satisfiability-preserving translation from L_H to L. We prove that for this class of logics, complexity, (uniform) interpolation, and finite axiomatization transfer from L to L_H. We also provide examples showing that, in general, none of complexity, decidability, the finite model property or the Beth property transfer.

1 Introduction

Nominals are a simple but useful addition to the modal language. They have found applications in various domains, including description logic [28] and logics of space [20]. Nominals increase the frame-definable power of the language: many modally undefinable frame properties, such as irrreflexivity, become definable when nominals are added. Adding nominals also gives rise to general completeness results covering frame properties not definable in the basic modal language, and leads to simpler proof systems by internalization of labeled deduction [4, 3]. From the perspective of correspondence theory, nominals are simply first-order constants: they denote individual elements of the domain.

In this paper, we investigate which properties (decidability, interpolation, etc.) are preserved when nominals are added to a modal logic. More precisely, we consider the extension of the basic modal language with a countably infinite set of nominals, called the *minimal hybrid language*. For each Kripke complete modal logic L, we define the hybrid companion logic L_H as the set of formulas of the minimal hybrid language valid on the frame class defined by L. We then address the following transfer question, for a range of properties $\mathfrak{P}$:[1]

If a modal logic L has $\mathfrak{P}$, does it follow that L_H has $\mathfrak{P}$ too?

We show that complexity, the finite model property, decidability, and the Beth property do not transfer. Some of these non-transfer results were already known before, but we present natural and generic counterexamples. On the other hand, we show that tabularity (i.e., completeness with respect to a single finite Kripke frame) *does* transfer. We also give positive transfer results for restricted classes of logics. For every modal logic L that has a master modality and admits filtration, we define a satisfiability preserving translation from L_H to L. Using this translation, we obtain transfer of complexity and (uniform) interpolation from L to L_H, as well as an axiomatization of L_H by adding one simple axiom scheme to the axiomatization

[1] Gargov and Goranko [15] ask the same question in the context of a richer hybrid language that includes the universal modality.

of L. We prove similar translation and transfer results for a class of logics without master modality that includes the basic modal logic $\mathbf{K}$.

The technique developed here allows us to derive short proofs for several results that were proved before by hand, e.g., the complexity of $\mathbf{K}_H$ [1] and the complexity and finite axiomatizability of $\mathbf{PDL}_H$ [24]. Furthermore, we prove that $\mathbf{K}_H$, $\mathbf{S5}_H$, $\mathbf{GL}_H$ and $\mathbf{Grz}_H$ have uniform interpolation over proposition letters. As far as we know, uniform interpolation has not been studied in the context of hybrid logic before. Finally, our results confirm the intuition that in many cases, adding nominals to a modal logic does not increase the complexity.

The paper is organized as follows: in Section 2 we recall basic facts and give the main definitions. In Section 3 we give examples that show that decidability, complexity, the finite model property and the Beth property do not transfer in general. In Section 4 we provide translations from hybrid logics to modal logics, which are used in Section 5 to show that, for certain classes of logics, complexity, (uniform) interpolation and finite axiomatization do transfer.

2 Preliminaries

2.1 Hybrid logic in a nutshell

Recall that the basic hybrid language is the result of extending the modal language with nominals and satisfaction operators. In the present part, we will be concerned with a less expressive hybrid language. The minimal hybrid language is an extension of the basic modal language with nominals. Formally, given a countable set of proposition letters PROP, and a countable set of nominals NOM, and a finite set of modalities MOD, the minimal hybrid language is defined as

$$\phi ::= p \mid i \mid \neg\phi \mid \phi \wedge \psi \mid \langle a \rangle \phi$$

where $p \in$ PROP, $i \in$ NOM and $a \in$ MOD. While the frames that we work with are the same as in ordinary modal logic, we put one extra condition on the models: each nominal must be true at a unique point in the model. In other words, a model for the hybrid language is a pair $(\mathfrak{F}, V)$, where $\mathfrak{F}$ is a frame and V is a valuation for $\mathfrak{F}$ such that $|V(i)| = 1$ for all $i \in$ NOM. Apart from this extra requirement, no changes are made to the semantics. In particular, the truth definition for nominals is the same as that for ordinary proposition letters:

$$(\mathfrak{F}, V), w \Vdash i \text{ iff } w \in V(i)$$

Validity of a hybrid formula on a frame is now defined by universal quantification over all hybrid valuations, i.e., valuations that assign singleton sets to the nominals. The singleton requirement on the valuation of nominals gives rise to new validities. For instance $(i \wedge \Diamond i \wedge \Box p) \to p$ is valid on all frames (if a point is reflexive and all its successors satisfy p, then the point itself satisfies p). Also, using nominals many frame properties are definable that were not definable in the basic modal language. For instance, irreflexivity is defined by $i \to \neg\Diamond i$.

Having nominals, it is very natural to further extend the language with satisfaction operators, allowing one to express that a formula holds at a point named by a nominal. However, in this part of the paper we will not consider these satisfaction operators.

Given a frame class F, *the modal (hybrid) logic of* F is simply the set of modal (hybrid) formulas that are valid on F. Conversely, given a modal or hybrid logic L, the *frame class defined by* L, denoted by $\mathsf{Fr}(L)$, is the class of frames that validate each formula of L. A modal or hybrid logic is said to be *Kripke complete* if it is the logic of some frame class.

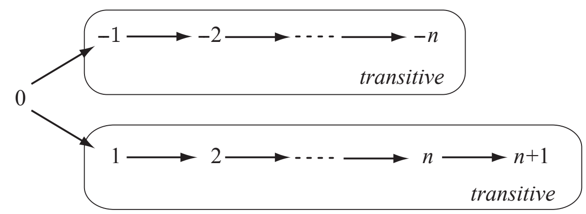

Figure 1: The frame $\mathfrak{F}_n$ used in the proof of Theorem 2.3

Definition 2.1 (Semantic hybrid companion) *For any modal logic L, let L_H be the hybrid logic of the frame class defined by L.*

We call L_H the *semantic hybrid companion* (or simply *hybrid companion*) of the modal logic L. Note that this is not the only possible way to define semantic hybrid companions for modal logics. If a modal logic L is complete for several frame classes, the hybrid logics of these classes need not be the same, and one could consider the hybrid logic of other frame classes than the one *defined* by L. Nevertheless, our choice seems a very natural one.

There is also a syntactic notion of hybrid companion.

Definition 2.2 (Syntactic hybrid companion) *For any modal logic L, let L^H be the smallest set of hybrid formulas containing all substitution instances of formulas in L (in the enriched languages with nominals) as well as*

$$\Diamond_1 \cdots \Diamond_n (i \wedge \phi) \to \Box_{n+1} \cdots \Box_m (i \to \phi) \qquad \text{(Nom)}$$

for each nominal i, hybrid formula ϕ and sequence of modalities $\Diamond_1, \ldots, \Diamond_m$, such that L^H is closed under Modus Ponens, Necessitation.

Note that, for logics that have a master modality, the infinite set of axioms in Definition 2.2 can be reduced to a single axiom scheme.

It is easy to see that $L^H \subseteq L_H$ (all instances of (Nom) are valid on all frames). The converse holds if L is a canonical modal logic [15, 8], but it does not hold in general:

Theorem 2.3 *There is a Kripke complete modal logic L for which L^H is not Kripke complete, and hence $L^H \neq L_H$.*

Proof: Let L be the logic of the frame class $\{\mathfrak{F}_n \mid n > 0\}$, where $\mathfrak{F}_n = (W, R)$ such that $W = \{-n, \ldots, 0, \ldots n+1\}$ and $R = \{(0,1), (0,-1)\} \cup \{(k, \ell) \mid 0 < k < \ell$ or $\ell < k < 0\}$, cf. Figure 1. By definition, L is Kripke complete and in fact has the finite model property. We will show that L^H is not Kripke complete.

For each $n \in \omega$, let δ_n be the formula $\Diamond^n \top \wedge \neg \Diamond^{n+1} \top$, expressing that the longest path from current world has length n. Furthermore, let γ be the **GL** axiom $\Box(\Box p \to p) \to \Box p$, expressing transitivity and converse wellfoundedness. Clearly, $\Box\gamma \in L$ and $\neg\Box\delta_n \in L$ for all $n \geq 1$. Now, it is not hard to see that in all frames validating these formulas, each element satisfying $\Diamond\Diamond\top$ has at least two distinct successors. Thus, each such frame validates the hybrid formula $\Diamond\Diamond\top \to \neg\Box i$. We will show that the latter formula is not derivable in L^H, thereby showing that L^H is Kripke incomplete.

Suppose, for the sake of contradiction, that $\Diamond\Diamond\top \to \neg\Box i$ were derivable in L^H. Fix any derivation, and let $\psi_1, \ldots, \psi_n$ be the (finitely many) instances of (Nom) used as axioms in this derivation. We will show that no derivation using only these instances of (Nom) as axioms can contain $\Diamond\Diamond\top \to \neg\Box i$ as a conclusion, thus establishing a contradiction.

Let k be any natural number greater than the modal depth of all ψ_i's, consider the frame $\mathfrak{F}_k$, and let V be the valuation for it given by $V(i) = \{-1, 1\}$ and $V(p) = V(j) = \{0\}$ for all proposition letters p and nominals j distinct from i. Note that V is not a standard hybrid valuation, since it makes i true at two points. Nevertheless, we claim that every formula derivable in L^H using only $\psi_1, \ldots, \psi_n$ as instances of (Nom) must be globally true on $(\mathfrak{F}_k, V)$. To see this, firstly, note that all substitution instances of formulas in L are valid on $\mathfrak{F}_k$ and hence globally satisfied in $(\mathfrak{F}_k, V)$. Secondly, $\psi_1, \ldots, \psi_n$ are globally satisfied in $(\mathfrak{F}_k, V)$ (the only points of the model satisfying i are 1 and -1, and by the construction of $\mathfrak{F}_k$ and V, they agree on all formulas of modal depth less than k). Finally, the set of formulas globally satisfied in $(\mathfrak{F}_k, V)$ is closed under Modus Ponens and Necessitation.

In particular, since $(\mathfrak{F}_k, V), 0 \not\models \Diamond\Diamond\top \to \neg\Box i$, it follows that $\Diamond\Diamond\top \to \neg\Box i$ cannot be derivable in L^H using only $\psi_1, \ldots, \psi_n$ as instances of (Nom), which contradicts our initial assumption. □

2.2 Filtrations

Filtrations are commonly used as a tool for establishing the finite model property and decidability for modal logics. It was observed by several people that if a modal logic admits filtration, then so does its hybrid companion (see below). Hence, if a modal logic is proved to have the finite model property (to be decidable) using filtrations, then its hybrid companion also has the finite model property (is decidable). Of course, this does not imply that the two logics have the same complexity. For instance, **KB**, the logic of symmetric frames, admits filtration, and therefore its hybrid companion $\mathbf{KB}_H$ has the finite model property and is decidable. Nevertheless, **KB** has a PSpace-complete satisfiability problem, whereas $\mathbf{KB}_H$ has an ExpTime-complete satisfiability problem, as we will show in Section 3.

In this paper, we will use filtrations in a different way: we will use them not to prove decidability or the finite model property, but to construct satisfiability preserving translations from certain hybrid companion logics L_H to the corresponding modal logics L. Subsequently, we will use these translations to show that L and L_H are in the same complexity class, that interpolation and uniform interpolation transfer from L to L_H, and that $L_H = L^H$. Note that admitting filtration is not a sufficient condition for the existence of such a translation: indeed, while **KB** admits filtration, clearly no polynomial translation from $\mathbf{KB}_H$ to **KB** can exist (assuming PSpace $\neq$ ExpTime).

Let us briefly recall the idea of filtration. For convenience of presentation we will restrict attention to uni-modal languages. Let $\mathfrak{M} = (W, R, V)$ be a model and let Σ be a set of formulas. Define an equivalence relation $\sim_\Sigma$ on W such that $w \sim_\Sigma v$ iff w and v agree on all formulas in Σ. Denote by $[w]$ the $\sim_\Sigma$-equivalence class containing w and let $W/_{\sim_\Sigma}$ be the set of all $\sim_\Sigma$-equivalence classes of W. Let V_Σ be the valuation for $W/_{\sim_\Sigma}$ given by $V_\Sigma(p) = \{[w] \mid w \models p\}$. By a *filtration of* $\mathfrak{M}$ *through* Σ we will mean any model $\mathfrak{M}' = (W/_{\sim_\Sigma}, R', V_\Sigma)$, with R' a binary relation over $W_{\sim_\Sigma}$. Often, Σ will be the set of subformulas of some formula ϕ, denoted by $Sub(\phi)$.

Definition 2.4 *A modal logic L* admits filtration *if for every formula ϕ there exists a finite set of formulas Σ_ϕ such that for every model $\mathfrak{M}$ based on an L-frame, if ϕ is satisfied at a point in $\mathfrak{M}$ then ϕ is satisfied at a point in some filtration of $\mathfrak{M}$ through Σ_ϕ based on an L-frame. A modal logic L* admits computable (polynomial) filtration *if it admits filtration and Σ_ϕ is computable from ϕ (in polynomial time). We say* *that a modal logic* admits simple filtration *if it admits filtration and* $\Sigma_\phi = Sub(\phi)$ *for all formulas* ϕ.

Note that since $|Sub(\phi)|$ is linear in the length of ϕ, every logic that admits simple filtration admits polynomial filtration. Also note that the above definitions are easily generalized to multi-modal logics.

The following folklore result was noted by several people (cf. e.g., [2, 15].)

Proposition 2.5 *If L admits (computable / polynomial / simple) filtration, then L_H admits (computable / polynomial / simple) filtration.*

Proof: We simply apply the usual filtration, treating nominals as proposition letters. All that needs to be checked is that the filtrated model is a hybrid model, in other words, that every nominal occurring in a given formula is true at exactly one point. Since each nominal is true at some point in the original model, it must also be true at some point in the filtrated model. Now, suppose that a nominal i is true at two points of the filtrated model, say $[w]$ and $[v]$. Then $w \models i$ and $v \models i$ in the original model, and so $w = v$, which implies that $[w] = [v]$. □

Many well-known logics admit PTime-computable filtration, including **K4**, **K45**, **KD45**, **S4**, **S5**, **K4.2**, **K4.3**, **S4.2**, **S4.3**, **K5**, **K4.1**, **S4.1** [10, §5.3], **S5** × **S5** [14], and **PDL** [13]. Each of these logics except **K5**, **K4.1**, **S4.1** and **PDL**, admits simple filtration. In the case of **PDL**, the filtration set Σ_ϕ is the Fisher-Ladner closure of ϕ. The logics **GL** and **Grz** also admit filtration. Since these logics will be referred to later on in the paper, we decided to discuss them explicitly.

Example 2.6 **GL** *admits simple filtration [4, Exercise in §4.8]: for any formula ϕ, let $\Sigma_\phi = Sub(\phi)$. For any model $\mathfrak{M} = (W, R, V)$ based on a* **GL**-*frame $\mathfrak{F}$, let $\mathfrak{M}_{\Sigma_\phi} = (W/_{\sim_{\Sigma_\phi}}, R_{\Sigma_\phi}, V_{\Sigma_\phi})$, where we let $[w]R_{\Sigma_\phi}[v]$ if the following two conditions hold:*

1. *for every $\Diamond\psi \in \Sigma_\phi$, $v \models \psi \vee \Diamond\psi$ implies $w \models \Diamond\psi$, and*
2. *there exists $\Diamond\psi \in \Sigma_\phi$ with $w \models \Diamond\psi$ and $v \not\models \Diamond\psi$.*

It can be shown that $\mathfrak{M}_{\Sigma_\phi}$ is again based on a (finite) **GL**-*frame, and for all $w \in W$ and $\psi \in Sub(\phi)$, $\mathfrak{M}_{\Sigma_\phi}, [w] \models \psi$ iff $\mathfrak{M}, w \models \psi$. In particular, if ϕ is satisfied at some point in $\mathfrak{M}$ then ϕ is also satisfied at some point in $\mathfrak{M}_{\Sigma_\phi}$.*

Example 2.7 **Grz** *admits polynomial filtration [6]: for any formula ϕ, let $\Sigma_\phi = Sub(\phi) \cup \{\Diamond(\neg\psi \wedge \Diamond\psi) : \Diamond\psi \in Sub(\phi)\}$. For any model $\mathfrak{M} = (W, R, V)$ based on a* **Grz**-*frame $\mathfrak{F}$, let $\mathfrak{M}_{\Sigma_\phi} = (W/_{\sim_{\Sigma_\phi}}, R_{\Sigma_\phi}, V_{\Sigma_\phi})$, where we let $[w]R_{\Sigma_\phi}[v]$ if $[w] = [v]$ or the following two conditions hold:*

1. *for every $\Diamond\psi \in \Sigma_\phi$, $v \models \psi \vee \Diamond\psi$ implies $w \models \Diamond\psi$, and*
2. *there exists $\Diamond\psi \in \Sigma_\phi$ with $w \models \Diamond\psi$ and $v \not\models \Diamond\psi$.*

It can be shown that $\mathfrak{M}_{\Sigma_\phi}$ is again based on a (finite) **Grz**-*frame, and for all $w \in W$ and $\psi \in Sub(\phi)$, $\mathfrak{M}_{\Sigma_\phi}, [w] \models \psi$ iff $\mathfrak{M}, w \models \psi$. In particular, if ϕ is satisfied at some point in $\mathfrak{M}$ then ϕ is also satisfied at some point in $\mathfrak{M}_{\Sigma_\phi}$.*

2.3 Thomason simulation

The Thomason simulation [27] associates to each bi-modal logic L a uni-modal logic L^s with almost the same properties. We will use it in Section 3 in order to find a uni-modal counterexample to the transfer of decidability and the finite model property. We will now briefly review this construction, focusing only on the aspects that are relevant for our purposes. For more details, the reader is referred to [19].

Given a bi-modal Kripke frame $\mathfrak{F} = (W, R_1, R_2)$, let $\mathfrak{F}^s = (W', R')$ be the uni-modal Kripke frame where W' consists of two disjoint copies of each $w \in W$ (denoted by w_1 and w_2, respectively) plus an extra element denoted by t, and $R = \{(w_k, v_k) \mid (w, v) \in R_k, k = 0, 1\} \cup \{(w_1, w_2), (w_2, w_1) \mid w \in W\} \cup \{(w_0, t) \mid w \in W\}$. Given a Kripke complete bi-modal logic L, let L^s to denote the uni-modal logic of $\{\mathfrak{F}^s \mid \mathfrak{F} \in Fr(L)\}$. It can be shown that every rooted L^s-frame is isomorphic to $\mathfrak{F}^s$ for some L-frame $\mathfrak{F}$.

This semantic construction is paralleled by a syntactic translation. Let us use α, β_1 and β_2 as shorthands for $\Box\bot$, $\Diamond\Box\bot$ and $\Diamond\top \wedge \neg\Box\Diamond\top$, respectively. For each formula $\phi(\Diamond_1, \Diamond_2)$, let ϕ^s be defined inductively as follows:

$$\begin{array}{lcl} p^s & = & \beta_1 \wedge p \\ (\neg\phi)^s & = & \beta_1 \wedge \neg(\phi^s) \\ \phi \wedge \psi & = & \phi^s \wedge \psi^s \\ \Diamond_1\phi^s & = & \beta_1 \wedge \Diamond\phi^s \\ \Diamond_2\phi^s & = & \beta_1 \wedge \Diamond(\beta_2 \wedge \Diamond(\beta_2 \wedge \Diamond(\beta_1 \wedge \phi^s))) \end{array}$$

Proposition 2.8 *For all bi-modal formulas ϕ and for all frames $\mathfrak{F}$, $\mathfrak{F} \models \phi$ iff $\mathfrak{F}^s \models \beta_1 \to \phi^s$.*

It follows from this result that if a Kripke-complete modal logic L is undecidable (or, lacks the finite model property), the same holds for L^s. As it happens, there is also a converse translation $(\cdot)^{\bar{s}}$ from uni-modal formulas to bi-modal formulas, such that $\mathfrak{F}^s \models \phi$ iff $\mathfrak{F} \models \phi^{\bar{s}}$. It follows that decidability and the finite model property are in fact *invariant* under passage from L to L^s, as are many other properties [19].

Theorem 2.9 *A Kripke-complete bi-modal logic L is decidable (has the finite model property) iff L^s is decidable (has the finite model property).*

These results can be extended to hybrid logics. Just as in the case of modal logics, given a Kripke complete bi-modal hybrid logic L we define L^s to be the uni-modal hybrid logic of the frame class $\{\mathfrak{F}^s \mid \mathfrak{F} \models L\}$. It is not hard to see that $(\cdot)^s$ and $(\cdot)_H$ then commute: for any Kripke complete bi-modal logic L, $(L_H)^s = (L^s)_H$.

The above formula translation is also quite straightforwardly extended to formulas containing nominals, by letting

$$i^s \quad = \quad \beta_1 \wedge (i \vee \Diamond(\beta_2 \wedge i)) \ .$$

Using this extended translation, Proposition 2.8 can be generalized to arbitrary hybrid formulas:

Proposition 2.10 *For all bi-modal hybrid formulas ϕ containing nominals $i_1, \ldots, i_n$, and for all frames $\mathfrak{F}$, $\mathfrak{F} \models \phi$ iff $\mathfrak{F}^s \models (\beta_1 \wedge \bigwedge_{k=1}^{n} \neg\Diamond(\alpha \wedge i)) \to \phi^s$.*

Proof: [$\Rightarrow$] Suppose $\mathfrak{F} \models \phi$, where $\mathfrak{F} = (W, R_1, R_2)$, and considere any valuation V for $\mathcal{F}^s$ and world u such that $\mathfrak{F}^s, V, u \models \beta_1 \wedge \bigwedge_{k=1}^{n} \neg\Diamond(\alpha \wedge i)$. We wil show that $\mathfrak{F}^s, V, u \models \phi^s$. First, note that, since $\mathfrak{F}^s, V, u \models \beta_1$, $u = w_1$ for some $w \in W$. Furthermore, note that each nominal occurring in ϕ is true at v_1 or v_2, for some $v \in W$. Let V' be the valuation for $\mathfrak{F}$ given by $V'(p) = \{w \mid w_1 \in V(p)\}$ for all proposition letters p and $V'(i) = \{w \mid V(i) = \{w_1\}$ or $V(i) = \{w_2\}\}$ for all nominals i. It follows from the above considerations that V is a well-defined hybrid valuation: it assigns a singleton set to each nominal. Furthermore, a straightforward inductive argument shows that, for all $v \in W$, $\mathfrak{F}, V, v \models \phi$ iff $\mathfrak{F}^s, v_1 \models \phi^s$. In particular, since ϕ is valid on $\mathfrak{F}$, we may conclude that $\mathfrak{F}^s, V, u \models \phi^s$.

[$\Leftarrow$] Suppose $\mathfrak{F}^s \models (\beta_1 \wedge \bigwedge_{k=1}^n \neg\Diamond(\alpha \wedge i)) \rightarrow \phi^s$, where $\mathfrak{F} = (W, R_1, R_2)$, and consider any valuation V for $\mathfrak{F}$ and any world $w \in W$. Let V' be the valuation for $\mathfrak{F}^s$ given by $V'(p) = \{v_1 \mid v \in V(p)\}$ for all proposition letters and nominals p. Clearly, V' is a well-defined hybrid valuation: it assigns singleton sets to the nominals. Furthermore, a straightforward inductive argument shows that, for all $v \in W$, $\mathfrak{F}, V, v \models \phi$ iff $\mathfrak{F}^s, v_1 \models \phi^s$. By construction, $\mathfrak{F}, V, w_1 \models \beta_1 \wedge \bigwedge_{k=1}^n \neg\Diamond(\alpha \wedge i)$, and hence $\mathfrak{F}^s, V, w_1 \models \phi^s$. It follows that $\mathfrak{F}, V, w \models \phi$. □

Theorem 2.11 *Let L be any Kripke complete bi-modal hybrid logic. If L is undecidable (lacks the finite model property) then L^s is also undecidable (also lacks the finite model property).*

Proof: Let L be any Kripke complete bi-modal hybrid logic, and let ϕ be any hybrid formula containing nominals $i_1, \ldots, i_n$. It follows from Proposition 2.10 that ϕ is satisfiable on a (finite) L-frame iff $\beta_1 \wedge \bigwedge_{k=1}^n \neg\Diamond(\alpha \wedge i) \wedge \phi^s$ is satisfiable on a (finite) L^s-frame. Since the latter formula can be effectively computed from ϕ, it follows that undecidability and lack of the finite model property transfer from L to L^s. □

As in the modal case, one could proceed to define analogous translations in the other direction, i.e., from L^s to L. a converse translation $(\cdot)^{\bar{s}}$ from uni-modal hybrid formulas to bi-modal hybrid formulas, such that $\mathfrak{F}^s \models \phi$ iff $\mathfrak{F} \models \phi^{\bar{s}}$. This translation could then be used to show that decidability and the finite model property are in fact *invariant* under passage from L to L^s, as are many other properties. We will not pursue this here, however, since the above results, in particular Theorem 2.11, suffice for present purposes.

3 General transfer results

The main question we will address in this paper is the following: *which properties of logics are preserved under passage from L to L_H?* Tabularity (i.e., completeness with respect to a single finite Kripke frame) is an example of a property that transfers.

Theorem 3.1 *If a modal logic L is tabular, then L_H is also tabular.*

Proof: Let L be the modal logic of some finite $\mathfrak{F}$, and consider the frame class $\mathsf{Fr}(L)$. It follows from Jónsson's lemma [18] that this frame class consists precisely of those frames that are isomorphic to disjoint unions of generated subframes of bounded morphic images of $\mathfrak{F}$. A finite frame has (up to isomophism) only finitely many bounded morphic images, and each of those is itself finite. Let the finite frames $\mathfrak{G}_1, \ldots, \mathfrak{G}_n$ be all the bounded morphic images of $\mathfrak{F}$. It follows that every frame in $\mathsf{Fr}(L)$ is isomorphic to a disjoint union of generated subframes of $\mathfrak{G}_1, \ldots, \mathfrak{G}_n$. In what follows, it will be convenient to assume without loss of generality that these frames are mutually disjoint and that $n > 1$.

In order to show L_H is tabular, it suffices to show that L_H is complete for some finite frame $\mathfrak{G} \in \mathsf{Fr}(L)$. Take as $\mathfrak{G}$ the disjoint union $\mathfrak{G}_1 \uplus \cdots \uplus \mathfrak{G}_n$. If a hybrid formula ϕ is L_H-consistent then, by definition, it is satisfied on some frame in $\mathfrak{F}' \in \mathsf{Fr}(L)$, under a valuation V and at a world w. By the above arguments, $\mathfrak{F}'$

is isomorphic to a disjoint union of generated subframes of $\mathfrak{G}_1, \ldots, \mathfrak{G}_n$. Let $\mathfrak{A}'$ be a component of $\mathfrak{F}'$ containing w. Then there is a generated subframe $\mathfrak{A}$ of some $\mathfrak{G}_k$ ($k \leq n$) such that $\mathfrak{A}$ is isomorphic to $\mathfrak{A}'$. Let $h : \mathfrak{A} \to \mathfrak{A}'$ be the relevant isomorphism. Fix a world u of $\mathfrak{G}$ not belonging to $\mathfrak{G}_k$ and define a valuation V' on $\mathfrak{G}$ as follows:

$$\begin{array}{rcl} V'(p) & = & \{v \in \mathfrak{A} \mid h(v) \in V(p)\} \\ V'(i) & = & \begin{cases} \{v\} & \text{if } V(i) = h(v) \text{ for some world } v \text{ of } \mathfrak{A} \\ \{u\} & \text{otherwise} \end{cases} \end{array}$$

An inductive argument shows that $(\mathfrak{G}, V'), h^{-1}(w) \models \phi$. It follows that L_H is complete for $\mathfrak{G}$, and hence is tabular. □

Note that the reason that we take all the bounded morphic images of $\mathfrak{F}$ is that hybrid formulas are not preserved under bounded morphic images. In other words, if a frame $\mathfrak{G}$ is a bounded morphic image of a frame $\mathfrak{F}$ and ϕ is a hybrid formula, then, unlike for modal formulas, the fact that $\mathfrak{F} \models \phi$ does not imply that $\mathfrak{G} \models \phi$. Also note that, even though Theorem 3.1 shows that tabularity transfers from L to L_H, the size of $\mathfrak{G}$ is in general exponential in the size of $\mathfrak{F}$. Thus, we can define a stronger notion, *n-tabularity*, that does not transfer, in general, from L to L_H for any given n. We say that a logic L is *n-tabular* if it is tabular and n is the least natural number such that L is complete with respect to a frame of size n.

Besides this positive transfer result, most results are negative. For a start, while the basic modal logic **K** has the Beth property, $\mathbf{K}_H$ lacks it. Let us briefly recall the definition of the Beth property. We will use $\models^{glo}$ to refer to the global entailment relation on models, i.e., $\Sigma \models^{glo}_L \phi$ means that for all models $\mathfrak{M}$ based on an L-frame, if $\mathfrak{M}$ globally satisfies all formulas in Σ then $\mathfrak{M}$ globally satisfies ϕ. For a set of formulas $\Sigma(p)$ containing the proposition letter p (and possible other proposition letters and nominals), we say that $\Sigma(p)$ implicitly defines p, on L-frames, if $\Sigma(p) \cup \Sigma(p') \models^{glo}_L p \leftrightarrow p'$. Here, p' is a proposition letter not occurring in Σ, and $\Sigma(p')$ is the result of replacing all occurrences of p by p' in $\Sigma(p)$. A (modal or hybrid) logic L is said to have the *Beth property* if whenever a set of formulas $\Sigma(p)$ defines a proposition letter p, on L-frames, then there is a formula θ in which p does not occur, such that $\Sigma \models^{glo}_L p \leftrightarrow \theta$. The relevant formula θ is called an *explicit definition* of p, relative to Σ and L.

Proposition 3.2 *$\mathbf{K}_H$ lacks the Beth property.*

Proof: Let $\Sigma = \{p \to i, j \wedge q \to \Diamond(i \wedge p), j \wedge \neg q \to \Diamond(i \wedge \neg p)\}$. In any model that globally satisfies Σ, p holds nowhere besides (possibly) at the point named by the nominal i, and it holds there if and only if q holds at the point named j. In other words, Σ implicitly defines p in terms of q, i and j.

Let $\mathfrak{F}$ be the frame $(\{w, v\}, \{(v, w)\})$, and let V, V' be valuations for $\mathfrak{F}$ such that $V(i) = V'(i) = \{w\}$, $V(j) = V'(j) = \{v\}$, $V(p) = V(q) = \emptyset$ and $V'(p) = \{w\}$ and $V'(q) = \{v\}$. Then $(\mathfrak{F}, V) \models \Sigma$ and $(\mathfrak{F}, V') \models \Sigma$, and $(\mathfrak{F}, V), w \not\models p$ and $(\mathfrak{F}, V'), w \models p$. Nevertheless, a simple bisimulation argument shows that w satisfies the same formulas in the language with i, j and q. It follows that there can be no explicit definition of p in terms of q, i and j. □

A second negative transfer result is due to Areces et al. [1], who show by means of a spy-point argument [5] that the satisfiability problem for the hybrid tense logic $\mathbf{K_t}_H$ is ExpTime-complete. This implies that complexity does not transfer in general, since the satisfiability problem for $\mathbf{K_t}$ is only PSpace-complete. Note that for the basic modal logic **K**, adding nominals does not increase the complexity: $\mathbf{K}_H$ is still in PSPACE, as was first shown by Schaerf [25]. The modal logic of symmetric frames **KB** provides a uni-modal counterexample to transfer of complexity. The satisfiability problem for **KB** is PSPACE-complete [11]. However,

Theorem 3.3 *The satisfiability problem for* $\mathbf{KB}_H$ *is* EXPTIME*-complete.*

Proof: The proof is again based on a spy-point argument. For any modal formula ϕ, let $\phi' = i \wedge \Diamond\neg i \wedge \Box\Box\Diamond i \wedge \Box\phi^{\neg i}$, where i is any nominal and $\phi^{\neg i}$ is obtained from ϕ by relativising all modalities with $\neg i$ (i.e., replacing all subformulas of the form $\Diamond\psi$ by $\Diamond(\neg i \wedge \psi)$ and replacing all subformulas of the form $\Box\psi$ by $\Box(\neg i \to \psi)$). One can easily see that if ϕ' holds at a world w in a symmetric model $\mathfrak{M}$ then ϕ holds globally in the submodel of $\mathfrak{M}$ generated by w, minus the world w itself. It follows that, on symmetric frames, ϕ' is satisfiable iff ϕ is globally satisfiable. The global satisfiability problem for modal formulas on the class of symmetric frames is EXPTIME-complete [11]. Hence, the satisfiability problem of $\mathbf{KB}_H$ is EXPTIME-hard. That the problem is inside EXPTIME follows from the fact that converse **PDL** with nominals is in EXPTIME [12]. To see this, with every hybrid formula ϕ we associate a formula $\phi^\sim$ in the language of converse **PDL** by replacing every occurrence of $\Diamond$ in ϕ by $\langle a \cup a^\sim \rangle$ and replacing every occurrence of $\Box$ in ϕ by $[a \cup a^\sim]$, for some fixed atomic program a. Then ϕ is satisfiable on symmetric frames iff $\phi^\sim$ is satisfiable, and since converse **PDL** with nominals is in EXPTIME, we obtain that $\mathbf{KB}_H$ is also in EXPTIME. □

Decidability and the finite model property do not transfer either. For decidability, this was first proved in [21, Theorem 14] in the context of description logic. That the finite model property does not transfer follows from known results on the description logic $\mathcal{ALCFIO}$ (cf. [23]). We will now give a simple example of a decidable modal logic with finite model property whose hybrid companion is both undecidable and lacks the finite model property. Let $\mathbf{K}_{23} \oplus \mathbf{S5}$ be the fusion of $\mathbf{K}_{23}$ and **S5**, where $\mathbf{K}_{23}$ is the normal modal logic axiomatized by the following two Sahlqvist axioms:

$$\textstyle\bigwedge_{1\leq k\leq 3} \Diamond_1 p_k \to \bigvee_{1\leq k<l\leq 3} \Diamond_1(p_k \wedge p_l) \qquad \text{(at most 2 successors)}$$
$$\textstyle\bigwedge_{1\leq k\leq 4} \Diamond_1\Diamond_1 p_k \to \bigvee_{1\leq k<l\leq 4} \Diamond_1\Diamond_1(p_k \wedge p_l) \qquad \text{(at most 3 two-step successors)}$$

Proposition 3.4 $\boldsymbol{K}_{23} \oplus \boldsymbol{S5}$ *has the finite model property and is decidable.*

Proof: $\mathbf{K}_{23}$ is decidable and has the finite model property [26]. The same holds for **S5** [4]. Since decidability and the finite model property are preserved under fusions [14], the result follows. □

Proposition 3.5 $(\boldsymbol{K}_{23} \oplus \boldsymbol{S5})_H$ *is undecidable and lacks the finite model property.*

Proof: We will provide a reduction from the global satisfiability problem of $\mathbf{K}_{23}$, which is known to be undecidable, to the satisfiability problem for $(\mathbf{K}_{23} \oplus \mathbf{S5})_H$.

For any uni-modal formula ϕ, let $\phi' = i \wedge \Box_2\Box_1\Diamond_2 i \wedge \Box_2\phi$. One can easily see that if ϕ' holds at a world w in an model $\mathfrak{M}$ based on a $(\mathbf{K}_{23} \oplus \mathbf{S5})$-frame, then ϕ holds globally in the submodel of $\mathfrak{M}$ generated by w. Conversely, if ϕ is globally satisfied in a model $\mathfrak{M} = (W, R, V)$ based on a $\mathbf{K}_{23}$-frame, then $(W, R, W^2, V'), w \models \phi'$ for all $w \in W$, where V' is the hybrid valuation that extends V by mapping i to $\{w\}$. It follows that ϕ is globally satisfiable on a (finite) $\mathbf{K}_{23}$-frame iff ϕ' is satisfiable on a (finite) $(\mathbf{K}_{23} \oplus \mathbf{S5})$-frame. Global satisfiability is undecidable for $\mathbf{K}_{23}$, and $\mathbf{K}_{23}$ lacks the global finite model property [26]. It follows that $(\mathbf{K}_{23} \oplus \mathbf{S5})_H$ is undecidable and lacks the finite model property. □

Note that the above proof is again based on a spy-point argument, and that the result would hold also if the **S5**-modality were replaced by a pair of tense modalities. Using the Thomason simulation, the above example can be turned into a uni-modal example of non-transfer of decidability and the finite model property: by Theorem 2.9, $(\mathbf{K}_{23} \oplus \mathbf{S5})^s$ has the finite model property and is decidable. As mentioned in Section 2.3, $((\mathbf{K}_{23} \oplus \mathbf{S5})^s)_H = ((\mathbf{K}_{23} \oplus \mathbf{S5})_H)^s$, and hence, by Theorem 2.11, $((\mathbf{K}_{23} \oplus \mathbf{S5})^s)_H$ lacks the finite model property and is undecidable.

The above results also imply that decidability does not transfer under taking fusions of hybrid logics, a phenomenon that has been observed and studied in [16].

In the remainder of this paper, we provide positive results for a class of logics that includes several well-known non-canonical logics, including **PDL**, **GL** and **Grz**. We show that for this class of logics, complexity, finite axiomatizability, interpolation and uniform interpolation transfer.

4 Translations from hybrid logics to modal logics

In this section, we define two translations from the minimal hybrid language to the basic modal language, and prove that they preserve satisfiability. The first translation applies to logics that have a master modality and admit filtration. The second translation applies to a class of logics that admit filtration but do not have a master modality.

4.1 Logics with a master modality

First, let us recall the definition of a master modality.

Definition 4.1 *A modal logic L has a master modality [4, p. 371] if there is a modal formula $\phi(p)$ containing only the proposition letter p such that for all models $\mathfrak{M}$ based on an L-frame and all worlds w, $\mathfrak{M}, w \models \phi(p)$ iff p is true somewhere in the submodel of $\mathfrak{M}$ generated by w (equivalently, if p is true at a point reachable from w in a finite number of steps). If a logic has a master modality, we will refer to it as $\diamond\!\!\!\!+$ (more precisely, we will use $\diamond\!\!\!\!+\,\psi$ as a shorthand for $\phi(\psi)$).*

Fact 4.2

1. *Every logic of bounded depth has a master modality.*
2. *Every extension of* **K4** *has a master modality.*
3. *Every extension of* **K5** *has a master modality.*
4. **PDL** *has a master modality.*[2]
5. *Every extension of* **K4** $\times$ **K4** *has a master modality.*
6. *Every extension of the tense logic* $\mathbf{K4}_t$ *with trichotomy has a master modality (where trichotomy is the axiom $Pp \wedge Pq \rightarrow P(p \wedge Pq) \vee P(q \wedge Pp) \vee P(p \wedge q)$).*

Proof:

1. Take $\phi = \bigvee_{0 \leq k \leq n} \diamond^k p$, where n is the bound on the depth.
2. Take $\phi = p \vee \diamond p$.
3. Take $\phi = p \vee \diamond p \vee \diamond\diamond p$
4. Take $\phi = \langle (\cup_i a_i)^* \rangle p$.
5. Take $\phi = p \vee \diamond_1 p \vee \diamond_2 p \vee \diamond_1 \diamond_2 p$.

[2] For convenience, we assume that the language contains only finitely many atomic programs. The results of this paper can easily be generalized to **PDL** with infinitely many atomic programs.

6. Take $\phi = p \vee Pp \vee PFp$

□

Note that **K** does not have a master modality (this can easily be shown by the fact that every modal formula has a finite modal depth, and hence can only talk about a small part of the generated submodel). Similarly, the basic tense logic $\mathbf{K}_t$, the tense logic of transitive frames $\mathbf{K4}_t$ and the logic **KB** of symmetric frames do not have a master modality.

The class of logics we will be working with is the class of logics that have a master modality and that admit filtration. Let L be a Kripke-complete modal logic that has a master modality and that admits filtration. Now we define a translation from the language of L_H to the language of L. That is, we translate every hybrid formula into a modal formula. For a hybrid formula $\phi(i_1, \ldots, i_n)$, let $\phi[\vec{i}/\vec{p_i}]$ denote the formula obtained from ϕ by uniformly replacing each nominal i_k by a distinct new proposition letter p_{i_k}.

Theorem 4.3 *Let L be a logic that has a master modality and that admits filtration. Let ϕ be any hybrid formula. Then ϕ is L_H-satisfiable iff the modal formula*

$$\phi^* = \phi[\vec{i}/\vec{p_i}] \ \wedge \bigwedge_{\substack{1\leq k\leq n \\ \psi\in\Sigma_{\phi[\vec{i}/\vec{p_i}]}}} \Big(\oplus(p_{i_k} \wedge \psi) \rightarrow \boxplus(p_{i_k} \rightarrow \psi) \Big)$$

is L-satisfiable, where $\Sigma_{\phi[\vec{i}/\vec{p_i}]}$ is the filtration set of $\phi[\vec{i}/\vec{p_i}]$.

Proof: [⇒] Suppose $(\mathfrak{F}, V), w \models \phi$ with $\mathfrak{F} \in \mathsf{Fr}(L)$. Let V' be any valuation that agrees with V on all proposition letters occurring in ϕ, and such that $V'(p_{i_k}) = V(i_k)$ for each nominal i_k. Clearly, $(\mathfrak{F}, V'), w \models \phi[\vec{i}/\vec{p_i}]$. The truth of the second conjunct of ϕ^* at w under V' follows directly from the fact that $V'(p_{i_k})$ is a singleton set for each $k = 1, \ldots, n$.

[⇐] Suppose $(\mathfrak{F}, V), w \models \phi^*$ with $\mathfrak{F} = (W, R) \in \mathsf{Fr}(L)$. Without loss of generality, we can assume that $\mathfrak{F}$ is generated by w (note that ϕ^* is a modal formula: it contains no nominals). Our task is to construct a hybrid model satisfying ϕ. First, we will filtrate $(\mathfrak{F}, V)$. Let $\Sigma = \Sigma_{\phi[\vec{i}/\vec{p_i}]}$. Since L admits filtration, there exists a model $\mathfrak{M} = (W/_{\sim_\Sigma}, R_\Sigma, V_\Sigma)$ such that $(W/_{\sim_\Sigma}, R_\Sigma) \in \mathsf{Fr}(L)$ and such that for all $v \in W$ and $\psi \in \Sigma$, $\mathfrak{M}, [v] \models \psi$ iff $(\mathfrak{F}, V), v \models \psi$. In particular, $\mathfrak{M}, [w] \models \phi[\vec{i}/\vec{p_i}]$.

Claim 1 *$V_\Sigma(p_{i_k})$ contains at most one point (for $k = 1, \ldots, n$).*

Proof of claim: Suppose $[v], [v'] \in V_\Sigma(p_{i_k})$. Then $v, v' \in V(p_{i_k})$, by the definition of V_Σ. Since $(\mathfrak{F}, V), w \models \oplus(p_{i_k} \wedge \psi) \rightarrow \boxplus(p_{i_k} \rightarrow \psi)$ for all $\psi \in \Sigma$, it follows that v, v' agree on formulas in Σ. Indeed, if $v \models \psi$ then $w \models \oplus(p_{i_k} \wedge \psi)$, so $w \models \boxplus(p_{i_k} \rightarrow \psi)$ and therefore $v' \models \psi$. Thus, $v \sim_\Sigma v'$ and so $[v] = [v']$. ⊣

If every p_{i_k} is true at *exactly* one point, then the proof is finished, since we can consider $(W/_{\sim_\Sigma}, R_\Sigma)$ to be a hybrid model for ϕ. In general, however, this need not be the case: p_{i_k} could be true nowhere. So, we need to ensure that for every p_{i_k} there is indeed a point where p_{i_k} is true. Let $\mathfrak{G}$ be the disjoint union of two isomorphic copies of $(W/_{\sim_\Sigma}, R_\Sigma)$. For convenience, we will use $[v]_1$ and $[v]_2$ to refer to the two distinct copies of a world $[v] \in W/_{\sim_\Sigma}$. Since $\mathsf{Fr}(L)$ is modally definable, it is closed under disjoint unions and hence, $\mathfrak{G} \in \mathsf{Fr}(L)$. Define the valuation V' for $(W/_{\sim_\Sigma}, R_\Sigma)$ so that $V'(p) = \{v_1 \mid v \in V_\Sigma(p)\}$ for each proposition letter p occurring in ϕ, and for each nominal $k = 1, \ldots, n$,

$$V'(p_{i_k}) = \begin{cases} \{[v]_1\} & \text{if } V_\Sigma(p_{i_k}) = \{[v]\} \\ \{[w]_2\} & \text{if } V_\Sigma(p_{i_k}) = \emptyset \end{cases}$$

Intuitively speaking, the only role of the second disjoint copy of $(W/_{\sim_\Sigma}, R_\Sigma)$ is to provide enough points so that we can make each p_{i_k} true somewhere, without affecting the truth of ϕ at $[w]_1$. Indeed, a simple bisimulation argument shows that $(\mathfrak{G}, V'), [w]_1 \models \phi[\vec{i}/\vec{p_i}]$.

By construction, V' assigns to each p_{i_k} a singleton set. Replacing each p_{i_k} by the corresponding i_k, we therefore obtain a hybrid model again, which furthermore satisfies ϕ at $[w]_1$. We conclude that ϕ is satisfiable on $\mathsf{Fr}(L)$. □

Corollary 4.4 *Let L be a complete modal logic that has a master modality and that admits filtration. Let ϕ be any hybrid formula with nominals $i_1, \ldots, i_n$. Then ϕ is L_H-valid iff the formula:*

$$\Big(\bigwedge_{\substack{1 \le k \le n \\ \psi \in \Sigma_{\neg\phi}}} \diamond\!\!\!\!+ (p_{i_k} \wedge \psi) \to \boxplus (p_{i_k} \to \psi) \Big) \to \phi[\vec{i}/\vec{p_i}]$$

is L-valid.

Proof: Suppose $L_H \vdash \phi$, for some formula ϕ with nominals $i_1, \ldots, i_n$. Then $\neg\phi$ is not L_H-satisfiable. Hence, by Theorem 4.3,

$$\Big(\bigwedge_{\substack{1 \le k \le n \\ \psi \in \Sigma_{\neg\phi}}} \diamond\!\!\!\!+ (p_{i_k} \wedge \psi) \to \boxplus (p_{i_k} \to \psi) \Big) \wedge \neg\phi[\vec{i}/\vec{p_i}]$$

is not L-satisfiable, which means that

$$L \vdash \Big(\bigwedge_{\substack{1 \le k \le n \\ \psi \in \Sigma_{\neg\phi}}} \diamond\!\!\!\!+ (p_{i_k} \wedge \psi) \to \boxplus (p_{i_k} \to \psi) \Big) \to \phi[\vec{i}/\vec{p_i}]$$

□

Remark 4.5 We remark here that many well-known logics are known to have a master modality and to admit polynomial filtration. We list some of them with references for the proofs: **K4**, **K45**, **KD45**, **S4**, **S5**, **K4.2**, **K4.3**, **S4.2**, **S4.3**, **K5**, **K4.1**, **S4.1** [10, §5.3]; **GL** and **PDL** [4, §4.8]; **S5** × **S5** [14]; **Grz** [6]. Moreover, all of these logics except **K5**, **K4.1**, **S4.1**, **PDL** and **Grz** admit simple filtration (cf. Definition 2.4).

4.2 Logics with shallow axioms

Now we show that even though the basic multi-modal logic $\mathbf{K}_n$ does not have a master modality, $\mathbf{K}_{nH}$ admits a satisfiability-preserving translation into $\mathbf{K}_n$. We call a modal formula *shallow* if every occurrence of a proposition letter is in the scope of at most one modal operator. We will show that the preservation result holds for extensions of $\mathbf{K}_n$ with shallow axioms. Note that every non-iterative axiom [22] is shallow, as well as every closed formula (i.e., a formula in which no proposition letters occur).

We will use $\diamond\!\!\!\cdot\,\psi$ as a shorthand for $\bigvee_{\diamond \in \mathrm{MOD}} \diamond\psi$, $\diamond\!\!\!\cdot^{\le n}\psi$ as a shorthand for $\bigvee_{0 \le k \le n} \underbrace{\diamond\!\!\!\cdot \ldots \diamond\!\!\!\cdot}_{k \text{ times}} \psi$.

For every formula ϕ let $md(\phi)$ be the modal depth of ϕ [4, Definition 2.28].

Theorem 4.6 *A hybrid formula ϕ is satisfiable iff the modal formula*

$$\phi^* \quad = \quad \phi[\vec{i}/\vec{p_i}] \quad \wedge \bigwedge_{\substack{1 \le k \le n \\ \psi \in Sub(\phi[\vec{i}/\vec{p_i}])}} \Big(\diamond\!\!\!\cdot^{\le md(\phi)} (p_{i_k} \wedge \psi) \to \boxdot^{\le md(\phi)} (p_{i_k} \to \psi) \Big)$$

is satisfiable.

Proof: The left to right implication is easy to prove. Now suppose that ϕ^* is satisfiable. Let $\mathfrak{M}, w \models \phi^*$, with $\mathfrak{M} = (\mathfrak{F}, V)$ and $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond \in \text{MOD}})$. Without loss of generality, we can assume that $\mathfrak{F}$ is generated by w. Let $R_{\diamondsuit} = \bigcup_{\Diamond \in \text{MOD}} R_\Diamond$. For every point $v \in W$, let $d_{\mathfrak{F}}(v)$ be the minimal number of $R_{\diamondsuit}$-steps in which v is reachable from the root w. Consider the equivalence relation $\sim_{Sub(\phi[\vec{i}/\vec{p_i}])}$. Two worlds stand in this equivalence relation iff they satisfy the same subformulas of $\phi[\vec{i}/\vec{p_i}]$. For any $\sim_{Sub(\phi[\vec{i}/\vec{p_i}])}$-equivalence class $[v]$, choose a representative $f[v] \in [v]$ such that for any $v' \in [v]$ we have $d_{\mathfrak{F}}(f[v]) \leq d_{\mathfrak{F}}(v')$. Note that while $f[w] = w$, these representatives are in general not unique. Also note that for every $v \in W$ and $\psi \in Sub(\phi[\vec{i}/\vec{p_i}])$, $\mathfrak{M}, v \models \psi$ iff $\mathfrak{M}, f[v] \models \psi$.

Let $W' = \{f[v] \mid v \in W\}$. For each $\Diamond \in \text{MOD}$, define the relation $R'_\Diamond$ on W' so that $f[u] R'_\Diamond f[v]$ iff there is a $v' \in [v]$ with $f[u] R_\Diamond v'$. Define a valuation V' on W' by letting $f[w] \in V'(p)$ iff $w \in V(p)$ for all $p \in Sub(\phi[\vec{i}/\vec{p_i}])$. Let $\mathfrak{F}' = (W', (R'_\Diamond)_{\Diamond \in \text{MOD}})$ and $\mathfrak{M}' = (\mathfrak{F}', V')$.

Claim 1 *For any $\psi \in Sub(\phi[\vec{i}/\vec{p_i}])$ and any point $v \in W$, $\mathfrak{M}, f[v] \models \psi$ iff $\mathfrak{M}', f[v] \models \psi$*

Proof of claim: By induction on the complexity of ψ. If ψ is a propositional letter, then the claim holds by the definition of V'. The Boolean cases are obvious. Finally, let $\psi = \Diamond\chi$, for some $\Diamond \in \text{MOD}$.

[$\Rightarrow$] Suppose that $\mathfrak{M}, f[v] \models \Diamond\chi$. Then there is a point $u \in W$ such that $f[v] R_\Diamond u$ and $\mathfrak{M}, u \models \chi$. Since $\chi \in Sub(\phi[\vec{i}/\vec{p_i}])$ and $u \sim_{Sub(\phi[\vec{i}/\vec{p_i}])} f[u]$, we have that $\mathfrak{M}, f[u] \models \chi$. By the induction hypothesis, it follows that $\mathfrak{M}', f[u] \models \chi$. Finally, we have that $f[v] R'_\Diamond f[u]$, by the definition of $R'_\Diamond$. Hence, $\mathfrak{M}', f[v] \models \Diamond\chi$.

[$\Leftarrow$] Suppose that $\mathfrak{M}', f[v] \models \Diamond\chi$. Then there is an $f[u] \in W'$ such that $f[v] R'_\Diamond f[u]$ and $\mathfrak{M}', f[u] \models \chi$. By the induction hypothesis, $\mathfrak{M}, f[u] \models \chi$. Also, by the definition of $R'_\Diamond$, there must be a $u' \in [u]$ such that $f[v] R_\Diamond u'$. Since $\chi \in Sub(\phi[\vec{i}/\vec{p_i}])$ and $u' \sim_{Sub(\phi[\vec{i}/\vec{p_i}])} f[u]$, it follows that $\mathfrak{M}, u' \models \chi$. We conclude that $\mathfrak{M}, f[v] \models \Diamond\chi$. $\dashv$

Let us define $d_{\mathfrak{F}'}$ similar to $d_{\mathfrak{F}}$. Note that $\mathfrak{F}'$ need not be point-generated anymore. For those $f[v] \in W'$ that are not reachable from the root $f[w] = w$, we let $d_{\mathfrak{F}'}(f[v]) = \infty$.

Claim 2 *$d_{\mathfrak{F}}(f[v]) \leq d_{\mathfrak{F}'}(f[v])$, for all $v \in W$*

Proof of claim: If $d_{\mathfrak{F}'}(f[v]) = \infty$, the claim obviously holds. Otherwise, the proof proceeds by induction on $d_{\mathfrak{F}'}(f[v])$. The base case, with $d_{\mathfrak{F}'}(f[v]) = 0$, only applies if $f[v] = w$, in which case the claim clearly holds. Next, suppose $d_{\mathfrak{F}'}(f[v]) = n + 1$. By definition, there must be a path of the form

$$f[w] = w \xrightarrow{R'_{\Diamond_1}} \cdots \xrightarrow{R'_{\Diamond_n}} f[u] \xrightarrow{R'_{\Diamond_{n+1}}} f[v]$$

for some $\Diamond_1, \ldots, \Diamond_{n+1} \in \text{MOD}$. It follows that $d_{\mathfrak{F}'}(f[u]) \leq n$, and hence by the induction hypothesis, $d_{\mathfrak{F}}(f[u]) \leq d_{\mathfrak{F}'}(f[u]) \leq n$. Since $f[u] R_{\Diamond_{n+1}}{}' f[v]$, by the definition of $R_\Diamond{}'$ we have that there is a $v' \in [v]$ such that $f[u] R_{\Diamond_{n+1}} v'$. This implies that $d_{\mathfrak{F}}(v') \leq n+1$. By the definition of f, we know that $d_{\mathfrak{F}}(f[v]) \leq d_{\mathfrak{F}}(v')$, because $v' \in [v]$. Therefore, $d_{\mathfrak{F}}(f[v]) \leq n + 1$. $\dashv$

Claim 3 *For all $k = 1 \ldots n$, there is at most one world $f[v] \in W'$ such that $d_{\mathfrak{F}'}(f[v]) \leq md(\phi)$ and $\mathfrak{M}', f[v] \models p_{i_k}$.*

Proof of claim: Suppose $\mathfrak{M}', f[v] \models p_{i_k}$ and $\mathfrak{M}', f[u] \models p_{i_k}$, with $d_{\mathfrak{F}'}(f[v]), d_{\mathfrak{F}'}(f[u]) \leq md(\phi)$. By Claim 2, $d_{\mathfrak{F}}(f[v]), d_{\mathfrak{F}}(f[u]) \leq md(\phi)$. Furthermore, $\mathfrak{M}, f[v] \models p_{i_k}$ and $\mathfrak{M}, f[u] \models p_{i_k}$. By our initial assumption, $\mathfrak{M}, w \models \phi^*$, hence $f[v] \sim_{Sub(\phi)} f[u]$, which implies that $f[v] = f[u]$. $\dashv$

From Claim 1, we immediately deduce that $\mathfrak{M}', w \models \phi[\vec{i}/\vec{p_i}]$. The valuation of $p_{i_1}, \ldots, p_{i_n}$ can be restricted to the worlds with depth $\leq md(\phi)$ without affecting the truth of $\phi[\vec{i}/\vec{p_i}]$ at w. In this way, we ensure that every p_{i_k} is true at at most one world. Finally, applying the same argument as in the proof of Theorem 4.3, we conclude that the original hybrid formula ϕ is satisfiable. □

Note that the length of ϕ^* is in general exponential in the length of ϕ, but that in case of uni-modal languages, it is polynomial.

Let L be a modal logic defined by finitely many shallow axioms, and for a hybrid formula $\phi(i_1, \ldots, i_n)$, let

$$\phi^* \quad = \quad \phi[\vec{i}/\vec{p_i}] \ \wedge \bigwedge_{\substack{1\leq k\leq n \\ \psi\in\Sigma}} \Big(\Diamond^{\leq md(\phi)}(p_{i_k} \wedge \psi) \rightarrow \boxdot^{\leq md(\phi)}(p_{i_k} \rightarrow \psi) \Big)$$

where Σ consists of all subformulas of ϕ plus all closed subformulas of the (finitely many) shallow axioms of L (recall that a modal formula is closed if it contains no proposition letters).

Theorem 4.7 *Let L be any complete modal logic axiomatized by finitely many shallow axioms. A hybrid formula ϕ is L_H-satisfiable iff ϕ^* is L-satisfiable.*

Proof: We use the same construction as in the proof of Theorem 4.6, but now we use a richer filtration set, that also includes all closed subformulas of the shallow axioms of L. It suffices to show that the constructed frame $\mathfrak{F}'$ is an L-frame. Let V' be a valuation on $\mathfrak{F}'$, and let $x \in W'$ be such that $(\mathfrak{F}', V'), x \models \phi$. Define V on $\mathfrak{F}$ such that $v \in V(p)$ iff $f[v] \in V'(p)$. We claim that for all shallow axioms χ of L and for all $v \in W$, $(\mathfrak{F}, V), f[v] \models \chi$ iff $(\mathfrak{F}', V'), f[v] \models \chi$.

This, we prove by induction on χ. Note that χ is shallow, and hence we may assume that χ is generated by the following recursive definition:

> $\chi ::= \top \mid p \mid \neg\chi \mid \chi_1 \wedge \chi_2 \mid \Diamond\psi$, where ψ is any Boolean combination of proposition letters and closed formulas.

The only non-trivial case in the induction is when χ is of the form $\Diamond\psi$ where ψ is a Boolean combination of proposition letters and closed formulas. In this case, we reason as follows.

[⇒] Suppose $(\mathfrak{F}, V), f[v] \models \Diamond\psi$. Then there is a $u \in W$ such that $f[v]R_\Diamond u$ and $(\mathfrak{F}, V), u \models \psi$. By the definition of V and the fact that all closed subformulas of ψ are in the filtration set, it follows that $(\mathfrak{F}', V'), f[u] \models \psi$. By definition of $R'_\Diamond$, $f[v]R'_\Diamond f[u]$. Hence, $(\mathfrak{F}', V'), f[v] \models \Diamond\psi$.

[⇐] Suppose $(\mathfrak{F}', V'), f[v] \models \Diamond\psi$. Then there is an $f[u] \in W'$ such that $(\mathfrak{F}', V'), f[u] \models \psi$ and $f[v]R'_\Diamond f[u]$. By definition of $R'_\Diamond$, there is a $u' \in [u]$ such that $f[v]R_\Diamond u'$. By the definition of V and the fact that all closed subformulas of ψ are in the filtration set, it follows that $(\mathfrak{F}, V), u' \models \psi$. Hence, $(\mathfrak{F}, V), f[v] \models \Diamond\psi$. □

This covers logics axiomatized using reflexivity ($\Box p \rightarrow p$), totality ($\Diamond\top$) and bounded width ($\text{bw}_n \equiv \Diamond p_1 \wedge \cdots \wedge \Diamond p_n \rightarrow \bigvee_{1\leq k<l\leq n} \Diamond(p_k \wedge p_l)$).

Note again that the length of ϕ^* is in general exponential in the length of ϕ, but that in case of uni-modal languages, it is polynomial.

Corollary 4.8 *Let L be a complete modal logic axiomatizable by finitely many shallow axioms. Let ϕ be any hybrid formula with nominals $i_1, \ldots i_n$. Then ϕ is L_H-valid iff the following formula is L-valid:*

$$\Big(\bigwedge_{\substack{1\leq k\leq n \\ \psi\in\Sigma}} \Diamond^{\leq md(\phi)}(p_{i_k} \wedge \psi) \rightarrow \boxdot^{\leq md(\phi)}(p_{i_k} \rightarrow \psi) \Big) \rightarrow \phi[\vec{i}/\vec{p_i}]$$

where Σ consists of all subformulas of ϕ plus all closed subformulas of the (finitely many) shallow axioms of L.

Proof: As for Corollary 4.4. □

5 Applications of the translations

From Theorem 4.3 and 4.7, together with the observation that for logics admitting polynomial filtration, the length of ϕ^* is polynomial in the length of ϕ, we obtain the following transfer results for complexity.

Corollary 5.1 *Let L be a complete modal logic satisfying one of the following conditions:*

(a) L has a master modality and admits polynomial filtration.

(b) L is a uni-modal logic defined by finitely many shallow axioms.

Then L_H-satisfiability is polynomially reducible to L-satisfiability.

Hence, for modal logics L satisfying the conditions of Corollary 5.1, complexity transfers in the following sense: if L is in (complete for) a complexity class C closed under polynomial reductions, then L_H is also in (complete for) C.[3] Note that Corollary 5.1 cannot be easily generalized: **KB**, the logic of symmetric frames, admits polynomial filtration, yet by Theorem 3.3, $\mathbf{KB}_H$ is ExpTime-complete, whereas **KB** is only PSpace-complete.

Next, we will discuss the issue of transfer of interpolation and uniform interpolation. For any hybrid formula ϕ, let $\mathbb{P}(\phi)$ and $\mathbb{N}(\phi)$ denote the set of proposition letters and nominals, respectively, occurring in ϕ.

Definition 5.2 (Interpolation for hybrid logics) *A hybrid logic L is said to have interpolation over proposition letters if for all formulas ϕ and ψ such that $L \vdash \phi \to \psi$, there is a formula θ such that $L \vdash \phi \to \theta$, $L \vdash \theta \to \psi$ and $\mathbb{P}(\theta) \subseteq \mathbb{P}(\phi) \cap \mathbb{P}(\psi)$.*

Note that according to this definition, θ might contain nominals occurring in ϕ but not in ψ or vice versa. It seems more natural to require also the nominals in the interpolant θ to occur both in ϕ and in ψ. However, it turns out that almost all hybrid logics lack this strong form of interpolation [7].

Recall that a modal logic *admits simple filtration* if it admits filtration and for every formula ϕ we have $\Sigma_\phi = Sub(\phi)$. For logics admitting simple filtration, interpolation transfers.

Theorem 5.3 *Let L be a complete modal logic satisfying one of the following conditions:*

(a) L has a master modality and admits simple filtration.

(b) L is defined by finitely many shallow axioms.

If L has interpolation, then L_H has interpolation over proposition letters.

[3]The familiar complexity classes NP, PSpace, ExpTime, NExpTime, 2ExpTime, etc. are all closed under polynomial reductions.

Proof: We only prove (a), since (b) is similar. Suppose $L_H \vdash \phi \to \psi$. Let $i_1, \ldots, i_n$ be the nominals occurring in the formula $\phi \to \psi$, and let $\Sigma = Sub(\neg(\phi \to \psi)[\vec{i}/\vec{p_i}])$ By Corollary 4.4, the following formula is L-valid.

$$\Big(\bigwedge_{\substack{1 \leq k \leq n \\ \chi \in \Sigma}} \Diamond\!\!\!+(p_{i_k} \wedge \chi) \to \boxplus(p_{i_k} \to \chi)\Big) \to (\phi[\vec{i}/\vec{p_i}] \to \psi[\vec{i}/\vec{p_i}]) \tag{1}$$

The antecedent of this formula says that *for all* $1 \leq k \leq n$*, if two worlds* w *and* w' *in the model both satisfy* p_{i_k}*, then* w *and* w' *satisfy exactly the same formulas in* Σ. Note that every formula in Σ is a Boolean combination of subformulas of $\phi[\vec{i}/\vec{p_i}]$ and $\psi[\vec{i}/\vec{p_i}]$. Hence, to say that w and w' satisfy the same formulas in Σ is equivalent to saying that they satisfy the same subformulas of $\phi[\vec{i}/\vec{p_i}]$ and $\psi[\vec{i}/\vec{p_i}]$. Therefore, the formula

$$\Big(\bigwedge_{\substack{1 \leq k \leq n \\ \chi \in Sub(\phi[\vec{i}/\vec{p_i}]) \cup Sub(\psi[\vec{i}/\vec{p_i}])}} \Diamond\!\!\!+(p_{i_k} \wedge \chi) \to \boxplus(p_{i_k} \to \chi)\Big) \to (\phi[\vec{i}/\vec{p_i}] \to \psi[\vec{i}/\vec{p_i}]) \tag{2}$$

is semantically equivalent to (1), and hence L-valid. It is easy to see that (2) is equivalent to

$$\begin{aligned} &\Big(\bigwedge_{\substack{1 \leq k \leq n \\ \chi \in Sub(\phi[\vec{i}/\vec{p_i}])}} \Diamond\!\!\!+(p_{i_k} \wedge \chi) \to \boxplus(p_{i_k} \to \chi)\Big) \wedge \phi[\vec{i}/\vec{p_i}] \to \\ &\qquad\qquad \Big(\bigwedge_{\substack{1 \leq k \leq n \\ \chi \in Sub(\psi[\vec{i}/\vec{p_i}])}} \Diamond\!\!\!+(p_{i_k} \wedge \chi) \to \boxplus(p_{i_k} \to \chi)\Big) \to \psi[\vec{i}/\vec{p_i}] \end{aligned} \tag{3}$$

Let θ be the modal interpolant for (3) in L. Note that apart from the proposition letters $p_{i_1}, \ldots p_{i_n}$, θ only contains proposition letters that occur both in ϕ and in ψ. Now, since L_H extends L and is closed under uniform substitution of formulas for proposition letters, we have:

$$L_H \vdash \Big(\bigwedge_{\substack{1 \leq k \leq n \\ \chi \in Sub(\phi)}} \Diamond\!\!\!+(i_k \wedge \chi) \to \boxplus(i_k \to \chi)\Big) \wedge \phi \to \theta[\vec{p_i}/\vec{i}]$$

$$L_H \vdash \theta[\vec{p_i}/\vec{i}] \to \Big(\bigwedge_{\substack{1 \leq k \leq n \\ \chi \in Sub(\psi)}} \Diamond\!\!\!+(i_k \wedge \chi) \to \boxplus(i_k \to \chi)\Big) \to \psi$$

Since $\Diamond\!\!\!+(i \wedge \chi) \to \boxplus(i \to \chi)$ is valid in hybrid logic for any i and χ, it follows that $L_H \vdash \phi \to \theta[\vec{p_i}/\vec{i}]$ and $L_H \vdash \theta[\vec{p_i}/\vec{i}] \to \psi$. Finally, as we mentioned above, all proposition letters occurring in $\theta[\vec{p_i}/\vec{i}]$ occur both in ϕ and in ψ. We conclude that $\theta[\vec{p_i}/\vec{i}]$ is an interpolant for $\phi \to \psi$. □

Definition 5.4 (Uniform interpolation for hybrid logics) *A hybrid logic L has uniform interpolation over proposition letters if for each formula ϕ and each finite set of proposition letters $P \subseteq \mathbb{P}(\phi)$, there is a formula ϕ_P such that*

- $\mathbb{P}(\phi_P) \subseteq P$*, and*
- *For all formulas ψ, if $\mathbb{P}(\psi) \cap \mathbb{P}(\phi) \subseteq P$ and $\mathbb{N}(\psi) \subseteq \mathbb{N}(\phi)$, then $L \vdash \phi \to \psi$ iff $L \vdash \phi_P \to \psi$.*

In contrast to what one might expect, the uniform interpolant ϕ_P does not apply to formulas ψ that contain nominals not occurring in ϕ.

Theorem 5.5 *Let L be a complete modal logic satisfying one of the following conditions:*

(a) L has a master modality and admits simple filtration.

(b) L is defined by finitely many shallow formulas.

If L has uniform interpolation, then L_H has uniform interpolation over proposition letters.

Proof: We only prove (a), since (b) is similar.

Let ϕ be a hybrid formula, $P \subseteq \mathbb{P}(\phi)$, and let $\mathbb{N}(\phi) = \{i_1, \ldots, i_k\}$. Let $P' = P \cup \{p_{i_1}, \ldots, p_{i_k}\}$. Let θ be a uniform interpolant in L over P' of the formula

$$\phi^* \quad = \quad \phi[\vec{i}/\vec{p_i}] \wedge \bigwedge_{\substack{1 \le k \le n \\ \chi \in Sub(\phi[\vec{i}/\vec{p_i}])}} \Diamond\!\!\!\!+\,(p_{i_k} \wedge \chi) \to \boxplus(p_{i_k} \to \chi))$$

We claim that $\theta[\vec{p_i}/\vec{i}]$ is a uniform interpolant in L_H of ϕ over P. Consider any hybrid formula ψ with $\mathbb{P}(\psi) \cap \mathbb{P}(\phi) \subseteq P$ and $\mathbb{N}(\psi) \subseteq \mathbb{N}(\phi)$. We will show that $L_H \vdash \phi \to \psi$ iff $L_H \vdash \theta[\vec{p_i}/\vec{i}] \to \psi$.

[$\Rightarrow$] Suppose $L_H \vdash \phi \to \psi$. Let $\Sigma = Sub(\neg(\phi \to \psi)[\vec{i}/\vec{p_i}])$, By Corollary 4.4, we have that the formula

$$\Big(\bigwedge_{\substack{1 \le k \le n \\ \chi \in \Sigma}} \Diamond\!\!\!\!+\,(p_{i_k} \wedge \chi) \to \boxplus(p_{i_k} \to \chi)\Big) \to \Big(\phi[\vec{i}/\vec{p_i}] \to \psi[\vec{i}/\vec{p_i}]\Big)$$

is L-valid. The same argument as in the proof of Theorem 5.3 shows that

$$L \vdash \Big(\phi[\vec{i}/\vec{p_i}] \wedge \bigwedge_{\substack{1 \le k \le n \\ \chi \in Sub(\phi[\vec{p_i}/\vec{i}])}} \Diamond\!\!\!\!+\,(p_{i_k} \wedge \chi) \to \boxplus(p_{i_k} \to \chi)\Big) \to$$

$$\Big(\bigwedge_{\substack{1 \le k \le n \\ \chi \in Sub(\psi[\vec{p_i}/\vec{i}])}} \Diamond\!\!\!\!+\,(p_{i_k} \wedge \chi) \to \boxplus(p_{i_k} \to \chi)\Big) \to \psi[\vec{i}/\vec{p_i}]$$

or, equivalently,

$$L \vdash \phi^* \to \Big(\big(\bigwedge_{\substack{1 \le k \le n \\ \chi \in Sub(\psi[\vec{p_i}/\vec{i}])}} \Diamond\!\!\!\!+\,(p_{i_k} \wedge \chi) \to \boxplus(p_{i_k} \to \chi)\big) \to \psi[\vec{i}/\vec{p_i}]\Big)$$

Since θ is a uniform interpolant for ϕ^*, it follows that

$$L \vdash \theta \to \Big(\big(\bigwedge_{\substack{1 \le k \le n \\ \chi \in Sub(\psi[\vec{p_i}/\vec{i}])}} \Diamond\!\!\!\!+\,(p_{i_k} \wedge \chi) \to \boxplus(p_{i_k} \to \chi)\big) \to \psi[\vec{i}/\vec{p_i}]\Big)$$

Since L_H extends L and is closed under uniform substitution of formulas for proposition letters, we have:

$$L_H \vdash \theta[\vec{p_i}/\vec{i}] \to \Big(\big(\bigwedge_{\substack{1 \le k \le n \\ \chi \in Sub(\psi)}} \Diamond\!\!\!\!+\,(i_k \wedge \chi) \to \boxplus(i_k \to \chi)\big) \to \psi\Big)$$

Since $\Diamond\!\!\!\!+\,(i \wedge \chi) \to \boxplus(i \to \chi)$ is valid in hybrid logic for any nominal i and formula χ, it follows that $L_H \vdash \theta[\vec{p_i}/\vec{i}] \to \psi$.

[$\Leftarrow$] Suppose $L_H \vdash \theta[\vec{p_i}/\vec{i}] \to \psi$. Since θ is a uniform interpolant for ϕ^*, $L \vdash \phi^* \to \theta$. L_H extends L and is closed under uniform substitution, hence

$$L_H \vdash \Big(\phi \wedge \bigwedge_{\substack{1\leq k\leq n\\ \chi\in Sub(\phi)}} \Diamond\!\!\!\!+(i_k \wedge \chi) \to \boxplus(i_k \to \chi)\Big) \to \theta[\vec{p_i}/\vec{i}]$$

Since $L_H \vdash \Diamond\!\!\!\!+(i \wedge \chi) \to \boxplus(i \to \chi)$ for any nominal i and formula χ, it follows that $L_H \vdash \phi \to \theta[\vec{p_i}/\vec{i}]$, and therefore, $L_H \vdash \phi \to \psi$.

□

It is known that **K**, **GL**, **S5** and **Grz** have uniform interpolation (see [29] and [17]). From Theorem 5.5 and the fact that **GL** and **S5** admit simple filtration, it follows immediately that $\mathbf{K}_H$, $\mathbf{S5}_H$ and $\mathbf{GL}_H$ have uniform interpolation over proposition letters. **Grz** does not admit simple filtration. Nevertheless, the same technique can be applied to $\mathbf{Grz}_H$ as well.

Theorem 5.6 $\mathbf{Grz}_H$ *has uniform interpolation over proposition letters.*

Proof: Consider again the proof of Theorem 5.5. The crux of the proof lies in the fact that the filtration set $Sub(\neg(\phi \to \psi))$ can be split up in two disjoint sets, such that every formula in the first set contains only symbols that occur in ϕ, and every formula in the second set contains only symbols that occur in ψ. As we will now show, the same holds for the filtration set of **Grz**. To see this, recall Example 2.7 and note that

$$\begin{aligned}
\Sigma_{\neg(\phi\to\psi)} &= Sub(\{\neg(\phi \to \psi)\} \cup \{\Diamond(\neg\chi \wedge \Diamond\chi) \mid \Diamond\chi \in Sub(\neg(\phi \to \psi))\}) \\
&= Sub(\{\neg(\phi \to \psi)\} \cup \{\Diamond(\neg\chi \wedge \Diamond\chi) \mid \Diamond\chi \in Sub(\phi) \cup Sub(\psi)\}) \\
&= \{\neg(\phi \to \psi), \phi \to \psi\} \cup Sub(\phi) \cup Sub(\{\Diamond(\neg\chi \wedge \Diamond\chi) \mid \Diamond\chi \in Sub(\phi)\}) \\
&\qquad \cup Sub(\psi) \cup Sub(\{\Diamond(\neg\chi \wedge \Diamond\chi) \mid \Diamond\chi \in Sub(\psi)\}) \\
&= \{\neg(\phi \to \psi), \phi \to \psi\} \cup \Sigma_\phi \cup \Sigma_\psi
\end{aligned}$$

Hence, every formula in $\Sigma_{\neg(\phi\to\psi)}$ is a Boolean combination of formulas in Σ_ϕ and Σ_ψ. The same argument as in the proof of Theorem 5.5 now shows that $\mathbf{Grz}_H$ has the uniform interpolation over proposition letters. □

Recall from Section 2.1 the definition of L^H, i.e., the axiomatic hybrid companion of the modal logic L.

Theorem 5.7 *Let L be a complete modal logic satisfying one of the following conditions:*

(a) L has a master modality and admits filtration.

(b) L is axiomatized by finitely many shallow modal formulas.

Then $L_H = L^H$.

Proof: We only prove (a), since (b) is similar.

That $L^H \subseteq L_H$ is clear (it suffices to observe that $\Diamond\!\!\!\!+(i \wedge \phi) \to \boxplus(i \to \phi)$ is valid, and that the inference rules preserves validity). In the remainder of the proof, we show that $L_H \subseteq L^H$.

Suppose $L_H \vdash \phi$, for some formula ϕ with nominals $i_1, \ldots, i_n$. Let $\Sigma = \Sigma_{\neg\phi[\vec{i}/\vec{p_i}]}$. Then by Corollary 4.4, the formula

$$\Big(\bigwedge_{\substack{1\leq k\leq n\\ \psi\in\Sigma}} \Diamond\!\!\!\!+(p_{i_k} \wedge \psi) \to \boxplus(p_{i_k} \to \psi)\Big) \to \phi[\vec{i}/\vec{p_i}]$$

is L-valid, and hence

$$L \vdash \Big(\bigwedge_{\substack{1 \leq k \leq n \\ \psi \in \Sigma}} \Diamond\!\!\!\!+ (p_{i_k} \wedge \psi) \to \boxplus(p_{i_k} \to \psi) \Big) \to \phi[\vec{i}/\vec{p_i}]$$

Since L^H contains all substitution instances of formulas in L, we have that

$$L^H \vdash \Big(\bigwedge_{\substack{1 \leq k \leq n \\ \psi \in \Sigma}} \Diamond\!\!\!\!+ (i_k \wedge \psi) \to \boxplus(i_k \to \psi) \Big) \to \phi$$

By definition, $L^H \vdash \Diamond\!\!\!\!+ (i \wedge \chi) \to \boxplus(i \to \chi)$ for all i and χ. Since L^H is closed under Modus Ponens, we conclude that $L^H \vdash \phi$. $\Box$

Remark 5.8 Suppose L is a logic that satisfies our conditions (has a master modality and admits filtrations). Furthermore, suppose L is complete for a frame class F. In general we cannot conclude from our results that L^H is complete for F. All we know is that L^H is complete for $\mathsf{Fr}(L)$. Consider the case of **GL**. As is well known, **GL** is not only complete for the class of transitive conversely well-founded frames (which it defines), but also for the class of finite transitive irreflexive trees (*finite trees* for short). By Theorem 5.7 we know that $\mathbf{GL}^H$ is complete for the class of transitive conversely well-founded frames. As it turns out, though, $\mathbf{GL}^H$ is *not* complete for the class of finite trees: the formula

$$\Diamond(p \wedge \Diamond i) \;\wedge\; \Diamond(q \wedge \Diamond i) \;\to\; \Diamond(p \wedge \Diamond q) \;\vee\; \Diamond(q \wedge \Diamond p) \;\vee\; \Diamond(p \wedge q)$$

is valid on finite trees but is not valid on the class of transitive conversely well-founded frames. Hence, it is not derivable in $\mathbf{GL}^H$. We conjecture that if this formula is added as an axiom to $\mathbf{GL}^H$, the resulting logic is complete for finite trees.

6 Conclusions

We addressed the question which properties transfer under passage from a modal logic to its (semantic) hybrid companion. We showed that tabularity transfers and the Beth property does not. Furthermore, we gave simple counterexamples to transfer of complexity, decidability and the finite model property.

Next, we provided satisfiability preserving translations from certain hybrid companion logics to the corresponding modal logics. Using these translations, we obtained transfer results concerning complexity, (uniform) interpolation and axiomatic completeness. These transfer results apply to modal logics with a master modality that admit filtration, and to logics axiomatized by shallow formulas, i.e., formulas in which every occurrence of a proposition letter is under the scope of at most one modal operator.

There are still many questions remaining. The study of this topic could be developed in at least three directions: (1) find other classes (or extend the class of logics we are working with) for which the translation works, (2) see which other properties do or do not transfer from L to L_H, (3) generalize these results to richer hybrid languages. Regarding the first point, our result concerning the logic of symmetric frames **KB** suggests that the class of logics to which the translation applies cannot be easily generalized. Concerning the second point, it was proved in Gargov and Goranko [15] and in [9] that canonicity transfers. Interesting remaining questions are whether interpolation and compactness transfer in general. With respect to the third point, generalizations of the present results to hybrid languages with @-operators can be found in [8].

Acknowledgments. The authors are very grateful to Johan van Benthem, Patrick Blackburn, Maarten Marx and Valentin Goranko for helpful discussions and valuable suggestions as well as to Loredana Afanasiev for carefully reading the paper. We would also like to thank the anonymous referees for their useful comments.

References

[1] Carlos Areces, Patrick Blackburn, and Maarten Marx. Hybrid logics: Characterization, interpolation and complexity. *Journal of Symbolic Logic*, 66(3):977–1010, 2001.

[2] Patrick Blackburn. Nominal tense logic. *Notre Dame Journal of Formal Logic*, 34:56–83, 1993.

[3] Patrick Blackburn. Internalizing labelled deduction. *Journal of Logic and Computation*, 10(1):137 – 168, 2000.

[4] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal logic*. Cambridge University Press, Cambridge, UK, 2001.

[5] Patrick Blackburn and Jerry Seligman. Hybrid languages. *Journal of Logic, Languages and Information*, 4(3):251–272, 1995.

[6] George Boolos. *The Logic of Provability*. Cambridge University Press, 1993.

[7] Balder ten Cate. Interpolation for extended modal languages. *Journal of Symbolic Logic*, 70(1):223–234, 2005.

[8] Balder ten Cate. *Model theory for extended modal languages*. PhD thesis, University of Amsterdam, 2005. ILLC Dissertation Series DS-2005-01.

[9] Balder ten Cate, Maarten Marx, and Petrucio Viana. Hybrid logics with Sahlqvist axioms. *Logic Journal of the IGPL*, 13(3):293–300, 2005.

[10] Alexander Chagrov and Michael Zakharyaschev. *Modal logic*. Oxford University Press, 1997.

[11] Cheng-Chia Chen and I-Peng Lin. The complexity of propositional modal theories and the complexity of consistency of propositional modal theories. In Anil Nerode and Yuri Matiyasevich, editors, *Proceedings of the Third International Symposium on Logical Foundations of Computer Science*, volume 813 of *Lecture Notes in Computer Science*. Springer, 1994.

[12] Giuseppe De Giacomo. *Decidability of Class-Based Knowledge Representation Formalisms*. PhD thesis, Università di Roma "La Sapienza", 1995.

[13] Michael J. Fisher and Richard E. Ladner. Propositional modal logic of programs. In *Proceedings of the ninth annual ACM symposium on theory of computing*, pages 286–294, 1977.

[14] Dov Gabbay, Ágnes Kurucz, Frank Wolter, and Michael Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications*, volume 148 of *Studies in Logic*. North-Holland, 2003.

[15] George Gargov and Valentin Goranko. Modal logic with names. *Journal of Philosophical Logic*, 22:607–636, 1993.

[16] Silvio Ghilardi and Luigi Santocanale. Algebraic and model-theoretic techniques for fusion decidability in modal logic. In Moshe Vardi and Andrei Voronkov, editors, *Logic for Programming, Artificial Intelligence and Reasoning (LPAR 03)*, volume 2850 of *Lecture Notes in Artificial Intelligence*, pages 152–166. Springer Verlag, 2003.

[17] Silvio Ghilardi and Marek Zawadowski. *Sheaves, Games and Model Completions*. Trends in logic. Kluwer Academic, 2002.

[18] Bjarni Jónsson. Algebras whose congruence lattices are distributive. *Math. Scand.*, 21:110–121, 1967.

[19] Marcus Kracht and Frank Wolter. Simulation and transfer results in modal logic – a survey. *Studia Logica*, 59:149–177, 1997.

[20] Oliver Kutz, Frank Wolter, Holger Sturm, Nobu-Yuki Suzuki, and Michael Zakharyaschev. Logics of metric spaces. *ACM Transactions on Computational Logic*, 4(2):260–294, 2003.

[21] Olivier Kutz, Carsten Lutz, Frank Wolter, and Michael Zakharyaschev. E-connections of abstract description systems. *Artificial Intelligence*, 156(1):1–73, 2004.

[22] David Lewis. Intensional logics without iterative axioms. *Journal of Philosophical Logic*, 3:457–466, 1974.

[23] Carsten Lutz. An improved nexptime-hardness result for the description logic $\mathcal{ALC}$ extended with inverse roles, nominals, and counting. LTCS-Report LTCS-04-07, Chair for Automata Theory, Institute for Theoretical Computer Science, Dresden University of Technology, Germany, 2004. See http://lat.inf.tu-dresden.de/research/reports.html.

[24] Solomon Passy and Tinko Tinchev. An essay in combinatory dynamic logic. *Information and computation*, 93:263–332, 1991.

[25] Andrea Schaerf. Reasoning with individuals in concept languages. *Data and Knowledge Engineering*, 13(2):141–176, 1994.

[26] Edith Spaan. *Complexity of Modal Logics*. PhD thesis, ILLC, University of Amsterdam, 1993.

[27] Steven K. Thomason. Reduction of tense logic to modal logic ii. *Theoria*, 41:154–169, 1975.

[28] Stephan Tobies. The complexity of reasoning with cardinality restrictions and nominals in expressive description logics. *Journal of Artificial Intelligence Research*, 12:199–217, 2000.

[29] Albert Visser. Uniform interpolation and layered bisimulation. In *Gödel 1996*, volume 6 of *Lecture Notes Logic*, pages 139–164, Berlin, 1996. Springer.