INSTANTIAL NEIGHBOURHOOD LOGIC

JOHAN VAN BENTHEM, NICK BEZHANISHVILI, SEBASTIAN ENQVIST, JUNHUA YU

ABSTRACT. This paper explores a new language of neighbourhood structures where existential information can be given about what kind of worlds occur in a neighbourhood of a current world. The resulting system of 'instantial neighbourhood logic' INL has a non-trivial mix of features from relational semantics and from neighbourhood semantics. We explore some basic model-theoretic behavior, including a matching notion of bisimulation, and give a complete axiom system for which we prove completeness by a new normal form technique. In addition, we relate INL to other modal logics by means of translations, and determine its precise SAT complexity. Finally, we discuss proof-theoretic fine-structure of INL in terms of semantic tableaux and some expressive fine-structure in terms of fragments, while discussing concrete illustrations of the instantial neighborhood language in topological spaces, in games with powers for players construed in a new way, as well as in dynamic logics of acquiring or deleting evidence. We conclude with some coalgebraic perspectives on what is achieved in this paper. Many of these final themes suggest follow-up work of independent interest.

1. INTRODUCTION

Neighbourhood semantics for modal logic [25, 29, 12, 21] is a generalization of both standard relational semantics and topological semantics, in which worlds are related to sets of worlds called 'neighbourhoods', and a modal formula $\Box\varphi$ holds if at least one neighbourhood has only points satisfying φ. The box modality thus encodes a quantifier pattern of the shape $\exists\forall$. There are many motivations for this generalization, from a desire to model weaker modal logics than the usual minimal system K to representing and reasoning about significant semantic structures that are not simply graph-like. However, when we extend a semantic realm for a language, there is always an inevitable issue of whether that language is still adequate for capturing all structural features of the richer models. In particular, neighbourhood models suggest that the content of neighbourhoods may be important, and for that, we need not just universal, but also existential information: what different kinds of worlds occur in given neighbourhoods? There are several motivations for making such a move to an appropriate richer language, as we shall see in this paper: ranging from topology to games, and from modeling notions of evidence to belief revision. Their common core is a new 'instantial neighbourhood modality' to be defined in the following section.

We will study the resulting extension INL of basic modal logic in quite some detail, and show that it has significant properties, many of them due to the fact that INL mixes features of neighbourhood modal logic with features from relational semantics. To set the scene, in Section 2, we discuss basics of the system: language, truth definition, bisimulation, and some typical model constructions such as tree unraveling. In Section 3 we prove a Hennessy-Milner result for bisimulation, and also give some applications to undefinability. Section 4 presents a sound minimal logic containing the key axioms and rules for reasoning with the neighbourhood modality. Section 5 follows up with a completeness proof based on normal forms and matching canonical models. Section 6 positions INL in between the basic neighbourhood modal logic and normal modal logics on relational models via translations that also allow us to determine the precise complexity of validity. Several further directions are then considered in Section 7, including further topics such as semantic tableaux for

INL as well as links with topology, game logics, dynamic logic of evidence change, and especially, promising extensions toward coalgebra. Section 9 gives our general conclusion.

2. A NEW NEIGHBOURHOOD LANGUAGE

2.1. **Basic definitions.** In this section we introduce a new system of 'instantial neighbourhood logic' INL that generalizes basic modal logic over neighbourhood models, using largely standard modal techniques, though with a number of non-trivial adaptations.

Let Prop be the set of propositional letters. We assume that the propositional language has primitive connectives $\neg, \wedge$, while $\bot, \vee, \rightarrow, \leftrightarrow$ are defined in the standard way. The language of INL extends the propositional language by a modal operator $\square$ creating new formulas

$$\square(\varphi_1, \ldots, \varphi_k; \psi)$$

for any INL-formula ψ and any (possibly empty) finite sequence $\varphi_1, \ldots, \varphi_k$ of INL-formulas. We allow this sequence to have length 0, in which case we write the formula as $\square\psi$.

Definition 2.1 (Modal depth)**.** *The* modal depth *of a formula is defined by induction as follows:* $md(p) = md(\bot) = 0, md(\varphi \wedge \psi) = max\{md(\varphi), md(\psi)\}, md(\neg\varphi) = md(\varphi), md(\square(\varphi_1, \ldots, \varphi_k; \psi)) = 1 + max\{md(\varphi_1), \ldots, md(\varphi_n), md(\psi)\}.$

Definition 2.2 (Neighbourhood frames and models)**.** *A neighbourhood frame is a pair $\mathfrak{F} = (W, N)$ where W is a non-empty set called a* domain*, and $N : W \rightarrow \mathcal{P}\mathcal{P}(W)$ is a map called a* neighbourhood function*, where $\mathcal{P}(W)$ is the powerset of W. A* valuation *on W is a map $V : \mathsf{Prop} \rightarrow \mathcal{P}W$.*

A triple $\mathfrak{M} = (W, N, V)$ is called a neighbourhood model *or a* neighbourhood model based on *(W, N) if (W, N) is a neighbourhood frame and V is a valuation on it.*

Definition 2.3 (Truth of INL-formulas)**.** *Let $\mathfrak{M} = (W, N, V)$ be a neighbourhood model, $w \in W$, and φ a formula. We inductively define when $\mathfrak{M}$ satisfies φ in a state w, written $\mathfrak{M}, w \models \varphi$:*

$\mathfrak{M}, w \models p$	*iff*	$w \in V(p),$
$\mathfrak{M}, w \models \neg\varphi$	*iff*	$\mathfrak{M}, w \not\models \varphi,$
$\mathfrak{M}, w \models \varphi \wedge \psi$	*iff*	$\mathfrak{M}, w \models \varphi$ *and* $\mathfrak{M}, w \models \psi,$
$\mathfrak{M}, w \models \square(\psi_1, ..., \psi_k; \varphi)$	*iff*	*there is $S \in N(w)$ such that for all $s \in S$ we have $\mathfrak{M}, s \models \varphi$ and for all $i \in \{1, ..., k\}$ there is $s_i \in S$ such that $\mathfrak{M}, s_i \models \psi_i$.*

Note that, when the sequence $\psi_1, ..., \psi_k$ is empty, the truth clause for the box becomes: $\mathfrak{M}, w \models \square\varphi$ iff there is $S \in N(w)$ such that for all $s \in S$ we have $\mathfrak{M}, s \models \varphi$. We denote the truth set $\{w \in W : \mathfrak{M}, w \models \varphi\}$ of φ by $V(\varphi)$. As an immediate illustration to give a feel for the truth clauses of the instantial neighbourhoods, consider the diagram shown in Figure 1. The diagram shows a neighbourhood model with a world w (the bottom one in the diagram) in which the formula $\square(\square p, \square\neg p; p)$ is true. (Here, we depict neighbourhoods of a world as "balloons" tied to the corresponding node in the diagram by a dotted path. The light grey shaded rectangle shows the truth set of the variable p.) What this formula means in informal terms is that there is a neighbourhood of the point of evaluation, in which p is true everywhere, and furthermore this neighbourhood contains one point with a neighbourhood in which p is true everywhere, and one point with a neighbourhood in which p is true nowhere.

We say that a formula φ is *satisfiable* in $\mathfrak{M} = (W, N, V)$ if there is $w \in W$ such that $\mathfrak{M}, w \models \varphi$. We say that φ is *satisfiable* if there is a neighbourhood model $\mathfrak{M} = (W, N, V)$ satisfying φ. We say that φ is valid in a neighbourhood frame (W, N) if for each model $\mathfrak{M}$ based on (W, N) and each $w \in W$ we have $\mathfrak{M}, w \models \varphi$, and φ is valid if it is valid in every neighbourhood frame.

Remark 2.4. *By way of positioning our choice of a semantic format, recall the standard semantics of modal operators in neighbourhood models:*

$$\mathfrak{M}, w \models \square\varphi \quad iff \quad V(\varphi) \in N(w),$$
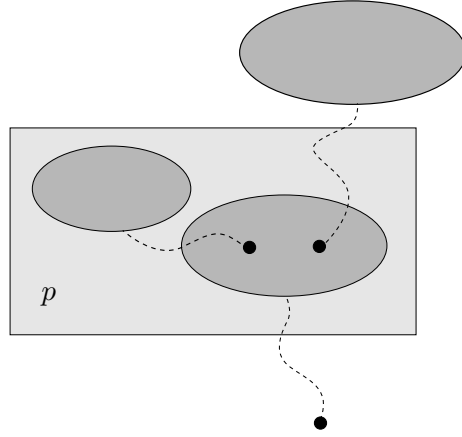$$\mathfrak{M}, w \models \Diamond\varphi \quad iff \quad V(\neg\varphi) \notin N(w).$$

FIGURE 1. Illustration of the modal truth clause

The logic of all neighbourhood models via the latter semantics is the system E, *which only contains the* congruence rule $\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$. *One can view our logic as specializing this semantics by requiring closure of neighborhood families under upward set inclusion, a stipulation that yields a stronger system*EM *with the* monotonicity rule $\frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$. *We refer to* [12] *and* [20] *for details.*

However, one can also view the above truth clause as an independent generalization of the monotonic modal semantics over neighbourhood models:

$$\mathfrak{M}, w \models \Box\varphi \quad iff \quad \text{there is } S \in N(w) \text{ such that for all } s \in S \text{ we have } \mathfrak{M}, s \models \varphi,$$
$$\mathfrak{M}, w \models \Diamond\varphi \quad iff \quad \text{for any } S \in N(w) \text{ there is } s \in S \text{ such that } \mathfrak{M}, s \models \varphi.$$

This fits our intended interpretations for INL *better (see our later discussion in Section 7 of topological spaces, games, and evidence models), and this is why we have chosen this format.*

Still, one could define a semantics for INL *more akin to that for* E, *with a truth clause*

$$\mathfrak{M}, w \models \Box(\psi_1, ..., \psi_k; \varphi) \quad iff \quad \text{there is } S \in N(w) \text{ such that } S = V(\varphi) \text{ and}$$
$$\text{for all } i \in \{1, ..., k\} \text{ there is } s_i \in S \text{ such that } \mathfrak{M}, s_i \models \psi_i.$$

The latter modality is expressible in an extension of the basic modal language in the standard neighbourhood semantics with an existential modality E over worlds, by the formula $\Box\varphi \wedge \bigwedge_{i=1}^{n} E(\psi_i \wedge \varphi)$. This formula will reappear in our discussion of special cases for INL *in Section 7.*

Bisimulations for neighbourhood models that fit our language are defined as follows:

Definition 2.5 (INL-Bisimulation). *Let $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ be neighbourhood models. A binary relation $Z \subseteq W \times W'$ is called a* INL-bisimulation *if for each $(w, w') \in Z$ (alternative notation $w \rightleftharpoons_Z w'$) we have:*

1. $\mathfrak{M}, w \models p$ *iff* $\mathfrak{M}', w' \models p$ *for each proposition letter p,*
2. $\forall S \in N(w) \; \exists S' \in N'(w') \begin{cases} \forall s' \in S' \exists s \in S(s \rightleftharpoons_Z s') \text{ and} \\ \forall s \in S \exists s' \in S'(s \rightleftharpoons_Z s') \end{cases}$
3. $\forall S' \in N'(w') \; \exists S \in N(w) \begin{cases} \forall s \in S \exists s' \in S'(s \rightleftharpoons_Z s') \text{ and} \\ \forall s' \in S' \exists s \in S(s \rightleftharpoons_Z s') \end{cases}$

Following standard terminology, states (or worlds – we will use both of these common terms interchangeably in this paper) w and w' are called INL-*bisimilar (written, $w \rightleftharpoons w'$) if there exists an* INL-*bisimulation Z such that $w \rightleftharpoons_Z w'$.*

2.2. **Warm-up results.** We begin by proving some basic semantic results about INL. Our first result is completely as expected: formulas of INL are invariant under INL-bisimulations.

Definition 2.6 (Modal Equivalence)**.** *Let $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ be neighbourhood models and let $w \in W$ and $w' \in W'$. We say that w and w' are* modally equivalent *(written, $w \leftrightsquigarrow w'$) if for any* INL-*formula α we have $\mathfrak{M}, w \models \alpha$ iff $\mathfrak{M}', w' \models \alpha$.*

Theorem 2.7. *Let $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ be neighbourhood models with $w \in W$ and $w' \in W'$. Then $w \rightleftharpoons w'$ implies $w \leftrightsquigarrow w'$.*

Proof. Suppose there is a bisimulation Z such that $w \rightleftharpoons_Z w'$. We show, by induction on the complexity of INL-formulas as defined earlier, that w and w' agree on all INL-formulas. The cases for proposition letters, $\neg$ and $\wedge$ are trivial. Now suppose $\mathfrak{M}, w \models \Box(\varphi_1, ..., \varphi_k; \psi)$. Then there is $U \in N(w)$ such that for all $u \in U$ we have $\mathfrak{M}, u \models \psi$ and for all $i \in \{1, \dots, k\}$ there is $u_i \in U$ such that $\mathfrak{M}, u_i \models \varphi_i$. Since $w \rightleftharpoons_Z w'$ and $U \in N(w)$, there is $U' \in N'(w')$ such that for all $u' \in U'$ there is $u \in U$ with $u \rightleftharpoons_Z u'$ and for all $u \in U$ there is $u' \in U'$ with $u \rightleftharpoons_Z u'$. Let v be an arbitrary state in U'. Because Z is a bisimulation, there exists $u \in U$ with $u \rightleftharpoons_Z v$. Since for all $u \in U$ we have $\mathfrak{M}, u \models \psi$, and the complexity of ψ is smaller than the complexity of $\Box(\varphi_1, ..., \varphi_k; \psi)$, by the induction hypothesis, we obtain that $\mathfrak{M}', v \models \psi$. Furthermore, for any $i \in \{1, \dots, k\}$, we have that $\mathfrak{M}, u_i \models \varphi_i$ for some $u_i \in U$, and as Z is a bisimulation, for each u_i there is $u'_i \in U'$ such that $u_i \rightleftharpoons_Z u'_i$. The same argument as above entails that for each $i \in \{1, \dots, k\}$ we have $\mathfrak{M}', u'_i \models \varphi_i$. In conclusion, we showed that $\mathfrak{M}', w' \models \Box(\varphi_1, ..., \varphi_k; \psi)$. The other direction is similar. Therefore, we obtain that $w \leftrightsquigarrow w'$. $\qquad\square$

Similarly to bounded morphisms (p-morphisms) in relational semantics, we may also consider special functions that preserve structure of neighbourhood models:

Definition 2.8. *Let $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ be neighbourhood models. Then a map $f : W \to W'$ is called a* bounded morphism *if, for all $u \in W$:*
- *$u \in V(p)$ iff $f(u) \in V'(p)$, for every proposition letter p,*
- *$N'(f(u)) = \{f[Z] \mid Z \in N(u)\}$, where $f[Z]$ is the image of Z for the map f.*

This definition gives the expected result:

Proposition 2.9. *Let $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ be neighbourhood models. Then a map $f : W \to W'$ is a bounded morphism from $\mathfrak{M}$ to $\mathfrak{M}'$ if and only if its graph is an* INL-*bisimulation.*

We have the following immediate corollary of Proposition 2.9 and Theorem 2.7.

Corollary 2.10. *Let $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ be neighbourhood models, and a map $f : W \to W'$ a bounded morphism from $\mathfrak{M}$ to $\mathfrak{M}'$. Then for each $w \in W$ and each* INL-*formula φ,*

$$\mathfrak{M}, w \models \varphi \quad \text{if and only if} \quad \mathfrak{M}', f(w) \models \varphi.$$

Next, we show that some of the basic model theoretic constructions familiar from standard relational semantics have simple counterparts in neighbourhood semantics. For this purpose we shall require the following notion:

Definition 2.11. *Given a neighbourhood model $\mathfrak{M} = (W, N, V)$, we define the* support *relation $S_{\mathfrak{M}} \subseteq W \times W$ by setting, for $u, v \in W$:*

$$(u, v) \in S_{\mathfrak{M}} \text{ if and only if } v \in \bigcup N(u)$$

We let $S_{\mathfrak{M}}^{\star}$ denote the reflexive-transitive closure of $S_{\mathfrak{M}}$.

Using this definition we can introduce an obvious analogue of the standard notion of generated submodels known from basic modal logic:

Definition 2.12. *Let* $\mathfrak{M} = (W, N, V)$ *be any neighbourhood model, and let* $u \in W$. *Then the point-generated submodel of* $\mathfrak{M}$ *at* u, *denoted* $\mathfrak{M}[u]$, *is the structure* (W', N', V') *where:*

- $W' = \{v \in W \mid (u, v) \in S_{\mathfrak{M}}^{\star}\}$,
- $N' = N{\upharpoonright}_{W'}$,
- $V' = V{\upharpoonright}_{W'}$.

More generally, we say that $\mathfrak{M}'$ is a *generated submodel of* $\mathfrak{M}$ if $W' \subseteq W$ and the inclusion map is a bounded morphism. The next proposition states that, as expected, point-generated submodels are generated submodels in this sense.

Proposition 2.13. *For any* $u \in W$, *the obvious inclusion map* $\iota : W' \to W$ *is a bounded morphism from* $\mathfrak{M}[u]$ *to* $\mathfrak{M}$. *Hence, for all formulas* φ, *we have:*

$$\mathfrak{M}, u \models \varphi \text{ iff } \mathfrak{M}[u], u \models \varphi$$

Next, we turn to tree unraveling. Here our goal is to show that INL has the "tree model property," i.e. every satisfiable formula is satisfiable in some tree-like model. The question is exactly what "tree-like" means in this context. Since trees are characterized by the property that there is some distinguished node, the root, from which there is a unique path to any other given node, the question we really need to answer here is what is a "path" in a neighbourhood model. A first answer is to say that a path in a model $\mathfrak{M}$ is any tuple $(w_0, ..., w_n)$ where, for each $i < n$, we have $(w_i, w_{i+1}) \in S_{\mathfrak{M}}$. This suggests the following definition:

Definition 2.14. *A model* $\mathfrak{M}$ *is said to be* tree-like *if the support relation* $S_{\mathfrak{M}}$ *is a tree.*

Definition 2.15. *Let* $\mathfrak{M} = (W, N, V)$ *be any neighbourhood model and let* $u \in W$. *We define the tree-unraveling of* $\mathfrak{M}$ *at* u, *denoted* $\mathfrak{M}_u^U$, *to be the structure* (W', N', V') *where:*

- W' *is the set of all finite sequences* $(v_0, ..., v_n)$ *over* W *with* $v_0 = u$ *and* $(v_i, v_{i+1}) \in S_{\mathfrak{M}}$ *for* $0 \leq i < n$.
- *Given a sequence* $(v_0, ..., v_n)$ *and* $Z' \subseteq W'$ *we set* $Z' \in N'(v_0, ..., v_n)$ *if and only if there is some* $Z \in N(v_n)$ *such that*

$$Z' = \{(v_0, ..., v_n, w) \mid w \in Z\}.$$

- *The valuation* V' *is defined by* $(v_0, ..., v_n) \in V'(p)$ *iff* $v_n \in V(p)$.

Clearly $\mathfrak{M}_u^U$ is a tree-like model, with root u. Furthermore, we have:

Proposition 2.16. *There is a bounded morphism* $f : \mathfrak{M}_u^U \to \mathfrak{M}$ *given by the assignment:*

$$(v_0, ..., v_n) \mapsto v_n$$

From this the desired tree model property follows immediately:

Proposition 2.17. *Any satisfiable formula is satisfiable at the root of some tree-like model.*

A second possibility is to say that a path in $\mathfrak{M}$ is any tuple

$$(w_0, Z_0, w_1, Z_1, ..., Z_{n-1}, w_n)$$

where, for each $i < n$, Z_i is a neighbourhood of w_i and $w_{i+1} \in Z_i$. Let us call such a tuple a *strong path from* w_0 to w_n in $\mathfrak{M}$.

Definition 2.18. *A model* $\mathfrak{M}$ *is said to be* strongly tree-like *if there is some* $w \in W$ *such that for every* $w' \in W$, *there is a unique strong path from* w *to* w' *in* $\mathfrak{M}$.

It is not hard to see that every strongly tree-like model is tree-like, but the converse implication does not hold. Generally, a tree-like model is strongly tree-like if and only if the neighbourhoods of any point are pairwise disjoint, i.e. there are no w, Z, Z' with $Z, Z' \in N(w)$ and $Z \cap Z' \neq \emptyset$. Under the constraint that neighbourhoods are pairwise disjoint in this sense, the two notions of tree-likeness are thus equivalent.

Now let us consider tree unraveling again.

Definition 2.19. *Let $\mathfrak{M} = (W, N, V)$ be any neighbourhood model and let $u \in W$. We define the* strong tree-unraveling *of $\mathfrak{M}$ at u, denoted $\mathfrak{M}_u^S$, to be the structure (W', N', V') where:*

- *W' is the set of all strong paths in $\mathfrak{M}$ beginning with u.*
- *Given a strong path $(v_0, Z_0, ..., Z_{n-1}, v_n)$ and $Z' \subseteq W'$ we set $Z' \in N'(v_0, Z_0, ..., Z_{n-1}, v_n)$ if and only if there is some $Z \in N(v_n)$ such that*

$$Z' = \{(v_0, Z_0, ..., Z_{n-1}, v_n, Z, w) \mid w \in Z\}.$$

- *The valuation V' is defined by $(v_0, Z_0, ..., Z_{n-1}, v_n) \in V'(p)$ iff $v_n \in V(p)$.*

The strong tree-unraveling of any model is strongly tree-like.

Proposition 2.20. *There is a bounded morphism $f : \mathfrak{M}_u^S \to \mathfrak{M}$ given by the assignment:*

$$(v_0, Z_0, ..., Z_{n-1}, v_n) \mapsto v_n$$

We now get the following strengthened version of the tree model property:

Proposition 2.21 (Strong tree model property)**.** *Any satisfiable formula is satisfiable at the root of some strongly tree-like model.*

Finally, we establish a finite-depth property for instantial neighbourhood logic:

Definition 2.22. *Let $\mathfrak{M} = (W, N, V)$ be any neighbourhood model, let $u \in W$, and let k be any integer. Then the* depth k point-generated submodel *of $\mathfrak{M}$ at u, denoted by $\mathfrak{M}[u, k]$, is defined to be the structure (W', N', V') where:*

- *W' is the set of all $v \in W$ such that there is a $S_{\mathfrak{M}}$-path from u to v of length $\leq k$.*
- *For $v \in W'$, we set $N'(v) = N(v)$ if there is a $S_{\mathfrak{M}}$-path from u to v of length $< k$, and $N'(v) = \emptyset$ otherwise. That is, $N'(v) = \emptyset$ if the shortest $S_{\mathfrak{M}}$-path from u to v has length k.*
- *$V' = V \upharpoonright_{W'}$.*

A fairly simple argument will establish the following:

Fact 2.23. *Let φ be any formula of modal depth $\leq k$. Then:*

$$\mathfrak{M}, u \models \varphi \text{ iff } \mathfrak{M}[u, k], u \models \varphi$$

Corollary 2.24 (Finite-height model property)**.** *Any satisfiable formula of modal depth k is satisfiable at the root of some tree-like model with an underlying tree of height at most k.*

3. Hennessy-Milner property on finite models

In this section we show that our notion of bisimulation makes sense, by proving a Hennessy-Milner theorem for INL. More precisely, we show that over finite models, the relation of modal equivalence is an INL-bisimulation. The restriction to finite models is not a best possible result relating modal equivalence of models to bisimulation, but it will serve the purposes of this paper.

Theorem 3.1. *Let $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ be finite neighbourhood models with $w \in W$ and $w' \in W'$. Then $w \leftrightsquigarrow w'$ iff $w \rightleftharpoons w'$.*

Proof. Our aim is to show that the relation $\leftrightsquigarrow$ is an INL-bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$. We show only one direction of the back-and-forth conditions, since the other direction can be proved using a symmetric argument.

First, pick any $w \in W$ and $w' \in W'$. We define a formula $\delta(w, w')$ as follows:

- If $w \leftrightsquigarrow w'$, set $\delta(w, w') = \top$.
- Otherwise, pick some formula φ such that $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}', w' \models \neg\varphi$ and set $\delta(w, w') = \varphi$.

So $\delta(w, w')$ either provides some witness that w and w' are *not* modally equivalent, or is set to $\top$ if they are modally equivalent. Now, for $w \in W$, set

$$\chi(w) := \bigwedge \{\delta(w, w') \mid w' \in W'\}$$

This conjunction is well defined because W' is a finite set. Note that $\mathfrak{M}, w \models \chi(w)$ for all $w \in W$. It is easy to prove the following claim:

Claim: For all $w \in W$ and all $w' \in W'$, we have:

$$w \leftrightsquigarrow w' \text{ iff } \mathfrak{M}', w' \models \chi(w).$$

Now suppose that $w \leftrightsquigarrow w'$, for $w \in W$ and $w' \in W'$. Let U be a neighbourhood of w, say $U = \{u_0, ..., u_k\}$. It is easy to see that we have:

$$\mathfrak{M}, w \models \Box(\chi(u_0), ..., \chi(u_k); \chi(u_0) \vee ... \vee \chi(u_k)).$$

Hence, we must also have

$$\mathfrak{M}', w' \models \Box(\chi(u_0), ..., \chi(u_k); \chi(u_0) \vee ... \vee \chi(u_k)).$$

This means there must be some neighbourhood U' of w' such that every formula $\chi(u_i)$ is true somewhere in U', and conversely every member of U' satisfies $\chi(u_i)$ for some $i \leq k$. By the Claim, it follows that the relation $\leftrightsquigarrow$ satisfies the appropriate back-and-forth conditions with respect to the sets U and U'. $\qquad\square$

Much more can be said about INL-bisimulations. For further notions and results, we refer to Section 7. The formulas that appear in our proof of the Hennessy-Milner theorem bear a strong resemblance to the normal forms for INL-formular that we etablish later in Section 5, which play a crucial role in the completeness proof for our axiomatization of INL. The proof also has a coalgebraic flavor, and this theme will be taken up in Section 7.5.

For now, however, we pursue another question. How does INL relate to more standard modal languages on neighbourhood models? To see this, we recall the notion of bisimulation for monotonic modal logic on neighbourhood models. This notion is folklore, but see [28] and [20] for details.

Definition 3.2 (Monotonic Bisimulation). *Let $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ be neighbourhood models. A binary relation $Z \subseteq W \times W'$ is called a* monotonic bisimulation *if for each $(w, w') \in Z$ (alternative notation $w \underline{\leftrightarrow}_Z w'$) we have:*

1. $\mathfrak{M}, w \models p$ *iff* $\mathfrak{M}', w' \models p$ *for each proposition letter p.*
2. $\forall S \in N(w) \, \exists S' \in N'(w') \forall s' \in S' \, \exists s \in S$ *such that* $(s \underline{\leftrightarrow}_Z s')$.
3. $\forall S' \in N'(w') \, \exists S \in N(w) \forall s \in S \, \exists s' \in S'$ *such that* $(s \underline{\leftrightarrow}_Z s')$.

It is easy to check that each INL-bisimulation is also a monotonic bisimulation. Below we will give an example that the converse, in general, is not true. Since monotonic bisimilarity entails modal equivalence for the basic modal language, this implies that over the neighbourhood models INL is more expressive than the standard modal language.

Example 3.3. We will now give an example of two neighbourhood models and states in these models which are bisimilar in the standard sense, but which are not INL-bisimilar. Start by choosing $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ where $W = \{w_0, w_1, w_2\}$, $N(w_0) = \{\{w_1\}, \{w_1, w_2\}\}$, $V(p) = \{w_1, w_2\}$, $V(q) = \{w_2\}$ and $W' = \{w'_0, w'_1\}$, $N'(w'_0) = \{w'_1\}$, $V'(p) = \{w'_1\}$, $V'(q) = \emptyset$.

Then it is easy to see that w_0 and $w_{0'}$ are monotonic bisimilar (consider a relation linking w_0 with w_0' and w_1 with w_1'), but not INL-bisimilar. To see this, by Theorem 3.1, it is sufficient to note that $\mathfrak{M}, w_0 \models \Box(q; p)$, but $\mathfrak{M}', w_0' \not\models \Box(q; p)$.
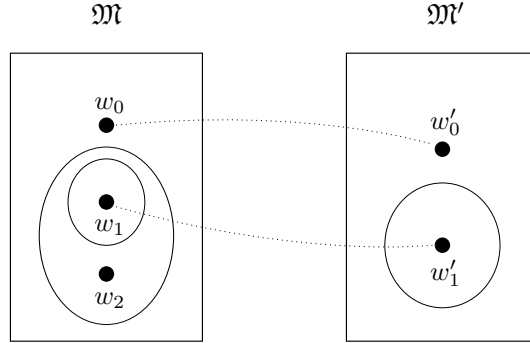

FIGURE 2. Bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$

From Example 3.3 we immediately derive

Proposition 3.4. *Over neighbourhood models,* INL *is not definable in the basic modal language.*

Remark 3.5. The interpretation of the box-operator in the basic neighborhood language has an existential-universal quantifier pattern, but in INL this fixed pattern is loosened up. In particular, we can define a universal-universal box $\boxtimes$ by the definition $\boxtimes\varphi = \neg\Box(\neg\varphi; \top)$. One might ask if the extension of the basic language with this operator yields the full expressive power of INL. However, following a discussion that took place after the submission of this paper, D. Klein, O. Roy, and the fourth author have shown that INL (even its "single-instance fragment", see Section 7.2) is not definable in the enhanced basic modal language with both $\Box$ and $\boxtimes$. Many further extensions of the basic modal language can be studied with the bisimulation techniques presented here.

In an opposite direction, with bisimulations for INL in place we can also easily prove limitative results on the expressive power of INL. As a straightforward example of this, we will show that the universal modality is not definable in the INL-language. Recall that the standard truth condition for the *global modalities* E (the existential version) and A (the universal version) reads as follows:

$$\mathfrak{M}, w \models E\varphi \quad \text{iff} \quad \text{there is } s \text{ in } \mathfrak{M} \text{ such that } \mathfrak{M}, s \models \varphi.$$
$$\mathfrak{M}, w \models A\varphi \quad \text{iff} \quad \text{for any } s \text{ in } \mathfrak{M} \text{ we have } \mathfrak{M}, s \models \varphi.$$

Example 3.6. Let $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ be neighbourhood models such that $W = \{w, v\}$, $N(w) = \emptyset$, $V(p) = \{v\}$, and $W' = \{w'\}$, $N(w') = \emptyset$, $V'(p) = \emptyset$. Then it is easy to see that $Z = \{(w, w')\}$ is an INL-bisimulation. On the other hand, $\mathfrak{M}, w \models Ep$, while $\mathfrak{M}, w' \not\models Ep$.

Thus, we arrive at the following proposition.

Proposition 3.7. *Over neighbourhood models, the universal modality is not definable in* INL*.*

4. Axiomatization and soundness

In this section we provide a Hilbert-style proof system for INL, prove soundness and derive some basic properties. Completeness of the axioms will be shown later in Section 5.

Our basic proof calculus has as axioms schemes all propositional tautologies, together with the following axiom schemes:

(R-Mon) $\Box(\gamma_1, ..., \gamma_j; \psi) \to \Box(\gamma_1, ..., \gamma_j; \psi \vee \chi),$

(L-Mon) $\Box(\gamma_1, ..., \gamma_j, \varphi; \psi) \to \Box(\gamma_1, ..., \gamma_j, \varphi \vee \chi; \psi),$

(Inst) $\Box(\gamma_1, ..., \gamma_j, \varphi; \psi) \to \Box(\gamma_1, ..., \gamma_j, \varphi \wedge \psi; \psi),$

(Norm) $\neg\Box(\bot; \psi),$

(Case) $\Box(\gamma_1, ..., \gamma_j; \psi) \to \Box(\gamma_1, ..., \gamma_j, \delta; \psi) \vee \Box(\gamma_1, ..., \gamma_j; \psi \wedge \neg\delta),$

(Weak) $\Box(\gamma_1, ..., \gamma_j, \varphi, \delta_1, ..., \delta_n; \psi) \to \Box(\gamma_1, ..., \gamma_j, \delta_1, ..., \delta_n; \psi),$

(Dupl) $\Box(\gamma_1, ..., \gamma_j, \delta_1, ..., \delta_n; \psi) \to \Box(\gamma_1, ..., \gamma_j, \varphi, \delta_1, ..., \delta_n; \psi),$ provided $\varphi \in \{\gamma_1, ..., \gamma_j, \delta_1, ..., \delta_n\}.$

The system has only two rules, modus ponens and substitution of equivalents:

(MP) $\dfrac{\alpha \to \beta \quad \alpha}{\beta},$

(RE) $\dfrac{\alpha \leftrightarrow \beta \quad \varphi}{\varphi[\alpha/\beta]}.$

where $\varphi[\alpha/\beta]$ is the result of possibly replacing some occurrences of α in φ by β.

Explanation. To understand these principles intuitively, note that some just express the set character of the finite sequence of instances at the start of our modalities, others state obvious upward monotonicity properties in all arguments, while we also have distribution over disjunction for instanced formulas. In addition, (Inst) expresses the compatibility of the universal condition with all instances, while (Case) shows how we can either add instances to a given modality witnessing a formula δ or strengthen the universal condition of the modality with the negation of δ.

The following simple lemma will simplify some of our proofs.

Lemma 4.1. *The following rules are admissible in the proof system:*

(TR) $\dfrac{\alpha \to \beta \quad \beta \to \gamma}{\alpha \to \gamma}$

(MT) $\dfrac{\alpha \to \beta \quad \neg\beta}{\neg\alpha}$

We write $\vdash \varphi$ to say that the formula φ is provable in this system, and $\Gamma \vdash \varphi$ as a shorthand for $\vdash \bigwedge \Gamma \to \varphi$ if Γ is a finite set of formulas. We are now ready to derive some basic schemes and admissible rule schemes of the logic INL.

Lemma 4.2. *The following theorem and rule schemes are provable ('admissible') in* INL.

(1) $\Box(\gamma_1, ..., \gamma_j, \theta, \varphi, \delta_1, ..., \delta_m; \psi) \to \Box(\gamma_1, ..., \gamma_j, \varphi, \theta, \delta_1, ..., \delta_m; \psi)$,

(2) $\neg\Box(\gamma_1, ..., \gamma_j, \bot, \delta_1, ..., \delta_m; \psi)$,

(3) $\neg\Box(\gamma; \bot)$,

(4) $\dfrac{\psi \to \chi}{\Box(\gamma_1, ..., \gamma_j; \psi) \to \Box(\gamma_1, ..., \gamma_j; \chi)}$,

(5) $\dfrac{\alpha \to \beta}{\Box(\gamma_1, ..., \gamma_j, \alpha; \psi) \to \Box(\gamma_1, ..., \gamma_j, \beta; \psi)}$,

(6) $\Box(\gamma_1, ..., \gamma_j, \alpha \vee \beta; \psi) \to \Box(\gamma_1, ..., \gamma_j, \alpha; \psi) \vee \Box(\gamma_1, ..., \gamma_j, \beta; \psi)$.

Proof. For (1), use (TR) to chain together the following implications:

$$\begin{array}{llll}
\Box(\gamma_1, ..., \gamma_j, \theta, \varphi, \delta_1, ..., \delta_m; \psi) & \vdash & \Box(\gamma_1, ..., \gamma_j, \varphi, \theta, \varphi, \delta_1, ..., \delta_m; \psi) & \text{(Dupl)} \\
\Box(\gamma_1, ..., \gamma_j, \varphi, \theta, \varphi, \delta_1, ..., \delta_m; \psi) & \vdash & \Box(\gamma_1, ..., \gamma_j, \varphi, \theta, \delta_1, ..., \delta_m; \psi) & \text{(Weak)}
\end{array}$$

For (2), just apply (MT) to the axiom $\neg\Box(\bot; \psi)$ and the following implication which is obtained by repeated applications of (Weak):

$$\Box(\gamma_1, ..., \gamma_j, \bot, \delta_1, ..., \delta_m; \psi) \to \Box(\bot; \psi)$$

We derive (3) as follows:

$$\begin{array}{llll}
\Box(\gamma; \bot) & \vdash & \Box(\gamma \wedge \bot; \bot) & \text{(Inst)} \\
 & \vdash & \gamma \wedge \bot \leftrightarrow \bot & \text{Classical logic} \\
\Box(\gamma; \bot) & \vdash & \Box(\bot, \bot) & \text{(RE)} \\
 & \vdash & \neg\Box(\bot; \bot) & \text{(Norm)} \\
 & \vdash & \neg\Box(\gamma; \bot) & \text{(MT)}
\end{array}$$

For (4), suppose that $\vdash \varphi \to \chi$. We derive the conclusion of the rule as a theorem as follows:

$$\begin{array}{llll}
\Box(\gamma_1, ..., \gamma_j; \varphi) & \vdash & \Box(\gamma_1, ..., \gamma_j; \varphi \vee \chi) & \text{(R-Mon)} \\
 & \vdash & \chi \leftrightarrow \varphi \vee \chi & \text{Classical logic + assumption} \\
\Box(\gamma_1, ..., \gamma_j; \varphi) & \vdash & \Box(\gamma_1, ..., \gamma_j; \chi) & \text{(RE)}
\end{array}$$

(5) can be derived similarly. Finally, for (6):

1. $\vdash \Box(\gamma_1, ..., \gamma_j, \alpha \vee \beta; \psi) \to \Box(\gamma_1, ..., \gamma_j, \alpha \vee \beta, \alpha; \psi) \vee \Box(\gamma_1, ..., \gamma_j, \alpha \vee \beta; \psi \wedge \neg\alpha)$ (Case)
2. $\vdash \Box(\gamma_1, ..., \gamma_j, \alpha \vee \beta, \alpha; \psi) \to \Box(\gamma_1, ..., \gamma_j, \alpha; \psi)$ (Weak)
3. $\vdash \Box(\gamma_1, ..., \gamma_j, \alpha \vee \beta; \psi \wedge \neg\alpha) \to \Box(\gamma_1, ..., \gamma_j, (\alpha \vee \beta) \wedge (\psi \wedge \neg\alpha); \psi \wedge \neg\alpha)$ (Inst), 2
4. $\vdash (\alpha \vee \beta) \wedge (\psi \wedge \neg\alpha) \to \beta$ Classical logic
5. $\vdash (\psi \wedge \neg\alpha) \to \psi$ Classical logic
6. $\vdash \Box(\gamma_1, ..., \gamma_j, (\alpha \vee \beta) \wedge (\psi \wedge \neg\alpha); \psi \wedge \neg\alpha) \to \Box(\gamma_1, ..., \gamma_j, \beta; \psi)$ (4),(5)
7. $\vdash \Box(\gamma_1, ..., \gamma_j, \alpha \vee \beta; \psi \wedge \neg\alpha) \to \Box(\gamma_1, ..., \gamma_j, \beta; \psi)$ (TR), 3
8. $\vdash \Box(\gamma_1, ..., \gamma_j, \alpha \vee \beta; \psi) \to \Box(\gamma_1, ..., \gamma_j, \alpha; \psi) \vee \Box(\gamma_1, ..., \gamma_j, \beta; \psi)$ 1,2,7, C.l.

$\square$

The proof system for INL presented above is sound and complete. Completeness will be proved in the next section. Here, we prove that the system is sound:

Theorem 4.3 (Soundness of INL)**.** *If* $\vdash \varphi$, *then* φ *is valid in the neighbourhood semantics.*

Proof. Since our two inference rules clearly preserve validity, we only have to check that all our axioms are valid. Most of the axioms in the above list are straightforward to check, so we give only the proofs for (Inst) and (Case).

For (Inst), we want to show that $\Box(\gamma_1, ..., \gamma_j, \varphi; \psi) \to \Box(\gamma_1, ..., \gamma_j, \varphi \wedge \psi; \psi)$ is a valid formula. Take an arbitrary model $\mathfrak{M} = (W, N, V)$ and some state $w \in W$. Assume $w \models \Box(\gamma_1, ..., \gamma_j, \varphi; \psi)$, then there is $S \in N(w)$ such that (1) ψ is satisfied at each state in S, (2) γ_i is satisfied at some state in S for each $i \in \{1, ..., j\}$, and (3) φ is satisfied at some state, say u in S. Since $u \in S$, by (1) we also have $u \models \psi$, and hence $u \models \varphi \wedge \psi$. Therefore, $w \models \Box(\gamma_1, ..., \gamma_j, \varphi \wedge \psi; \psi)$.

For (Case), we want to show validity of the formula

$$\Box(\gamma_1, ..., \gamma_j; \psi) \to \Box(\gamma_1, ..., \gamma_j, \delta; \psi) \vee \Box(\gamma_1, ..., \gamma_j; \psi \wedge \neg\delta)$$

Again take an arbitrary model $\mathfrak{M} = (W, N, V)$ and state $w \in W$. Assume that $w \models \Box(\gamma_1, ..., \gamma_j; \psi)$. Then there is $S \in N(w)$ such that (1) ψ is satisfied at each state in S, (2) γ_i is satisfied at some state in S for each $i \in \{1, ..., j\}$. Now, assume furthermore that the formula $\Box(\gamma_1, ..., \gamma_j, \delta; \psi)$ is *not* satisfied at w. Then it must be the case that δ is satisfied nowhere in S, so $\neg\delta$ is true everywhere in S, giving $w \models \Box(\gamma_1, ..., \gamma_j, \psi \wedge \neg\delta)$. From the assumption that $w \models \Box(\gamma_1, ..., \gamma_j; \psi)$ we have now derived that

$$w \models \Box(\gamma_1, ..., \gamma_j, \delta; \psi) \vee \Box(\gamma_1, ..., \gamma_j; \psi \wedge \neg\delta),$$

which concludes the argument. $\qquad\square$

5. Normal forms and completeness

5.1. **A normal form theorem.** Our goal in this section is to prove a completeness theorem for the axiom system presented above. The proof will proceed via a normal form theorem.

Throughout the section, we assume a fixed well-ordering over the set of all formulas of INL. With this in mind, the following notation will be useful: given a finite set of formulas Γ and a formula φ, we denote by $\Box(\Gamma; \varphi)$ the formula $\Box(\psi_1, ..., \psi_n; \varphi)$ where $\psi_1, ..., \psi_n$ is a list of all the formulas in Γ, without repetitions, in the order dictated by our fixed well-ordering over the language.

Lemma 5.1. *For any formula of the form $\Box(\psi_1, ..., \psi_n; \varphi)$, we have:*

$$\vdash \Box(\psi_1, ..., \psi_n; \varphi) \leftrightarrow \Box(\{\psi_1, ..., \psi_n\}; \varphi)$$

Proof. Use (Weak) and (Dupl) to remove repetitions, and use (1) to rearrange the formulas $\psi_1, ..., \psi_n$ in the order consistent with the fixed well-order over the language. $\qquad\square$

As a first application this leads to the following lemma:

Lemma 5.2. *Let $i \in \omega$ and let P be a finite set of propositional variables. Then there are, up to provable equivalence, only finitely many formulas of modal depth $\leq i$ and variables in P.*

Proof. A straightforward induction on the modal depth of a formula. The crucial step is to show that there are only finitely many formulas up to provable equivalence of the form $\Box(\psi_1, ..., \psi_n; \varphi)$ where $\psi_1, ..., \psi_n, \varphi$ are all of depth $\leq k$. But by the previous lemma, each such formula is equivalent to a formula of the form $\Box(\Gamma; \varphi)$ where φ is of depth $\leq k$ and Γ is a finite *set* of formulas of depth $\leq k$. It is now easy see that the number of formulas up to provable equivalence of the form $\Box(\psi_1, ..., \psi_n; \varphi)$ where $\psi_1, ..., \psi_n, \varphi$ are all of modal depth $\leq k$ is bounded by $2^m \times m$, where m is the number of formulas of modal depth $\leq k$ up to provable equivalence. $\qquad\square$

Definition 5.3. *Let P be a finite set of propositional variables, and let i be any natural number. An i-type over P is a consistent set of formulas Γ of modal depth $\leq i$ and variables in P such that, for every formula ψ of depth $\leq i$ and variables in P, we have $\psi \in \Gamma$ or $\neg\psi \in \Gamma$. In other words: an i-type over P is a maximal consistent set of formulas of modal depth $\leq i$ and variables in P.*

We denote the set of i-types over P by $T(i, P)$. We can associate a formula with each i-type τ as follows: for each formula in τ, pick the least formula that is provably equivalent to it. The collection of all these formulas is then finite by Lemma 5.2, so we can take the conjunction of all of them and denote it by $\hat{\tau}$. If Γ is a set of i-types then we write $\hat{\Gamma} = \{\hat{\tau} \mid \tau \in \Gamma\}$.

Lemma 5.4. *Let φ be any formula with variables from P and modal depth at most i. Then:*

$$\vdash \varphi \leftrightarrow \bigvee\{\hat{\tau} \mid \tau \in T(i, P) \,\&\, \varphi \in \tau\}$$

We are now ready for our normal form theorem:

Theorem 5.5. *Let $\varphi, \psi_1, ..., \psi_k$ be formulas with variables in P and all of modal depth $\leq i$. Then the formula $\Box(\{\psi_1, ..., \psi_k\}, \varphi)$ is provably equivalent to a formula of the form:*

$$\bigvee\{\Box(\hat{\Gamma}; \bigvee\hat{\Gamma}) \mid \Gamma \in F\}$$

where F is a family of sets of i-types over P such that, for each $\Gamma \in F$, the following facts hold:
  (1) *For each $j \in \{1, ..., k\}$ there is some $\tau \in \Gamma$ with $\psi_j \in \tau$.*
  (2) *For each $\tau \in \Gamma$ we have $\varphi \in \tau$.*

Proof. We know that $\Box(\psi_1, ..., \psi_k; \varphi)$ is provably equivalent to $\Box(\{\psi_1, ..., \psi_k\}; \varphi)$. Note that we can assume that $\Box(\{\psi_1, ..., \psi_k\}; \varphi)$ is consistent, since otherwise it is provably equivalent to $\bigvee\emptyset$, which is a disjunction of the right shape. We perform the reduction in two steps.

Claim 1: The formula $\Box(\{\psi_1, ..., \psi_k\}; \varphi)$ is provably equivalent to a formula of the form:

$$\bigvee\{\Box(\hat{\Gamma}; \varphi) \mid \Gamma \in F\}$$

where F is a family of sets of i-types over P such that conditions (1) and (2) hold.

Claim 2: Any formula of the form $\Box(\hat{\Gamma}; \varphi)$, where Γ is a set of i-types over P each containing φ, is provably equivalent to a disjunction of formulas of the form $\Box(\hat{\Phi}; \bigvee\hat{\Phi})$, where each Φ is a set of i-types over P for which conditions (1) and (2) hold.

The theorem obviously follows from these two claims, so we prove them one by one.

For Claim 1: First, we use Lemma 5.4 together with the uniform substitution rule (RE) to rewrite the formula $\Box(\{\psi_1, ..., \psi_k\}; \varphi)$ as:

$$\Box(\{\bigvee\hat{\Psi}_1, ..., \bigvee\hat{\Psi}_k\}; \varphi)$$

where, for each $j \in \{1, ..., k\}$, we denote by Ψ_j the set $\{\tau \in T(i, P) \mid \psi_j \in \tau\}$. At this point, we can repeatedly apply the theorem scheme (6) to obtain the equivalent disjunction

$$\bigvee\{\Box(\{\hat{\tau}_1, ..., \hat{\tau}_k\}; \varphi) \mid \tau_1 \in \Psi_1 \,\&\, ... \,\&\, \tau_k \in \Psi_k\}$$

Clearly condition (1) holds for each set $\{\tau_1, ..., \tau_k\}$. All that is left to do to finish Claim 1 is to remove those disjuncts of the form $\Box(\{\hat{\tau}_1, ..., \hat{\tau}_k\}; \varphi)$ where for some $j \in \{1, ..., k\}$, $\varphi \notin \tau_j$. But for each such disjunct, we have $\neg\varphi \in \tau_j$, so by (Weak) and (L-Mon):

$$\vdash \Box(\{\hat{\tau}_1, ..., \hat{\tau}_k\}; \varphi) \to \Box(\neg\varphi; \varphi)$$

But then we can apply (Inst) together with (Norm) to show that the disjunct provably entails $\bot$. Hence, the whole disjunction is provably equivalent to the smaller disjunction where this disjunct is removed, by basic propositional logic. With this observation, we are finished with Claim 1.

For Claim 2: Let $\Box(\hat{\Gamma}; \varphi)$ be a formula of the shape described in the premise of the claim. Again, we apply Lemma 5.4 to rewrite the formula as $\Box(\hat{\Gamma}; \bigvee\hat{\Phi})$ where $\Phi = \{\tau \in T(i, P) \mid \varphi \in \tau\}$. Generally, call a formula of the shape $\Box(\hat{\Gamma}; \bigvee\hat{\Phi})$ where Γ and Φ are sets of i-types over P such that $\Gamma \subseteq \Phi$ an *almost normal box formula*. For any almost normal box formula $\Box(\hat{\Gamma}; \bigvee\hat{\Phi})$, let the *error degree* of $\Box(\hat{\Gamma}; \bigvee\hat{\Phi})$ be defined as the size of the set $\Phi \setminus \Gamma$. Since there are only finitely many

i-types over P, the error degree of an almost normal box formula is always a finite ordinal, which equals 0 if and only if $\Gamma = \Phi$. So, to prove Claim 2, it now suffices to show:

Claim 3: Any almost normal box formula $\Box(\hat{\Gamma}; \bigvee \hat{\Phi})$ of error degree m, where $m > 0$, is provably equivalent to a disjunction of almost normal box formulas of error degree $m - 1$. Also, for each disjunct $\Box(\hat{\Gamma}'; \bigvee \hat{\Phi}')$ of this disjunction, we have $\Gamma \subseteq \Gamma'$ and $\Phi' \subseteq \Phi$.

To prove Claim 3, let $\Box(\hat{\Gamma}; \bigvee \hat{\Phi})$ be an almost normal box formula of error degree m. Since $m > 0$ there is some $\tau \in \Phi \setminus \Gamma$. Using the axiom (Case) we find:

$$\vdash \Box(\hat{\Gamma}; \bigvee \hat{\Phi}) \leftrightarrow (\Box(\hat{\Gamma} \cup \{\hat{\tau}\}; \bigvee \hat{\Phi}) \vee \Box(\hat{\Gamma}; \bigvee \hat{\Phi} \wedge \neg \hat{\tau}))$$

More precisely, the direction from left to right follows from (Case), and the converse direction follows using classical logic, (4) and (Weak). But clearly, the formula $\bigvee \hat{\Phi} \wedge \neg \hat{\tau}$ is provably equivalent to $\bigvee(\hat{\Phi} \setminus \{\hat{\tau}\})$ just by propositional logic. So the whole disjunction on the right side takes the shape:

$$\Box(\hat{\Gamma} \cup \{\hat{\tau}\}; \bigvee \hat{\Phi}) \vee \Box(\hat{\Gamma}; \bigvee(\hat{\Phi} \setminus \{\hat{\tau}\}))$$

Each of these two disjuncts is an almost normal box formula of error degree $m - 1$, and so we are done with the proof of Claim 3, and the proof of the theorem is complete. $\qquad\square$

5.2. **The model construction.** Fix a finite set of propositional variables P. We construct a canonical model $\mathfrak{M} = (W, N, V)$ as follows. Let W be the (disjoint) union of all the type sets $T(i, P)$ for $i \in \omega$. The neighbourhood structure N is now defined by setting $N(\tau) = \emptyset$ for $\tau \in T(0, P)$ and, for $\tau \in T(i, P)$ with $i > 0$:

$$N(\tau) = \{\Gamma \subseteq T(i - 1, P) \mid \Box(\hat{\Gamma}; \bigvee \hat{\Gamma}) \in \tau\}$$

Finally, for $p \in P$ we set $V(p) = \{\tau \in W \mid p \in \tau\}$.

Lemma 5.6 (Truth lemma). *Let φ be a formula with propositional variables only in the set P, and of modal depth at most i. Then for any $\tau \in T(i, P)$, we have:*

$$\mathfrak{M}, \tau \models \varphi \text{ iff } \varphi \in \tau$$

Proof. The proof proceeds by induction on i. The case for $i = 0$ is simple and left to the reader.

Suppose the lemma holds for i, and consider an $i + 1$-type τ. It suffices to show that the lemma holds for all formulas of the form $\Box(\psi_1, ..., \psi_k; \varphi)$, where $\psi_1, ..., \psi_k$ and φ are all of depth i. A simple argument will then extend the claim to arbitrary formulas of depth $i + 1$.

First, suppose that $\Box(\psi_1, ..., \psi_k; \varphi) \in \tau$. By Theorem 5.5, this formula is then equivalent to a disjunction of the form

$$\bigvee \{\Box(\hat{\Gamma}; \bigvee \hat{\Gamma}) \mid \Gamma \in F\}$$

where F is a family of sets of i-types over P satisfying conditions (1) and (2). So at least one of these disjuncts $\Box(\hat{\Gamma}; \bigvee \hat{\Gamma})$ must be in τ since it is an $i + 1$-type, which means that $\Gamma \in N(\tau)$. Using conditions (1) and (2) for Γ, together with the induction hypothesis applied to all i-types, we now easily show that $\mathfrak{M}, \tau \models \Box(\psi_1, ..., \psi_k; \varphi)$.

Conversely, suppose that $\mathfrak{M}, \tau \models \Box(\psi_1, ..., \psi_k; \varphi)$. Using the induction hypothesis for i-types, we see that there is a set of i-types $\Gamma \in N(\tau)$ such that $\varphi \in \bigcap \Gamma$ and, for every $j \in \{1, ..., k\}$, there is some $\tau_j \in \Gamma$ with $\psi_j \in \tau_j$. But then $\Box(\hat{\Gamma}; \bigvee \hat{\Gamma}) \in \tau$. To show that $\Box(\psi_1, ..., \psi_k; \varphi) \in \tau$ it now suffices to show the following:

$$\vdash \Box(\hat{\Gamma}; \bigvee \hat{\Gamma}) \rightarrow \Box(\psi_1, ..., \psi_k; \varphi)$$

But since $\varphi \in \bigcap \Gamma$ we clearly have $\vdash \bigvee \hat{\Gamma} \rightarrow \varphi$, and if $\psi_j \in \tau_j$ then $\vdash \hat{\tau}_j \rightarrow \psi_j$. We now only have to apply (4), (5) and (Weak) to obtain the required implication, and so we are done. $\qquad\square$

Theorem 5.7 (Completeness). *For any formula φ, if $\models \varphi$ then $\vdash \varphi$.*

Proof. Suppose $\nvdash \varphi$, so that $\neg\varphi$ is consistent. Let i be the modal depth of φ and let P be the propositional variables that occur in φ. Construct the model $\mathfrak{M}$ as above. By Lemma 5.4, there is an i-type τ with $\neg\varphi \in \tau$, hence $\varphi \notin \tau$, and by the Truth Lemma, we have $(\mathfrak{M}, \tau) \nvDash \varphi$. Therefore $\nvDash \varphi$, which completes the proof. $\qquad\square$

This completeness proof was inspired by normal form techniques from [15, 26], suitably adapted to modal neighbourhood semantics. We have been unable to find a more standard Henkin construction to establish our main result, and leave this as an open problem.

Finally, our proof method clearly produces finite counter-examples to non-derivable formulas φ, whose size can be effectively computed from that of φ. Therefore we have established an effective finite model property, and in particular:

Corollary 5.8 (Decidability)**.** *Derivability and validity for the logic* INL *are decidable.*

6. Translations and complexity

Our new logic extends earlier systems of modal logic, but what is the precise formal relation? The answer lies in a number of faithful translations. In this section we show how to embed the basic modal logic K and the monotonic neighbourhood logic EM into INL. We also embed INL into the bimodal logic $K \oplus K$ and eventually even into K itself.

All the proofs to follow are based on semantic arguments, and could be used as semantic validity reductions. However, as all the aforementioned logics are complete with respect to the corresponding (relational or neighbourhood) semantics, we formulate our results in proof-theoretic terms.

Recall that K is the basic modal logic. For the next translation it is easier to use the $\Diamond$-presentation of the basic modal language. Later in this section we will also use $\Box$-presentations. We define a translation δ from the basic modal language into the language of INL as follows:

- $\delta(p) = p$ for each propositional variable p.
- δ preserves Boolean connectives.
- $\delta(\Diamond\varphi) = \Box(\delta(\varphi); \top)$.

Theorem 6.1. *For each formula α of basic modal language we have:*
$$K \vdash \alpha \text{ iff } INL \vdash \delta(\alpha).$$

Proof. Suppose $K \nvdash \alpha$. Then by the completeness of K with respect to relational models, there is a relational model $\mathfrak{M} = (W, R, V)$ and $w \in W$ such that $\mathfrak{M}, w \nvDash \alpha$. For each $x \in W$ we let $N(x) = \{R[x]\}$. Then $\mathfrak{M}' = (W, N, V)$ is a neighbourhood model. It is easy to show by induction on the complexity of formulas that for each β in the basic modal language we have

$$\mathfrak{M}, w \models \beta \text{ iff } \mathfrak{M}', w \models \delta(\beta).$$

So $\mathfrak{M}', w \nvDash \delta(\alpha)$, and $INL \nvdash \delta(\alpha)$.

Conversely, suppose $INL \nvdash \delta(\alpha)$. Then by the completeness of INT with respect to neighbourhood models, there is $\mathfrak{M} = (W, N, V)$ and $w \in W$ such that $\mathfrak{M}, w \nvDash \delta(\alpha)$. We define R on W by setting for each $x, y \in W$, xRy if $y \in U$ for some $U \in N(x)$. Then $\mathfrak{M}'' = (W, R, V)$ is a relational model. Now, it is easy to show by induction on the complexity of formulas that, for each β in the basic modal language, we have this equivalence: [1]

$$\mathfrak{M}'', w \models \beta \text{ iff } \mathfrak{M}, w \models \delta(\beta).$$

Thus, $\mathfrak{M}'', w \nvDash \alpha$ and $K \nvdash \alpha$. $\qquad\square$

Next, recall that EM is the basic monotonic neighbourhood logic. We consider the following translation τ from the basic modal language into the language of INL:

---

[1]We emphasize that the crux of this construction is as follows: the rather 'flat' relation R defined above works thanks to the very special nature of the translated formulas that the model has to preserve.

- $\tau(p) = p$ for each propositional variable p,
- τ preserves Boolean connectives,
- $\tau(\Box\varphi) = \Box(\emptyset; \tau(\varphi))$.

Lemma 6.2. *For any neighbourhood model* $\mathfrak{M} = (W, N, V)$ *and* $w \in W$ *we have*
$$\mathfrak{M}, w \models \alpha \text{ iff } \mathfrak{M}, w \models \tau(\alpha).$$

Proof. An easy induction on the complexity of α. $\square$

Theorem 6.3. *For each formula* α *of basic modal logic we have:*
$$\mathsf{EM} \vdash \alpha \text{ iff } \mathsf{INL} \vdash \tau(\alpha).$$

Proof. Suppose $\mathsf{EM} \nvdash \alpha$. Then by the completeness of EM with respect to monotonic neighbourhood models, there is a monotonic neighbourhood model $\mathfrak{M} = (W, N, V)$ and $w \in W$ such that $\mathfrak{M}, w \nvDash \alpha$. By Lemma 6.2 $\mathfrak{M}, w \nvDash \tau(\alpha)$. So $\mathsf{INL} \nvdash \tau(\alpha)$. The converse direction is similar and follows from the completeness of INL with respect to neighbourhood models. $\square$

It is important to see that the above result is not trivial. It is clear that there is an embedding of EM in INL. But the crucial further point is that this embedding is also faithful.

Next, let $\mathsf{K} \oplus \mathsf{K}$ be the fusion of K with itself. We define the following translation of the language of INL into the bimodal language with two unary modalities.[2] Let

- $\sigma(p) = p$ for each propositional variable p.
- σ commutes with the Boolean connectives.
- $\sigma(\Box(\psi_1, \ldots, \psi_n; \varphi)) = \Diamond_1((\Diamond_2\sigma(\psi_1) \land \cdots \land \Diamond_2\sigma(\psi_n)) \land \Box_2\sigma(\varphi))$.

The idea behind this translation, that goes back to the approach in Kracht and Wolter [23], is that neighbourhood models can be represented as bimodal Kripke models, where neighbourhoods are introduced as worlds by themselves that can 'contain' old-style worlds.

Theorem 6.4. *For each formula* α *in the language of* INL *we have:*
$$\mathsf{INL} \vdash \alpha \text{ iff } \mathsf{K} \oplus \mathsf{K} \vdash \sigma(\alpha).$$

Proof. Suppose $\mathsf{INL} \nvdash \alpha$. Then by the completeness of INL with respect to neighbourhood models, there is a neighbourhood model $\mathfrak{M} = (W, N, V)$ and $w \in W$ such that $\mathfrak{M}, w \nvDash \alpha$. We let $X = W \cup \mathcal{P}(W)$. For each $x, y \in X$ we set xR_1y if $x \in W$, $y \in \mathcal{P}(W)$ and $y \in N(x)$ and xR_2y if $x \in \mathcal{P}(W)$, $y \in X$ and $y \in x$. We also let $V'(p) = V(p)$. Then $\mathfrak{M}' = (X, R_1, R_2, V')$ is a birelational model of bimodal logic. Moreover, by an easy induction on the complexity of formulas one can show that for each formula β in the language of INL we have

$$\mathfrak{M}, w \models \beta \text{ iff } \mathfrak{M}', w \models \sigma(\beta).$$

Therefore, $\mathfrak{M}', w \nvDash \sigma(\alpha)$ and $\mathsf{K} \oplus \mathsf{K} \nvdash \sigma(\alpha)$.

Conversely, suppose $\mathsf{K} \oplus \mathsf{K} \nvdash \sigma(\alpha)$. Then by the completeness of $\mathsf{K} \oplus \mathsf{K}$ with respect to birelational models, there is a birelational model $\mathfrak{M} = (W, R_1, R_2, V)$ and $w \in W$ such that $\mathfrak{M}, w \nvDash \sigma(\alpha)$. For each $x \in W$ we let $N(x) = \{U : U = R_2[y]$ for some $y \in W$ with $xR_1y\}$. Then $\mathfrak{M}' = (W, N, V)$ is a neighbourhood model. Moreover, by an easy induction on the complexity of formulas one can show that for each formula β in the language of INL we have

$$\mathfrak{M}', w \models \beta \text{ iff } \mathfrak{M}, w \models \sigma(\beta).$$

Thus, $\mathfrak{M}', w \nvDash \alpha$ and so $\mathsf{INL} \nvdash \alpha$.

$\square$

---

[2]A similar translation for the basic neighbourhood language is well-known, and it has been used extensively, e.g., in Parikh [27], van Benthem [5], and Kracht and Wolter [23].

The preceding translation is in line with the natural two-sorted nature of neighbourhood models when viewed as generalized Henkin models for second-order logic. However, we can do better in a way that may be surprising to the reader.

Consider the special case of σ when $\Diamond_1 = \Diamond_2$, and write this modality simply as $\Diamond$, while still using σ for the translation. Then σ embeds the language of INL into the uni-modal language.

Theorem 6.5. *For each formula α in the language of* INL *we have:*

$$\mathsf{INL} \vdash \alpha \text{ iff } \mathsf{K} \vdash \sigma(\alpha).$$

Proof. The proof is similar to the proof of Theorem 6.4. For the right to left direction, with any neighbourhood model $\mathfrak{M} = (W, N, V)$, as above, we associate a relational model (X, R, V'), where $R = R_1 \cup R_2$. [3] Conversely, given a relational model (W, R, V) we construct a neighbourhood model (W, N, V), as above, by assuming $R_1 = R_2 = R$. The rest is a matter of routine verification. $\square$

Translations go only so far. The preceding result may look like equating INL with the language of normal modal logic, but this is not quite right. The special semantic and proof-theoretic behavior of instantial neighbourhood logic only comes out when we study its concrete presentation.

Still, syntactic translations often do give useful information. For instance, the above observations allow us to determine the precise computational complexity of our new neighbourhood logic.

Corollary 6.6. *The complexity of the satisfiability problem for* INL *is* PSpace-*complete.*

Proof. We "sandwich" INL between two PSpace-complete logics. That the satisfiability problem for INL is in PSpace follows from Theorems 6.4 or 6.5, since satisfiability for K and $\mathsf{K} \oplus \mathsf{K}$ is in PSpace [11, 17]. That it is PSpace-hard follows from Theorem 6.1, since K is PSpace-hard [11]. $\square$

However, not every special feature of our logic follows automatically by translations. For instance, our results on bisimulation and on axiomatic completeness required additional serious labor.

7. Further directions

This paper has presented the modal basics of the system INL of instantial neighbourhood logic. Even so, we have not exhausted all standard topics that are usually raised in modal logic.

One such topic is correspondence theory for special INL axioms on neighbourhood frames, which would require an extension of the correspondence analysis in [21] and [20] building on [23]. [4] Another standard topic would be algebraic analysis and extensions of modal algebras with operations that are monotonic in some arguments and distributive in others. See [13] and the references therein for generalizations of modal algebra in this direction.

However, in this final section, we list some further directions that may be slightly less obvious, with illustrations of what lies ahead, although we cannot pursue these directions in depth in the short compass of this paper. We start with a few concretizations of INL, and after that, we consider language redesign, as well as other more general issues.

7.1. **Semantic tableaux.** We have given a semantic and a proof-theoretic analysis of INL, even including its decidability. But the true combinatoric behavior of a logical system often comes out more concretely with a semantic tableau system that allows for controlled search of counter-examples and proofs, and even for automated deduction analysis. Virtues of tableaux also include providing cut-free Gentzen calculi for logics, and their attendant combinatorial methods.

There is in fact a complete tableau calculus for INL, which will be made available in our follow-up work. However, given the usual elaborate book-keeping details for tableau calculi, including a full statement would seriously disrupt the balance of this paper, and hence we have decided to just state

---

[3]Here, technically, we consider worlds and sets of worlds as disjoint sets of objects, to avoid confusion.

[4]For instance, a Sahlqvist theorem now gets harder because of the two-quantifier $\Diamond\Box$ logical form of antecedents with INL modalities: even one iteration will produce modality sequences going beyond the usual Sahlqvist syntax.

some general observations here on what such a system achieves. Accordingly, to understand what follows here, the reader should have some understanding of classical and modal tableau systems (cf. [6, 16, 19]).

Modal tableau rules and validity reductions. Tableaux for modal logics search for counterexamples using standard decomposition rules for the Boolean operators, but the heart of what they do is the tableau rule that describes decomposition of modalities. Typically, at a tableau node where all Boolean rules have been applied, one has some proposition letters to be made true and others to be made false, while some modalized formulas are to be made true and some others false. In this case, the tableau rule creates a subtree for a counterexample, since more than one accessible world or neighbourhood may need to be created for the world corresponding to the current node. In the description of the starting nodes for the successors in the subtree, one layer of modalities has disappeared, so syntactic complexity goes down, and termination of tableaux is assured.

The background for this syntactic decomposition procedure are strong reduction principles for validity of modal sequents, with antecedents read conjunctively, and consequents disjunctively as usual. We state these principles for the related simpler systems of modal logic over neighbourhood models or over relational models (cf. [6]).

Proposition 7.1. *Let $p_1, \ldots p_n, q_1, \ldots, q_m$ be propositional variables and let $\varphi_1, \ldots, \varphi_n \ \psi_1, \ldots, \psi_m$ be modal formulas. Then*

(1) *Over relational models, $p_1, \ldots p_n, \Box\varphi_1, \ldots, \Box\varphi_n \models q_1, \ldots, q_m, \Box\psi_1, \ldots, \Box\psi_m$ iff $\vec{p} \cap \vec{q} \neq \emptyset$, or there is some $i \leq m$ such that $\vec{\varphi} \models \psi_i$.*

(2) *Over neighbourhood models, $p_1, \ldots p_n, \Box\varphi_1, \ldots, \Box\varphi_n \models q_1, \ldots, q_m, \Box\psi_1, \ldots, \Box\psi_m$ iff $\vec{p} \cap \vec{q} \neq \emptyset$, or there are $i \leq m$ and $j \leq n$ such that $\varphi_j \models \psi_i$.*

Note what this says. For the basic normal modal logic, modal consequences only hold when, right underneath the top boxes, the antecedents together imply at least one consequent. For the basic modal logic on neighbourhood models, something even more startling holds: a modalized consequence holds only if, right underneath the top boxes, some antecedent implies some consequent.[5] A similar type of reduction principle holds for modal neighborhood logic in its standard minimal version discussed earlier, but we do not spell this out here.

From reduction principles to tableau rules. These validity reductions have an immediate connection to tableau rules. Read in one direction, they drive a tableau rule for finding counter-examples. If the modalized sequent consequence is invalid, then the negation of its reduction statement provides a tableau decomposition rule. But also, if a tableau closes, and we can assume inductively that the sequents in the reduction statement are already derivable, then the original modalized sequent is derivable by a simple Gentzen-style rule. Using similar considerations as for the above simpler cases, one can find a reduction principle for our system INL, reducing validity of modal statements to a Boolean combination of validity statements for sequents having a modal depth at least 1 less than that of the original sequent. Because of its complexity (we have to mix the styles of thinking of the two reduction principles stated above), we forego a explicit statement here.

System properties. The resulting modal tableau system, suitably defined, has the usual features. Tableaus are finitely terminating, a formula is valid iff it has a closed tableau (with the negation of the formula placed at the top), suitably defined subforests with open branches induce counter-examples for the top-most formulas, and closed tableaus correspond to cut-free proofs in sequent calculus for INL, allowing for constructive proofs of basic system properties such as interpolation.

---

[5]That is, modalized antecedents cannot even work together to produce significant conclusions, an extreme form on 'non-distribution' of modalities over conjunctions.

7.2. **Specializing to concrete settings, reductions and expansions.** The system INL presented in this paper is a general modal logic over arbitrary neighbourhood structures. Yet, it is also of interest to see what becomes of it in more concrete settings. We consider a few such cases.

Ordinary relational semantics. As we have seen already in our Section 6 on translations, every standard relational model $\mathfrak{M} = (W, R, V)$ induces a neighbourhood structure as follows. For each world w, we set $N(w) = \{R[w]\}$. In these special neighbourhood models, our INL modality becomes definable as follows, where on the right-hand side, the box modality is standard relational.

Proposition 7.2. *Let $\mathfrak{M}$ be as above. For all $w \in W$ and all INL-formulas $\varphi_1, \ldots, \varphi_n, \psi$, we have*

$$\mathfrak{M}, w \models \Box(\varphi_1, \ldots, \varphi_n; \psi) \Leftrightarrow \mathfrak{M}, w \models \Box\psi \wedge \bigwedge_{i=1}^{n} \Diamond\varphi_i$$

Proof. If $\mathfrak{M}, w \models \Box(\varphi_1, \ldots, \varphi_n; \psi)$ then there is some neighborhood Z of w such that $\mathfrak{M}, v \models \psi$ for all $v \in Z$, and such that there are $u_1, ..., u_n \in Z$ with $\mathfrak{M}, u_i \models \varphi_i$ for each $i \in \{1, ..., n\}$. But in fact we must have $Z = R[w]$, and it follows that $\mathfrak{M}, w \models \Box\psi \wedge \bigwedge_{i=1}^{n} \Diamond\varphi_i$. The converse direction is straightforward. $\square$

Hence, over relational models with neighborhoods as above, the language INL is definable in the basic modal language. The converse occurred in Section 6 (proof of Theorem 6.1), whence:

Proposition 7.3. *Over relational models, INL is expressively equivalent with basic modal logic.*

We could also reformulate the definability results in this subsection in terms of syntactic translations between logics, whose definition will be clear from our text.

Topological semantics. Next we consider another well-known source for neighbourhood models: topological spaces. We recall that a topological space (X, τ) can be seen as a neighbourhood frame if for each $w \in X$ we let $N(w) = \{U \subseteq X : U \text{ is an open neighbourhood of } w\}$, where an *open neighbourhood* of a point w is an open set containing w.

Proposition 7.4. *Over topological spaces, INL is expressively equivalent with the basic modal logic of $\Box$ enriched with the existential modality E.*

Proof. The existential modality E in INL is defined as in Proposition 7.3, while $\Box\varphi = \Box(\emptyset; \varphi)$. Conversely, in the modal logic of $\Box$ and E over topological spaces, the INL modality $\Box(\varphi_1, \ldots, \varphi_n; \psi)$ is definable as

$$E(\varphi_1 \wedge \Box\psi) \wedge \cdots \wedge E(\varphi_n \wedge \Box\psi) \wedge \Box\psi.$$

It is easy to show that this reduction holds, and this time, the main reason behind it is the fact that topological opens are closed under unions – while also the whole space is an open set. [6] $\square$

We note that the basic modal logic enriched with the existential modality is a natural logic for describing topological structure [1], and INL may be viewed as an interesting notational variant. Another way of seeing the close connection is this observation that we state without proof:

Proposition 7.5. *On topological models, the INL-bisimulations of this paper are precisely the total topo-bisimulations in the sense of* [1].

Yet other specializations. Neighbourhood structures occur in many places. INL can also be studied in areas such as agency, including logics for belief revision based on partial plausibility orderings (Girard [18]), or logics for players' powers in games (Parikh [27], van Benthem [7]).

In the latter case, a game induces a neighbourhood structure as follows. Neighbourhoods of the root in a game are those sets Z of outcomes that a player can force the game to end in by playing one

---

[6]A fortiori, it follows that, over monotonic models, INL is expressively equivalent to its single-instance fragment.

of her strategies, i.e. such that for some strategy σ for player i, every "σ-guided match" (a history where the relevant player plays σ) produces an outcome in Z. The neighbourhoods associated with a game and player are then referred to as the *powers* of that player in the given game.

With this interpretation, neighbourhoods are closed upwards under subsethood, and furthermore the set of *all* possible outcomes should always be a neighbourhood since any strategy forces it. So by similar reasoning as in the previous section on the topological semantics, extending the language of game logic in the style of INL would amount to the same increase in expressive power as adding a global modality.

But there is another way to view games as neighbourhood structures, closer to the usual "strategic forms" of games employed in game theory. Instead of looking at powers in the above monotonic sense, we may be interested in what the strategies of the players look like in terms of their spread of possible outcomes. Then we arrive at a more fine-grained notion of power in a game, that also describes the "control" that *other players* have over the available outcomes. Let an *instantiated* power of Player i in a game G be a set Z of outcomes such that i has a strategy σ for which the following "back-and-forth"-conditions hold: every σ-guided match has an outcome in Z, and furthermore every member of Z is an outcome of some σ-guided match.

Clearly, instantiated powers are not generally upwards closed, and with this interpretation we get a rich extension of game logic by allowing instantial modalities $(G)(\psi_1, .., \psi_k; \varphi)$, interpreted as ranging over instantiated powers of some distinguished player in the game G.

We are currently exploring this "instantial game logic" in [8], focusing mainly on a language called IGL in which games are built up from atomic games using operators: test ($?$), angelic and demonic choice ($\cup$ and $\cap$), composition ($\circ$) and iteration (the Kleene star $(-)^\star$). This logic is quite well behaved, with a doubly exponential time upper bound on the satisfiability problem. We also have a sound and strongly complete system of axioms for the star-free fragment of the logic, as a natural extension of the axioms presented for INL here with suitable, and non-trivial, laws for game-building operations, and a sound and weakly complete system of axioms obtained by adding a fixpoint axiom and an induction rule for the Kleene star.

We see this system as exploring a general issue raised in van Benthem [7], the systematic investigation of natural game equivalences and their natural corresponding logical languages.

Finally, we are also working on an instantial neighbourhood-based game logic with an unrestricted dual operator $(-)^\partial$, interpreted as "role switch". Some statements we can express in this language are quite subtle, witness the following formula (suggested to us by an anonymous referee):

$$(G)((G^\partial)\neg\texttt{Win}; \texttt{Win})$$

where $\texttt{Win}$ is an added propositional constant saying that the outcome is a win for player I. At first glance, this formula may appear to express a contradiction, but what it really says is that Player I has a winning strategy in a first iteration of the game G, such that in one of the possible outcomes of this strategy, Player II has a winning strategy in the *next* iteration of the game G. [7]

Remark 7.6. *We do not want to suggest that logics of agency are our only or even main paradigm for* INL. *For instance, neighbourhood structures also arise naturally in the form of* hypergraphs, *that is, families of subsets of a domain, where each set is a "generalized arrow"* [10, 31]. [8] *In this setting, once more, it is not obviously the case that we can reduce the language* INL *to other modal logics, and the* INL *fragment of hypergraph theory may be well-worth exploring.*

---

[7]By contrast, the formula $(G)(\neg\texttt{Win}; \texttt{Win})$ is a contradiction just by the basic axioms of INL: it says that Player I has a winning strategy of which one of the possible outcomes is a loss for Player I.

[8]Hypergraphs as generalized graphs have their origins in mathematics, but they have also been proposed for modeling 'hyperintensionality' in philosophy: cf. the program proposed in Leitgeb [24].

7.3. **Redesigning the basic language.** The language of INL is not sacrosanct, and there are further options. In fact, our preceding discussion of game logic already considered an extension with algebraic game terms. But it is also of interest to go down to fragments of the full language.

Fragments. A more linguistic way of specializing the logic is by looking at fragments. While the full language of INL allows arbitrary finite sets of instances, a natural restriction would be to having just one *single instance*, in a format $\Box(\varphi;\psi)$. It is easy to see that the main proofs in this paper specialize to this fragment, so we have completeness, and even Pspace-complexity, since the lower bound reduction that we gave only uses the single-instance fragment.

Of slightly greater interest is the matching notion of bisimulation Z. Here we have to modify Definition 2.5 as follows.

Definition 7.7. *Given two neighbourhood models* $\mathfrak{M} = (W, N, V)$ *and* $\mathfrak{M}' = (W', N', V')$, *a relation* $Z \subseteq W \times W'$ *is said to be a* single-instance bisimulation *if, whenever* wZw':

1. $\mathfrak{M}, w \models p$ *iff* $\mathfrak{M}', w' \models p$ *for each proposition letter* p,
2. $\forall S \in N(w) \forall t \in S \; \exists S' \in N'(w') \begin{cases} \forall s' \in S' \exists s \in S(sZs') \text{ and} \\ \exists t' \in S'(tZt') \end{cases}$
3. $\forall S' \in N'(w') \forall t' \in S' \; \exists S \in N(w) \begin{cases} \forall s \in S \exists s' \in S'(sZs') \text{ and} \\ \exists t \in S(tZt') \end{cases}$

We can easily prove an invariance result for the single-instance fragment of INL with respect to single-instance bisimulations, and we leave the details to the reader. With this invariance result in place, we can prove an undefinability result showing that INL has strictly greater expressive power than the single-instance fragment:

Proposition 7.8. *The single-instance fragment cannot define the two-instance formula* $\Box(p, q; \top)$.

Proof. Consider the models $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ such that $W = \{w, v, u\}$, $W' = \{w', v', u'\}$, $N(w) = \{\{v\}, \{u\}\}$, $N'(w') = \{\{v', u'\}, \{v'\}, \{u'\}\}$, $V(p) = \{v\}$, $V(q) = \{u\}$, $V'(p) = \{v'\}$, and $V'(q) = \{u'\}$ (see Figure 3 below). We set $N(v) = N(u) = N'(v') = N'(u') = \emptyset$. Then it is easy to see that w and w' are single-instance INL-bisimilar (consider a relation linking w with w', v with v' and u with u'). Consequently they satisfy the same formulas in the single-instance of INL. However, they are not INL-bisimilar. In particular, it is easy to see that $\mathfrak{M}', w' \models \Box(p, q; \top)$, but $\mathfrak{M}, w \not\models \Box(p, q; \top)$. $\qquad\square$
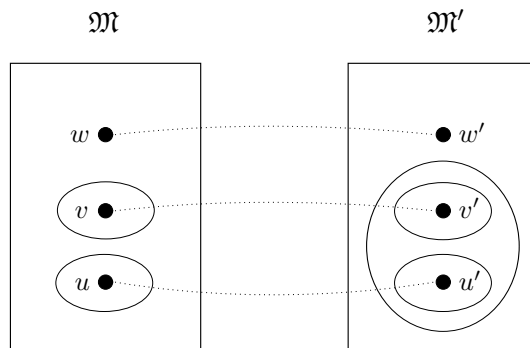

FIGURE 3. A single-instance bisimulation

Similarly, we could define the "k-instance fragment" of INL for each positive integer k, and the proof we gave above could be modified without trouble to prove that INL does not collapse into any of its k-instance fragments. In other words, the k-instance fragments of INL form a strict hierarchy.

Language extensions. Going in the opposite direction, as in the case of normal modal logics over relational models, various language extensions make sense. In particular, it is easy to see that our bisimulations preserve much more than just our base language. We can also add Kleene iteration, or even a full propositional dynamic logic version of our language with a family of atomic neighbourhood relations and complex operations over these. Van Benthem [3], [7] have several results in this setting, including a discussion of operations forming new neighbourhood relations that are "safe" for basic neighbourhood simulations. It would be of interest to extend this analysis to our extended INL bisimulations. Another line of extension would be the study of a modal mu-calculus over an INL base, which would be a natural extension of current work on mu-calculi over basic neighbourhood models [27, 14].

There are also less obvious extensions. For instance, we can translate INL into a *two-sorted first-order logic*, with one sort for points and one sort for neighbourhoods, plus two distinguished relations: one for the neighbourhood relation, and one for the (abstract) element relation. This framework would allow us to formulate a preservation theorem complementing our bisimulation analysis, showing how INL is the bisimulation-invariant fragment of a suitable first-order logic for neighbourhood models.[9] For monotonic modal logic on neighbourhood models this has been done by Pauly (see [20] for an exposition of his unpublished proof), for standard neighbourhood semantics this result has been proved in [21].

Dynamics of model change A further important language extension goes right back to the original motivation for introducing INL. Van Benthem and Pacuit [9] introduces "evidence models" where the neighbourhood relation links worlds to sets of worlds that are the output of some device generating evidence. This evidence can be of any sort, no structural closure conditions are imposed, and it is even allowed, for instance, that two evidence sets are disjoint: the sources then contradict each other. The original static evidence logic had a standard neighbourhood modality describing what pieces of evidence support what propositions, as well as a universal modality standing for the knowledge that the agent has about the universe of all currently still possible worlds.

But in inquiry, evidence is in constant flux: it can be added, deleted, or modified. Van Benthem and Pacuit [9] therefore introduce 'dynamic modalities' that refer to what is true in a model after evidence has changed. In particular, they study three basic operations for such a change.

First of all, an event $!\varphi$ of *hard information* that φ is the case intersects all available evidence sets with the truth set of φ and keeps all non-empty ones. A dynamic modality $[!\varphi]\psi$ then says in the current model $(\mathfrak{M}, s)$ that φ is true at the current world s after this change has taken place, producing an updated model $\mathfrak{M}!\varphi$. Now the heart of an axiomatization for such a dynamic modality is a "recursion law" that states what new static neighbourhood modalities will hold after the update. And significantly, this recursion law leads to an extension of the basic neighbourhood language with a conditionalized neighbourhood modality:

$$[!\varphi]\Box\psi \leftrightarrow \Box^\varphi[!\varphi]\psi.$$

$\Box^\varphi[!\varphi]\psi$ says that there was evidence compatible with φ such that ψ holds after the $!\varphi$ update. But $\Box^\varphi\alpha$ is of course a formula from the single-instance fragment of INL: $\Box(\varphi; \alpha)$. Indeed, the following can be shown (cf. [9] for a precise statement of the relevant recursion law):

Proposition 7.9. *The closure of the basic neighbourhood language under $[!\varphi]$ is contained in the single-instance fragment of INL, and this fragment is closed under $[!\varphi]$.*

---

[9]There might even be natural *guarded fragments* on neighbourhood models in this first-order setting.

We have not been able to determine yet whether the single-instance fragment of INL is in fact the closure of the basic neighbourhood language under the modality $[!\varphi]$.

A similar analysis can be given for a second basic operation $+\varphi$ of *soft information* which adds the set of φ-worlds as an evidence set to the current model.

However, the origins of full INL in this setting arise with the third basic dynamic operation $-\varphi$, that of *retracting* a proposition φ. What this does is remove all evidence sets from the model that are contained in the truth set of the formula φ. This time, when we write a recursion law for the basic neighbourhood modality $[-\varphi]\Box\psi$, we need to state the existence of evidence sets in the original model that are compatible with $-\varphi$, and that validate $[-\varphi]\psi$:

$$[-\varphi]\Box\psi \leftrightarrow (E\neg\varphi \to \Box^{\neg\varphi}[-\varphi]\psi).$$

The formula to the right-hand side here is in the single-instance fragment of INL. But this time, when we apply the dynamic retraction modality once more to the right-hand side, we need an equivalent formula with a neighborhood modality recording two instances. In the notation of [9], the general recursion law now looks as follows:

$$[-\varphi]\Box^{\alpha}_{\psi}\chi \leftrightarrow (\neg A\phi \to \Box^{[-\phi]\alpha}_{[-\varphi]\overline{\psi},\neg\varphi}[-\varphi]\chi)$$

where the box formulas indicate that the relevant neighborhood contains instances for the whole finite sequence $\overline{\psi}$, while all its worlds satisfy α. The connection with INL-formulas will be clear, and in general we get:

Proposition 7.10. *The closure of the basic neighbourhood language under $[-\varphi]$ is contained in* INL*, and* INL *is closed under applying the dynamic modality $[-\varphi]$.*

Again, we have an open problem:

Question 7.11. *Is* INL *equal to the closure of the basic neighbourhood language under the dynamic retraction modality?*

This would be of great interest, since it would motivate a static modal language like INL in terms of dynamic considerations of model change. However, we have no proof for this assertion, and our current conjecture is that the answer to the question is negative. In that case, dynamic extensions of natural modal neighborhood logics might well be *sui generis*.

7.4. **Further model theory and infinitary** INL **languages.** Our analysis of INL-bisimulation can be extended in a standard style, involving yet one more language extension. This time consider an infinitary INL, allowing arbitrary set conjunctions and disjunctions, and arbitrary sets of instances in the instantial part of the INL-modality. The following results can be proved by straightforward adaptations of results for modal logic over relational models, keeping in mind our analysis of INL-bisimulations in Section 3.

Proposition 7.12. INL*-bisimulation preserves infinitary* INL*-formulas on all pointed models.*

Proposition 7.13. *For each model $\mathfrak{M}, s$, there exists an infinitary* INL*-formula $\delta(\mathfrak{M}, s)$ that defines $\mathfrak{M}, s$ up to* INL*-bisimulation.*

The proof of this result is a standard construction of so-called Scott formulas [22, 2] in standard semantics for first-order logic or modal logic. In particular, going along the ordinals, in the successor step $\alpha+1$, we enumerate INL descriptions up to depth α for all neighbourhoods of the current world s, and add a statement that no other neighbourhoods exist.[10]

---

[10]The size of the set of instances can be bounded by the cardinality of the model. The limit fixed-point argument producing the total description is as usual.

Discussion. There are some interesting connections here with our completeness proof in Section 5. The normal form of disjunctions of complete types that we used there may be compared to the following statement in the present setting. Every infinitary INL-formula is equivalent to a disjunction of Scott formulas describing available types in the model. But note that the inductive step used in this model description is not of the form used earlier, but 'one level up'. Scott sentences describe what sort of neighbourhoods are available, they do not confine themselves to describing what is exactly the case in one neighbourhood. The proper setting for understanding these analogies and differences may be the coalgebraic perspective of Section 7.5 below. [11]

7.5. **Co-algebraic perspectives.** Instantial neighbourhood semantics has strong connections with coalgebra. First of all, readers familiar with universal coalgebra may have noticed that neighbourhood frames are coalgebras for the double covariant powerset functor, $\mathcal{P} \circ \mathcal{P} : \mathbf{Set} \to \mathbf{Set}$, and more importantly our definition of a bounded morphism $f : \mathfrak{M} \to \mathfrak{M}'$ of neighbourhood models requires precisely that the map f is a coalgebra morphism for the underlying neighbourhood frames. The usual equation for coalgebra morphisms takes the form:

$$N' \circ f = \mathcal{P}\mathcal{P}f \circ N$$

where N, N' are the neighbourhood maps of the two neighbourhood frames, now viewed as coalgebras of the form $N : W \to \mathcal{P}\mathcal{P}W$ and $N' : W' \to \mathcal{P}\mathcal{P}W'$. The usual commutative square depicting this condition is displayed below:

$$
\begin{array}{ccc}
\mathcal{P}\mathcal{P}W & \xrightarrow{\ \mathcal{P}\mathcal{P}f\ } & \mathcal{P}\mathcal{P}W' \\
{\scriptstyle N}\Big\uparrow & & \Big\uparrow{\scriptstyle N'} \\
W & \xrightarrow[\ f\ ]{} & W'
\end{array}
$$

Our definition of INL-bisimulations now simply follows as a consequence of this coalgebraic analysis: the functor $\mathcal{P} \circ \mathcal{P}$ preserves weak pullback squares, and so comes equipped with a relation lifting known as the Barr extension. Generally, for a set functor T, the Barr extension $\overline{\mathsf{T}}$ of T assigns to every binary relation $R \subseteq X \times Y$ the "lifted" relation $\overline{\mathsf{T}}R \subseteq \mathsf{T}X \times \mathsf{T}Y$ given by:

$$(\alpha, \beta) \in \overline{\mathsf{T}}R \text{ iff, for some } \gamma \in \mathsf{T}R : \mathsf{T}\pi_X(\gamma) = \alpha \text{ and } \mathsf{T}\pi_Y(\gamma) = \beta$$

where $\pi_X : R \to X$ and $\pi_Y : R \to Y$ are the projection maps. A T-bisimulation between T-coalgebras (X, f) and (Y, g) is then defined to be a relation $R \subseteq X \times Y$ such that:

$$(u, v) \in R \ \Rightarrow\ (f(u), g(v)) \in \overline{\mathsf{T}}R$$

For the special case of $\mathsf{T} = \mathcal{P} \circ \mathcal{P}$, this condition amounts to precisely the back-and-forth conditions used in our definition of bisimulations. We refer to [30] for more details.

Turning to the language, it is easy to see that the box modalities we use to construct formulas $\Box(\gamma_1, ..., \gamma_k; \psi)$ correspond to $k + 1$-ary predicate liftings for $\mathcal{P} \circ \mathcal{P}$ in the usual sense. More interestingly, since $\mathcal{P} \circ \mathcal{P}$ preserves weak pullback squares we may also introduce a Moss-style nabla modality whose semantics is based on the Barr extension for $\mathcal{P} \circ \mathcal{P}$. This language is defined so that, for every finite collection $\Gamma_1, ..., \Gamma_k$ of finite sets of formulas, we may construct the formula

$$\nabla\{\Gamma_1, ..., \Gamma_k\}$$

---

[11]For finite models, we can do better. The Scott construction stops at some finite stage. But there are alternative descriptions. One describes the given model in terms of unique proposition letters for each world, encoding the accessibilities between them, and prefixing it all by one PDL iteration modality, or alternatively, a greatest fixed-point operator, [4]. We can also do this with existential second-order quantifiers over the world-defining p-predicates. In the modal case, this reduces to mu-calculus form by the Janin-Walukiewicz theorem. Is there an analogue for this result in neighbourhood semantics? And how would such an analysis extend to infinite models?

which then gets the following semantics: given a neighbourhood model $\mathfrak{M} = (W, N, V)$ and $w \in W$, we have $\mathfrak{M}, w \models \nabla\{\Gamma_1, ..., \Gamma_k\}$ if and only if:

Forth: For every $Z \in N(w)$ there is some Γ_i such that:
- For every $u \in Z$ there is some $\psi \in \Gamma_i$ with $u \models \psi$
- For every $\psi \in \Gamma_i$ there is some $u \in Z$ with $u \models \psi$

Back: For every Γ_i there is some $Z \in N(w)$ such that:
- For every $u \in Z$ there is some $\psi \in \Gamma_i$ with $u \models \psi$
- For every $\psi \in \Gamma_i$ there is some $u \in Z$ with $u \models \psi$

Exploring this language further could be an interesting task for future research, but we will not pursue it further here.

7.6. **Instantial neighborhood semantics without monotonicity.** The reader may wonder what happens if we interpret our instantial neighborhood modalities in the usual neighborhood semantics for weaker 'classical' modal logics, where monotonicity is not built in. One appealing truth clause for a formula $\Box(\psi_1, .., \psi_n; \varphi)$ would then be: $\mathfrak{M}, w \models \Box(\psi_1, ..., \psi_n; \varphi)$ iff there is some neighborhood Z of w such that $Z = V(\varphi)$, and $V(\psi_i) \cap Z \neq \emptyset$ for each $i \in \{1, ..., n\}$. What is the expressive power of this language? A quick, partial answer is that this language translates into the basic modal language (interpreted with the non-monotone neighborhood semantics) extended with a global existential modality. The formula $\Box(\psi_1, ..., \psi_n; \varphi)$ can be defined as:

$$\Box\varphi \wedge E(\psi_1 \wedge \varphi) \wedge ... \wedge E(\psi_n \wedge \varphi)$$

If we add a constraint that the domain of a model is always a neighborhood of each world, then the two languages are simply equivalent: we can then define the global modality $E\varphi$ as $\Box(\varphi; \top)$. More can be said about this alternative set-up, whose lack of monotonicity also has some technical drawbacks, for instance, with fixed-point extensions for INL-style logics of computation or games. For the moment, we just flag this alternative road here.

8. Conclusions

In this paper we introduced *instantial neighbourhood logic* INL. This logic provides a new modal language and new semantics to reason about neighbourhood frames and models. We defined INL-bisimulations and showed that these bisimulations preserve the truth of INL-formulas and that for finite models the converse is also true. We also provided a sound and complete axiomatization for INL on neighbourhood frames. By embedding INL into well-known modal logics we showed that it is Pspace-complete. We also reviewed a number of other connections of INL with existing logical formalisms and outlined a number of future research directions where instantial neighborhood logic makes sense and provides fresh perspectives.

9. Acknowledgment

References

[1] M. Aiello, J. van Benthem, and G. Bezhanishvili. Reasoning about space: The modal way. *J. Log. Comput.*, 13(6):889–920, 2003.

[2] J. Barwise. *Admissible sets and structures.* Springer-Verlag, Berlin-New York, 1975.

[3] J. van Benthem. *Exploring Logical Dynamics.* Studies in Logic, Language and Information. CSLI Publications & Cambridge University Press, 1996.

[4] J. van Benthem. Dynamic odds & ends. ILLC Prepublication Series Report ML-1998-08, 1998.

[5] J. van Benthem. Logic games are complete for game logics. *Studia Logica*, 75(2):183–203, 2003.

[6] J. van Benthem. *Modal logic for open minds*. CSLI Publications, 2010.

[7] J. van Benthem. *Logic in games*. The MIT Press, Cambridge MA, 2014.

[8] J. van Benthem, N. Bezhanishvili, and E. Enqvist. A game logic for instantiated powers. In preparation.

[9] J. van Benthem and E. Pacuit. Dynamic logics of evidence-based beliefs. *Studia Logica*, 99(1-3):61–92, 2011.

[10] C. Berge. *Hypergraphs: Combinatorics of finite sets*. North-Holland, 1989.

[11] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, 2001.

[12] B. Chellas. *Modal logic*. Cambridge University Press, Cambridge-New York, 1980. An introduction.

[13] W. Conradie, S. Ghilardi, and A. Palmigiano. Unified correspondence. In *Johan van Benthem on logic and information dynamics*, volume 5 of *Outst. Contrib. Log.*, pages 933–975. Springer, Cham, 2014.

[14] S. Enqvist, F. Seifan, and Y. Venema. Expressiveness of the modal mu-calculus on monotone neighborhood structures. *CoRR*, abs/1502.07889, 2015.

[15] K Fine. Normal forms in modal logic. *Notre Dame J. Formal Logic*, 16:229–237, 1975.

[16] M. Fitting. *Proof Methods for Modal and Intuitionistic Logics*. Reidel, Dordrecht, 1983.

[17] D. M. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-dimensional modal logics: theory and applications*, volume 148 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 2003.

[18] P. Girard. *Modal Logic for Belief and Preference Change. For ordering.* PhD thesis, Stanford University, 2008.

[19] R. Goré. Tableau methods for modal and temporal logics. In *Handbook of tableau methods*, pages 297–396. Kluwer Acad. Publ., Dordrecht, 1999.

[20] H. H. Hansen. Monotonic modal logics. Master's Thesis, ILLC, University of Amsterdam, 2003.

[21] H. H. Hansen, C. Kupke, and E. Pacuit. Neighbourhood structures: Bisimilarity and basic model theory. *Logical Methods in Computer Science*, 5(2), 2009.

[22] J. Keisler. *Model theory for infinitary logic. Logic with countable conjunctions and finite quantifiers.* North-Holland Publishing Co., Amsterdam-London, 1971. Studies in Logic and the Foundations of Mathematics, Vol. 62.

[23] M. Kracht and F. Wolter. Normal monomodal logics can simulate all others. *J. Symb. Log.*, 64(1):99–138, 1999.

[24] H. Leitgeb. New foundations of hyperintensional semantics? In preparation, 2015.

[25] R. Montague. Universal grammar. *Theoria*, 36:373–398, 1970.

[26] L. Moss. Finite models constructed from canonical formulas. *J. Philos. Logic*, 36(6):605–640, 2007.

[27] R. Parikh. The logic of games and its applications. In *Topics in the theory of computation (Borgholm, 1983)*, volume 102 of *North-Holland Math. Stud.*, pages 111–139. North-Holland, Amsterdam, 1985.

[28] M. Pauly. *Logic for Social Software*. PhD thesis, ILLC, University of Amsterdam, 2001.

[29] D. Scott. Advice on modal logic. In *Philosophical Problems in Logic. Some Recent Developments*, pages 143–173. Reidel, Dordrecht, 1970.

[30] Y. Venema. Algebras and coalgebras. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of modal logic*, volume 3 of *Studies in Logic and Practical Reasoning*, pages 331–426. Elsevier, 2007.

[31] V. I. Voloshin. *Introduction to Graph and Hypergraph Theory*. Nova Science Publishers, Inc.,, 2009.

Johan van Benthem: Institute for Logic, Language and Computation, University of Amsterdam, P.O. Box 94242, 1090 GE Amsterdam, The Netherlands, Department of Philosophy, Stanford University, USA, and Changjiang Scholar, Tsinghua University, China, J.vanBenthem@uva.nl

Nick Bezhanishvili: Institute for Logic, Language and Computation, University of Amsterdam, P.O. Box 94242, 1090 GE Amsterdam, The Netherlands, N.Bezhanishvili@uva.nl

Sebastian Enqvist: Institute for Logic, Language and Computation, University of Amsterdam, P.O. Box 94242, 1090 GE Amsterdam, The Netherlands, sebastian.enqvist@fil.lu.se

Junhua Yu: Department of Philosophy, Tsinghua University, Beijing, 100084, China, junhua.yu.5036@outlook.com