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Varieties of two-dimensional cylindric algebras. Part I: Diagonal-free case

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Abstract. We investigate the lattice $\Lambda(\mathbf{Df}_2)$ of all subvarieties of the variety $\mathbf{Df}_2$ of two-dimensional diagonal-free cylindric algebras. We prove that a $\mathbf{Df}_2$-algebra is finitely representable iff it is finitely approximable, characterize finite projective $\mathbf{Df}_2$-algebras, and show that there are no non-trivial injectives and absolute retracts in $\mathbf{Df}_2$. We prove that every proper subvariety of $\mathbf{Df}_2$ is locally finite, and hence $\mathbf{Df}_2$ is hereditarily finitely approximable. We describe all six critical varieties in $\Lambda(\mathbf{Df}_2)$, which leads to a characterization of finitely generated subvarieties of $\mathbf{Df}_2$. Finally, we describe all square representable and rectangularly representable subvarieties of $\mathbf{Df}_2$.

1. Introduction

The paper deals with the varieties of two-dimensional diagonal-free cylindric algebras. The variety $\mathbf{Df}_2$ of all two-dimensional diagonal-free cylindric algebras is widely studied by many authors. As a general reference we will use the fundamental work by Henkin, Monk and Tarski [9]. Among many other things, it is known that $\mathbf{Df}_2$ is finitely approximable, that every two-dimensional diagonal-free cylindric algebra is representable, that the equational theory of $\mathbf{Df}_2$ is decidable, and that $\mathbf{Df}_2$ is not locally finite. In spite of this, little research has pursued investigation of the lattice $\Lambda(\mathbf{Df}_2)$ of all subvarieties of $\mathbf{Df}_2$. The purpose of this paper is to fill in this gap and provide answers to some basic questions about the subvarieties of an otherwise widely studied variety. In particular, we prove that a $\mathbf{Df}_2$-algebra is finitely representable iff it is finitely approximable, characterize finite projective $\mathbf{Df}_2$-algebras, and show that there are no non-trivial injectives and absolute retracts in $\mathbf{Df}_2$. It is also proved that every proper subvariety of $\mathbf{Df}_2$ is locally finite, implying that $\mathbf{Df}_2$ is hereditarily finitely approximable. A characterization of finitely generated varieties of $\mathbf{Df}_2$-algebras is provided by describing all six critical subvarieties of $\mathbf{Df}_2$. It is also shown that $\mathbf{Df}_2$ is the only non-finitely

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generated subvariety of $\mathbf{Df}_2$ which is square representable, a necessary and sufficient condition for a finitely generated variety in $\Lambda(\mathbf{Df}_2)$ to be square representable is given, and rectangularly and finitely rectangularly representable subvarieties of $\mathbf{Df}_2$ are described.

The paper is organized as follows. In §2 we recall some basic facts on $\mathbf{Df}_1$-algebras – also known as Halmos' monadic algebras – and $\mathbf{Df}_2$-algebras. In particular, we will see that both $\mathbf{Df}_1$ and $\mathbf{Df}_2$ have nice topological representations, and that they both posses such nice algebraic properties as congruence-distributivity, the congruence extension property, semi-simplicity and finite approximability. The difference between $\mathbf{Df}_1$ and $\mathbf{Df}_2$ will also be underlined. In §3 we show that though every $\mathbf{Df}_2$-algebra is representable, only finitely approximable $\mathbf{Df}_2$-algebras (including all free and projective $\mathbf{Df}_2$-algebras) are finitely representable. We also prove that there are no non-trivial injectives and absolute retracts in $\mathbf{Df}_2$, and characterize finite projective $\mathbf{Df}_2$-algebras. §4 proves that though $\mathbf{Df}_2$ is not locally finite, every proper subvariety of $\mathbf{Df}_2$ is locally finite, implying that $\mathbf{Df}_2$ is hereditarily finitely approximable. A rough picture of the lattice $\Lambda(\mathbf{Df}_2)$ will also be given. In §5 we characterize finitely generated subvarieties of $\mathbf{Df}_2$ by describing all six critical varieties of $\mathbf{Df}_2$-algebras. Finally, in §6 we describe square representable and rectangularly representable subvarieties of $\mathbf{Df}_2$.

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2. Preliminaries

2.1. $\mathbf{Df}_1$. Recall that a $\mathbf{Df}_1$-*algebra* is a couple $(B, \exists)$, where B is a Boolean algebra, and $\exists$ is an unary operator on B satisfying the following three conditions:

$$a \leq \exists a$$

$$\exists 0 = 0$$

$$\exists(\exists a \wedge b) = \exists a \wedge \exists b.$$

$\mathbf{Df}_1$-algebras are widely known as Halmos' *monadic algebras*. In order to make our notations uniform, we have chosen to call them $\mathbf{Df}_1$-algebras. We denote the variety of all $\mathbf{Df}_1$-algebras by $\mathbf{Df}_1$. It is well-known that for every $(B, \exists) \in \mathbf{Df}_1$, the set $B_0 = \{a \in B : \exists a = a\} = \{\exists a : a \in B\}$ of all fixed points of $\exists$ forms a *relatively* *complete* subalgebra of B (that is, for every $a \in B$, the set $\{b \in B_0 : a \leq b\}$ has a least element), and that every $(B, \exists) \in \mathbf{Df}_1$ can be represented as a couple (B, B_0), where B_0 is a relatively complete subalgebra of B.

Also recall that the lattice of congruences of $(B, \exists) \in \mathbf{Df}_1$ is isomorphic to the lattice of $\exists$-ideals of $(B, \exists)$, which in turn is isomorphic to the lattice of ideals of B_0. Here an ideal $I \subseteq B$ is said to be an $\exists$-*ideal* if $a \in I$ implies $\exists a \in I$. Hence, $\mathbf{Df}_1$ is congruence-distributive and has the congruence extension property.

Further, $(B, B_0) \in \mathbf{Df}_1$ is a simple algebra iff $B_0 = \{0, 1\}$ (which will subsequently be denoted by $\mathbf{2}$), and every $\mathbf{Df}_1$-algebra is a subdirect product of simple $\mathbf{Df}_1$-algebras. Hence, the class of subdirectly irreducible $\mathbf{Df}_1$-algebras coincides with the class of simple $\mathbf{Df}_1$-algebras, meaning that $\mathbf{Df}_1$ is a *semi-simple* variety.

Furthermore, every finitely generated $\mathbf{Df}_1$-algebra is finite. Hence, $\mathbf{Df}_1$ is *locally finite*. Consequently, $\mathbf{Df}_1$ is generated by its finite simple algebras, that is $\mathbf{Df}_1 = \mathbf{Var}\{(\mathbf{2}^n, \mathbf{2})\}_{n\in\omega}$, where $\mathbf{Var}(K)$ denotes the variety generated by a class K. In other words, $\mathbf{Var}(K) = \mathbf{HSP}(K)$, where $\mathbf{H}$, $\mathbf{S}$, and $\mathbf{P}$ denote the operations of taking homomorphic images, subalgebras and direct products, respectively. Finally, the lattice of all subvarieties of $\mathbf{Df}_1$ is an increasing chain $\mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots$ which converges to $\mathbf{Df}_1$, where $\mathcal{V}_n = \mathbf{Var}(\mathbf{2}^n, \mathbf{2})$ for every $n \in \omega$ (for a proof of these and other related results we refer to Halmos [8], Bass [1], Monk [17], and Kagan and Quackenbush [12]).

We also recall from Halmos [8] a topological representation of $\mathbf{Df}_1$-algebras. A *Stone space* is a 0-dimensional, compact and Hausdorff space. Simultaneously closed and open sets of a Stone space X will be called *clopens*, and we denote the set of all clopens of X by $CP(X)$. For an arbitrary binary relation R on X, $x \in X$ and $A \subseteq X$, let $R(x) = \{y \in X : xRy\}$, $R^{-1}(x) = \{y \in X : yRx\}$, $R(A) = \bigcup_{x\in A} R(x)$ and $R^{-1}(A) = \bigcup_{x\in A} R^{-1}(x)$. We call $R(x)$ the R-*saturation* of x, and $R(A)$ the R-*saturation* of A. R is said to be a *clopen* relation, if $A \in CP(X)$ implies $R^{-1}(A) \in CP(X)$. R is said to be *point-closed* if $R(x)$ is a closed set, for every $x \in X$. Note that if R is an equivalence relation, then $R(x) = R^{-1}(x)$ and $R(A) = R^{-1}(A)$.

A *Halmos space* is a couple (X, E), where X is a Stone space, and E is a point-closed and clopen equivalence relation on X. Given two Halmos spaces X and X', a function $f \colon X \to X'$ is said to be a *Halmos morphism* if f is continuous, and $fE(x) = E'f(x)$, for every $x \in X$. Denote the category of Halmos spaces and Halmos morphisms by $\mathbf{HS}$. Then we have that $\mathbf{Df}_1$ is dual (dually equivalent) to $\mathbf{HS}$. In particular, every $(B, \exists) \in \mathbf{Df}_1$ can be represented as $(CP(X), E)$, for the corresponding Halmos space (X, E). We recall that (X, E) can be constructed as follows: X is the set of all ultrafilters of B, $\phi(a) = \{x \in X : a \in x\}$, $\{\phi(a)\}_{a\in B}$ is a base for topology, and xEy iff $(\exists a \in x \Leftrightarrow \exists a \in y)$, for any $a \in B$.

Having this duality at hand, we can obtain dual descriptions of important algebraic concepts of $\mathbf{Df}_1$. For example, open E-saturated subsets of X correspond to $\exists$-ideals, and hence to congruences of $(B, \exists)$, while E-saturated clopens correspond to elements of B_0. Subsequently, the dual spaces of simple algebras are Halmos spaces with the *trivial* E, that is xEy for any $x, y \in X$.

2.2. $\mathbf{Df}_2$.

Definition 2.1. An algebra $(B, \exists_1, \exists_2)$ is said to be a *two-dimensional diagonal-free cylindric algebra* if $(B, \exists_1)$ and $(B, \exists_2)$ are $\mathbf{Df}_1$-algebras, and for any $a \in B$ we have:

$$\exists_1\exists_2 a = \exists_2\exists_1 a.$$

Denote the variety of all two-dimensional diagonal-free cylindric algebras by $\mathbf{Df}_2$.

For any $(B, \exists_1, \exists_2) \in \mathbf{Df}_2$, let $B_1 = \{\exists_1 a : a \in B\}$ and $B_2 = \{\exists_2 a : a \in B\}$. We also let $B_0 = B_1 \cap B_2$. It is easy to check that all the three algebras are indeed $\mathbf{Df}_2$-algebras, and are actually subalgebras of $(B, \exists_1, \exists_2)$. It also should be clear that $B_0 = \{\exists_1\exists_2 a : a \in B\} = \{\exists_2\exists_1 a : a \in B\} = \{a \in B : a = \exists_1 a = \exists_2 a\}$, and that actually it is a relatively complete subalgebra of B. Hence $(B, B_0) \in \mathbf{Df}_1$. Moreover, B_0 is responsible for congruences of $(B, \exists_1, \exists_2)$. Indeed, call an ideal $I \subseteq B$ a $\mathbf{Df}_2$*-ideal*, if $a \in I$ implies $\exists_1 a, \exists_2 a \in I$. Now similarly to the case of $\mathbf{Df}_1$ we have the following:

Theorem 2.2. *There exists a lattice isomorphism between the lattice of congruences of $(B, \exists_1, \exists_2)$, the lattice of $\mathbf{Df}_2$-ideals of $(B, \exists_1, \exists_2)$, and the lattice of ideals of B_0. Hence, $\mathbf{Df}_2$ is congruence-distributive and has the congruence extension property.*

Proof. This is a routine adaptation of a similar result for $\mathbf{Df}_1$. □

As a direct consequence of this theorem, we obtain a characterization of subdirectly irreducible and simple $\mathbf{Df}_2$-algebras, which is similar to that of $\mathbf{Df}_1$:

Theorem 2.3. (See [9])

(1) *$(B, \exists_1, \exists_2)$ is subdirectly irreducible iff $(B, \exists_1, \exists_2)$ is simple iff $B_0 = \mathbf{2}$.*
(2) *$\mathbf{Df}_2$ is semi-simple.*

Proof. (1) $(B, \exists_1, \exists_2)$ is simple $\Rightarrow$ $(B, \exists_1, \exists_2)$ is subdirectly irreducible $\Rightarrow$ B_0 is subdirectly irreducible $\Rightarrow$ $B_0 = \mathbf{2}$ $\Rightarrow$ $(B, \exists_1, \exists_2)$ is simple.

(2) follows from (1). □

2.3. Topological representation. The dual spaces of $\mathbf{Df}_2$-algebras are also obtained as easy extensions of Halmos spaces. A triple (X, E_1, E_2) is said to be a $\mathbf{Df}_2$-*space*, if (X, E_i) is a Halmos space for $i = 1, 2$, and $E_1E_2(x) = E_2E_1(x)$ for every $x \in X$. Given two $\mathbf{Df}_2$-spaces X and X', a function $f\colon X \to X'$ is said to be a $\mathbf{Df}_2$-*morphism*, if f is a Halmos morphism for both E_1 and E_2. We denote the category of $\mathbf{Df}_2$-spaces and $\mathbf{Df}_2$-morphisms by $\mathbf{DS}$. Then we have the following easy extension of Halmos' result:

Theorem 2.4. $\mathbf{Df}_2$ *is dual to* $\mathbf{DS}$. *In particular, every* $\mathbf{Df}_2$*-algebra can be represented as* $(CP(X), E_1, E_2)$ *for the corresponding* $\mathbf{Df}_2$*-space* (X, E_1, E_2).

Proof. (Sketch) Everything goes similarly to Halmos' proof. The only additional fact that has to be verified is the following. In the dual space of $(B, \exists_1, \exists_2)$, $E_1E_2(x) = E_2E_1(x)$ for every $x \in X$; and conversely, in every $\mathbf{Df}_2$-space we have $E_1E_2(A) = E_2E_1(A)$ for every $A \in CP(X)$. The latter is obvious, since $E_i(A) = \bigcup_{x\in A} E_i(x)$ and E_i commutes with $\bigcup$ for $i = 1, 2$. We only show the former claim.

Suppose (X, E_1, E_2) is the dual space of $(B, \exists_1, \exists_2)$. Then $E_1E_2(A) = E_2E_1(A)$ for every $A \in CP(X)$. Now since $\{x\} = \bigcap\{A : A \in CP(X) \ \&\ x \in A\}$, and the family $\{A \in CP(X) : x \in A\}$ is downward directed[1], then by the Esakia lemma [6][2] both E_1 and E_2 commute with the intersection. Hence

$$\begin{aligned} E_1E_2(x) &= E_1E_2\big(\bigcap\{A : A \in CP(X) \ \&\ x \in A\}\big) \\ &= \bigcap\{E_1E_2(A) : A \in CP(X) \ \&\ x \in A\} \\ &= \bigcap\{E_2E_1(A) : A \in CP(X) \ \&\ x \in A\} \\ &= E_2E_1\big(\bigcap\{A : A \in CP(X) \ \&\ x \in A\}\big) = E_2E_1(x). \qquad \square \end{aligned}$$

As an easy corollary we obtain that the category Fin$\mathbf{Df}_2$ of finite $\mathbf{Df}_2$-algebras is dual to the category Fin$\mathbf{DS}$ of finite $\mathbf{Df}_2$-spaces with the discrete topology. Hence, every finite $\mathbf{Df}_2$-algebra is represented as the algebra $(P(X), E_1, E_2)$, where $P(X)$ denotes the power set of X for the corresponding finite $\mathbf{Df}_2$-space (X, E_1, E_2).

2.4. Duality. Now, similarly to $\mathbf{Df}_1$, we can obtain dual descriptions of algebraic concepts of $\mathbf{Df}_2$-algebras. To obtain the dual description of $\mathbf{Df}_2$-ideals we need the following definition: $A \subseteq X$ is said to be *saturated* if it is simultaneously E_1 and E_2-saturated.

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[1] A family $\mathcal{F} = \{A_i\}_{i\in I}$ is said to be *downward directed*, if $A_i, A_j \in \mathcal{F}$ implies that there is $A_k \in \mathcal{F}$ such that $A_k \subseteq A_i \cap A_j$.

[2] Which asserts that for a compact X, a point-closed quasi order R on X, and a downward directed family $\mathcal{F}$ of closed subsets of X, we have

$$R^{-1}\bigcap_{A\in\mathcal{F}} A = \bigcap_{A\in\mathcal{F}} R^{-1}A.$$

Theorem 2.5. (1) *For every* $\mathbf{Df}_2$*-algebra* $\mathcal{B} = (B, \exists_1, \exists_2)$ *and its dual* $\mathcal{X} = (X, E_1, E_2)$*, the lattice of* $\mathbf{Df}_2$*-ideals of* $\mathcal{B}$ *is isomorphic to the lattice of open saturated sets of* $\mathcal{X}$*.*

(2) *Congruences of* $\mathcal{B}$ *correspond to open saturated sets of* $\mathcal{X}$*.*

(3) *Elements of* B_0 *correspond to saturated clopens of* $\mathcal{X}$*.*

Proof. This is a routine adaptation of a similar result for $\mathbf{Df}_1$-algebras. □

To obtain the dual description of subalgebras of $\mathbf{Df}_2$-algebras we need the following definition. A partition R of X is said to be *separated* if from $\neg(xRy)$ it follows that there exists an R-saturated clopen A such that $x \in A$ and $y \notin A$. A separated partition is called *correct* if it satisfies an additional condition: $RE_i(x) \subseteq E_iR(x)$ for every $x \in X$ and $i = 1, 2$. Note that since E_1, E_2 and R are equivalence relations, R is correct iff it is separated and $RE_i(x) = E_iR(x)$ for every $x \in X$ and $i = 1, 2$. For any $\mathbf{Df}_2$-space $\mathcal{X}$, order the set of all correct partitions of X by the set-theoretical inclusion.

Theorem 2.6. *The lattice of subalgebras of* $\mathcal{B} \in \mathbf{Df}_2$ *is dually isomorphic to the lattice of correct partitions of its dual* $\mathcal{X}$*.*

Proof. This is a routine adaptation of a similar dual characterization of subalgebras of $\mathbf{Df}_1$-algebras. □

For any $\mathbf{Df}_2$-space $\mathcal{X} = (X, E_1, E_2)$ and a correct partition R, denote by $\mathcal{X}/R$ the quotient space of $\mathcal{X}$ by R. That is $\mathcal{X}/R = (X/R, (E_1)_R, (E_2)_R)$, where $X/R = \{R(x) : x \in X\}$, topology on X/R is the quotient topology, that is the opens of $\mathcal{X}/R$ are up to homeomorphism the R-saturated opens of $\mathcal{X}$, and $R(x)(E_i)_R R(y)$ iff there are $x' \in R(x)$ and $y' \in R(y)$ with $x'E_iy'$ for $i = 1, 2$.

To obtain the dual description of simple $\mathbf{Df}_2$-algebras, we need the following:

Definition 2.7. (See [9]) A $\mathbf{Df}_2$-space $\mathcal{X}$ is said to be a *component* if $E_1E_2(x) = X$ for every $x \in X$.

Now suppose $\mathcal{B}$ is a $\mathbf{Df}_2$-algebra and $\mathcal{X}$ is its dual. From Theorems 2.3 and 2.4 it follows that $\mathcal{B}$ is simple iff X and $\emptyset$ are the only saturated clopens in $\mathcal{X}$. On the other hand, $\mathcal{X}$ is a component iff X and $\emptyset$ are the only saturated sets in $\mathcal{X}$. It should be clear that if $\mathcal{X}$ is a component, then X and $\emptyset$ are the only saturated clopens in $\mathcal{X}$. Conversely, suppose X and $\emptyset$ are the only saturated clopens in $\mathcal{X}$, but $\mathcal{X}$ is not a component. Then there exists $x \in X$ such that $E_1E_2(x) \neq X$. Now since $E_1E_2(x) = \bigcap\{E_1E_2(A) : A \in CP(X) \ \& \ x \in A\}$ and $E_1E_2(x) \neq X$, there exists $A \in CP(X)$ such that $x \in A$ and $E_1E_2(A) \neq X$. Hence, $E_1E_2(A)$ is a saturated clopen different from X and $\emptyset$. The obtained contradiction proves that $\mathcal{X}$ is a component, and we arrive at the following:

Theorem 2.8 (See [9])**.** *$\mathcal{B}$ is simple iff its dual $\mathcal{X}$ is a component.* □

Now suppose a $\mathbf{Df}_2$-algebra $\mathcal{B} = (B, \exists_1, \exists_2)$ and its dual $\mathcal{X}$ are given. Since B_1, B_2, and B_0 are relatively complete subalgebras of B, they correspond to point-closed and clopen equivalence relations on $\mathcal{X}$. It should be clear that B_1 corresponds to E_1, B_2 to E_2, and B_0 to $E_1 \circ E_2$, where $E_1 \circ E_2$ denotes the composition of E_1 and E_2. In case $\mathcal{B}$ is simple, we have $x(E_1 \circ E_2)y$, for any $x, y \in X$, and hence $E_1 \circ E_2$ is trivial.

Note that there is yet another equivalence relation on $\mathcal{X}$, which naturally arises from E_1 and E_2. Define E_0 by putting xE_0y iff xE_1y and xE_2y. In other words, $E_0 = E_1 \cap E_2$. From the very definition of E_0 it follows that E_0 is a point-closed relation. However, in general E_0 is not a clopen equivalence relation, since $E_0(A)$ may be different from $E_1(A) \cap E_2(A)$, if A is not a singleton. In spite of this, it is easy to show that E_0 is a correct partition of $\mathcal{X}$ (it directly follows from the inclusions $E_0 \subseteq E_i$ for $i = 1, 2$).

Denote by $\mathcal{A}$ the $\mathbf{Df}_2$-algebra generated by the set $\{\exists_1 a \wedge \exists_2 a : a \in B\}$. Clearly $\mathcal{A}$ is a subalgebra of $\mathcal{B}$. Let $\mathcal{X}$ be the dual of $\mathcal{B}$.

Lemma 2.9. *If the number of E_0-equivalence classes of $\mathcal{X}$ is finite, then $\mathcal{A}$ is the subalgebra of $\mathcal{B}$ corresponding to the correct partition E_0.*

Proof. Obviously for every clopen A of $\mathcal{X}$, $E_1(A) \cap E_2(A)$ is a E_0-saturated clopen of $\mathcal{X}$. Hence, $\mathcal{A}$ is a subalgebra of the subalgebra of $\mathcal{B}$ corresponding to the correct partition E_0.

Conversely, since the number of E_0-equivalence classes of $\mathcal{X}$ is finite, every E_0-equivalence class $E_0(x)$ of $\mathcal{X}$ is a clopen. Indeed, $E_0(x)$ is closed, and its complement is a finite union of closed sets, hence is closed as well. Therefore, every E_0-equivalence class has the form $E_1(A) \cap E_2(A)$ for a clopen $A = E_0(x)$. Hence, every $E_0(x)$ belongs to $\mathcal{A}$. Now since every E_0-saturated clopen is a finite union of E_0-equivalence classes, every E_0-saturated clopen is also an element of $\mathcal{A}$. Therefore, $\mathcal{A}$ is the subalgebra of $\mathcal{B}$ corresponding to the correct partition E_0. □

It follows that if $\mathcal{B}$ is finite, then we have a nice algebraic description of the subalgebra of $\mathcal{B}$ corresponding to E_0. Unfortunately, Lemma 2.9 may be false if the number of E_0-equivalence classes of $\mathcal{X}$ is infinite.

Call the sets of the form $E_i(x)$ *E_i-clusters* $(i = 0, 1, 2)$. Let C_i denote E_1-clusters, and C^j denote E_2-clusters of $\mathcal{X}$. Also let $\{C_i\}_{i\in I}$ and $\{C^j\}_{j\in J}$ be the families of all E_1- and E_2-clusters of X, respectively. Then we have the following easy but useful characterization of components:

Lemma 2.10. *The following two conditions are equivalent:*

(1) *$\mathcal{X}$ is a component;*

(2) $C_i \cap C^j \neq \emptyset$ *for any* $i \in I$ *and* $j \in J$.

Proof. This is easy. □

2.5. Finite approximability. Though several "good" properties of $\mathbf{Df}_1$ (such as semi-simplicity, congruence-distributivity, the congruence extension property, topological representation) are preserved by $\mathbf{Df}_2$, there are some properties (such as local finiteness, and the amalgamation property) which $\mathbf{Df}_2$ does not preserve. It was already noticed by Tarski that $\mathbf{Df}_2$ is not locally finite (see, e.g., Henkin, Monk and Tarski [9], Halmos [8], and below). That $\mathbf{Df}_2$ does not have the amalgamation property was first noticed by Comer [4] (see also Sain [18] and Marx [16]). On the other hand, it is known that $\mathbf{Df}_2$ is *finitely approximable.*

Definition 2.11. A variety $\mathbf{V}$ is said to be *finitely approximable* if it is generated by its finite members.

Different frame-theoretical proofs of this can be found in Segerberg [19], Shehtman [20] and Marx [15]. Here we give a sketch of an algebraic proof.

Theorem 2.12. $\mathbf{Df}_2$ *is finitely approximable.*

Proof. (Sketch) Suppose a polynomial $P(a_1, \dots, a_n)$ is not equal to 1 in a $\mathbf{Df}_2$-algebra $\mathcal{B} = (B, \exists_1, \exists_2)$. Consider the finite set $Sub(P)$ of all subpolynomials of $P(a_1, \dots, a_n)$. Generate by $Sub(P)$ a subalgebra $\mathcal{B}(Sub(P))$ of the $\exists_1$-reduct of $\mathcal{B}$. Since the $\exists_1$-reduct of $\mathcal{B}$ is a $\mathbf{Df}_1$-algebra and $\mathbf{Df}_1$ is locally finite, $\mathcal{B}(Sub(P))$ is finite too. Define $\exists_2'$ on $\mathcal{B}(Sub(P))$ by putting

$$\exists_2' a = \text{The least element of } \{b \in \mathcal{B}(Sub(P)) : a \leq b \ \& \ b \in B_2\}.$$

It can be shown that $(\mathcal{B}(Sub(P)), \exists_1, \exists_2')$ is a $\mathbf{Df}_2$-algebra (though, in general, it is not a subalgebra of $\mathcal{B}$) and $P(a_1, \dots, a_n)$ is not equal to 1 in it. Hence $\mathbf{Df}_2$ is finitely approximable. □

3. Representable $\mathbf{Df}_2$-algebras

3.1. Finitely representable algebras. For any cardinals κ and ι, define on the cartesian product $\kappa \times \iota$ two equivalence relations E_1 and E_2 by putting

$$(i_1, i_2) E_1 (j_1, j_2) \text{ iff } i_2 = j_2; \ (i_1, i_2) E_2 (j_1, j_2) \text{ iff } i_1 = j_1,$$

for $i_1, j_1 \in \kappa$ and $i_2, j_2 \in \iota$.

Definition 3.1. (See [9]) Call $(\kappa \times \iota, E_1, E_2)$ a *rectangle*, and $(\kappa \times \kappa, E_1, E_2)$ a *square.*

Obviously $(P(\kappa \times \iota), E_1, E_2)$ and $(P(\kappa \times \kappa), E_1, E_2)$ are $\mathbf{Df}_2$-algebras, where $P(\kappa \times \iota)$ denotes the power set of $\kappa \times \iota$.

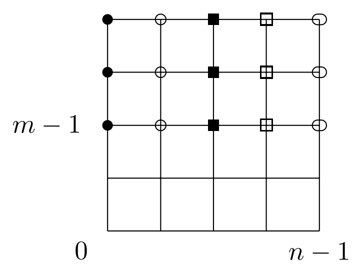

FIGURE 1

Definition 3.2. (See [9]) Call $(P(\kappa \times \iota), E_1, E_2)$ a *rectangular algebra*, and $(P(\kappa \times \kappa), E_1, E_2)$ a *square algebra*. Denote by **Rect** the class of all rectangular algebras, and by **Sq** the class of all square algebras. Also let *Fin***Rect** and *Fin***Sq** denote the classes of all finite rectangular and finite square algebras, respectively.

It is obvious that $\mathbf{Sq} \subset \mathbf{Rect}$ and Fin**Sq**⊂Fin**Rect**.

Definition 3.3. Call a $\mathbf{Df}_2$-algebra $\mathcal{B}$ *rectangularly representable* if $\mathcal{B} \in \mathbf{SP}\ (\mathbf{Rect})$, and *square representable* if $\mathcal{B} \in \mathbf{SP}(\mathbf{Sq})$.

The following theorem is well-known:

Theorem 3.4 (See [9]). *Every* $\mathbf{Df}_2$*-algebra is both rectangularly and square representable.* □

Now we will turn to the question: what is a necessary and sufficient condition for a $\mathbf{Df}_2$-algebra to be representable by finite rectangular (square) algebras?

It turns out that the class of $\mathbf{Df}_2$-algebras representable by finite rectangular algebras coincides with the class of $\mathbf{Df}_2$-algebras representable by finite square algebras. Indeed, we have the following:

Lemma 3.5. *For every finite rectangle* $\mathcal{X}$*, there exists a finite square* $\mathcal{Y}$ *and a correct partition* R *of* $\mathcal{Y}$ *such that* $\mathcal{X}$ *is isomorphic to the quotient of* $\mathcal{Y}$ *by* R*.*

Proof. Suppose $X = n \times m$ and $n > m$. Consider $n \times n$ and define R on it by identifying the points $(k, m-1)$ and $(k, m+j)$, $j \in n-m$ and $k \in n$ (see Figure 1, where the points of the same color are identified). It is routine to check that R is a correct partition of $n \times n$, and that the quotient of $n \times n$ by R is isomorphic to $n \times m$. □

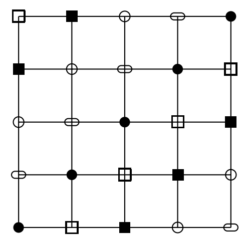

Figure 2

Therefore, by Theorem 2.6, for every finite rectangular algebra $\mathcal{B}$ there exists a finite square algebra $\mathcal{B}'$ such that $\mathcal{B}$ is a subalgebra of $\mathcal{B}'$. Hence, for any $\mathbf{Df}_2$-algebra $\mathcal{B}$, $\mathcal{B} \in \mathbf{SP}(\mathrm{Fin}\mathbf{Rect})$ iff $\mathcal{B} \in \mathbf{SP}(\mathrm{Fin}\mathbf{Sq})$). We simply call these algebras *finitely representable*.

It turns out that finitely representable algebras are closely related to finitely approximable ones.

Definition 3.6. (Malcev [14]) An algebra is said to be *finitely approximable* if it is a subdirect product of its finite homomorphic images.

Our goal is to prove that a $\mathbf{Df}_2$-algebra $\mathcal{B}$ is finitely representable iff it is finitely approximable. For this we need some auxiliary lemmas.

Suppose a component $\mathcal{X} = (X, E_1, E_2)$ is given. $\mathcal{X}$ is said to be a *bicluster* if $E_1(x) = E_2(x) = X$ for any $x \in X$. $\mathcal{X}$ is said to be *uniform* if every E_0-cluster of $\mathcal{X}$ has the same cardinality.

Lemma 3.7. *For every finite bicluster $\mathcal{X}$, there exists a finite square $\mathcal{Y}$ and a correct partition R of $\mathcal{Y}$ such that $\mathcal{X}$ is isomorphic to the quotient of $\mathcal{Y}$ by R.*

Proof. Suppose X consists of n points. Consider the square $(n \times n, E_1, E_2)$. Define an equivalence relation R on $n \times n$ by putting

$$(k, m)R(k', m') \text{ iff } k - m \equiv k' - m' \pmod{n}.$$

This means that every R-equivalence class contains one and only one point from every E_i-cluster (see Figure 2, where the points of the same color are identified). Since $RE_i(k, m) = n \times n = E_iR(k, m)$ for $k, m \in n$ and $i = 1, 2$, we have that R is a correct partition of $n \times n$. It should be clear now that the quotient of $n \times n$ by R is isomorphic to $\mathcal{X}$. $\square$

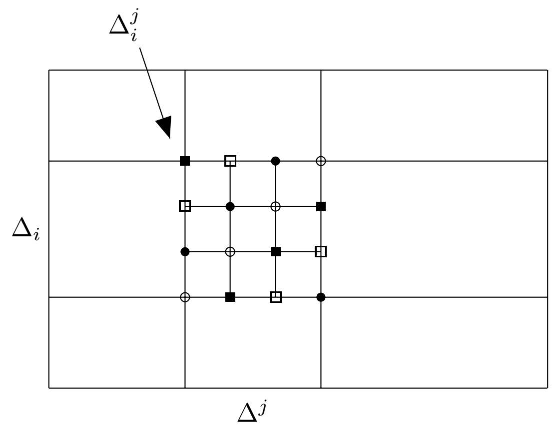

FIGURE 3

Lemma 3.8. *For every finite uniform component $\mathcal{X}$, there exists a finite rectangle $\mathcal{Y}$ and a correct partition R of $\mathcal{Y}$ such that $\mathcal{X}$ is isomorphic to the quotient of $\mathcal{Y}$ by R.*

Proof. Let $\{C_i\}_{i=1}^n$ and $\{C^j\}_{j=1}^m$ be the classes of all E_1- and E_2-clusters of X, respectively. Also let C_i^j denote the E_0-cluster $C_i^j = C_i \cap C^j$. Since $\mathcal{X}$ is uniform, the cardinality of every C_i^j is the same. Let it be $k > 0$. Consider the rectangle $nk \times mk$. Let Δ^j be $[jk \times mk] \setminus [(j-1)k \times mk]$ and Δ_i be $[nk \times ik] \setminus [nk \times (i-1)k]$. Also let $\Delta_i^j = \Delta_i \cap \Delta^j$. Call the $k \times k$ square Δ_i^j the (i,j)-square. Define a partition R on $nk \times mk$ by sewing each square Δ_i^j into a bicluster as in the proof of Lemma 3.7 (see Figure 3). It follows that R is a correct partition, and that the quotient of $nk \times mk$ is isomorphic to $\mathcal{X}$. $\square$

Lemma 3.9. *For every finite component $\mathcal{X}$, there exists a finite uniform component $\mathcal{Y}$ and a correct partition R of $\mathcal{Y}$ such that $\mathcal{X}$ is isomorphic to the quotient of $\mathcal{Y}$ by R.*

Proof. Let C_i, C^j and C_i^j denote the same as in the proof of Lemma 3.8. Also let $|C_i^j| = k_i^j$ (obviously all $k_i^j > 0$) and $k = \max k_i^j$. Consider the uniform component $\mathcal{Y}$ which is obtained from $\mathcal{X}$ by changing every E_0-cluster of $\mathcal{X}$ into a E_0-cluster containing k points. Define a partition R of $\mathcal{Y}$ identifying in each E_0-cluster of $\mathcal{Y}$ $k-(k_i^j+1)$ points (see Figure 4, where filled circles represent the identified points). It should be clear that R is a correct partition of $\mathcal{Y}$, and that the quotient of $\mathcal{Y}$ by R is isomorphic to $\mathcal{X}$. $\square$

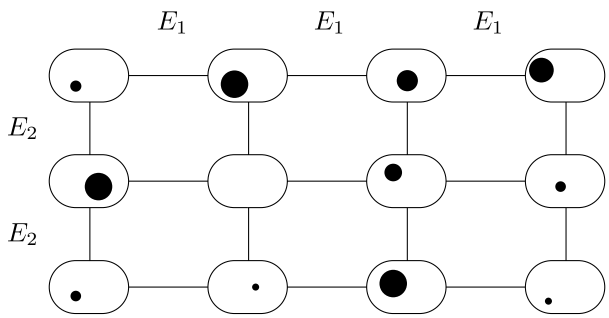

FIGURE 4

From Theorems 2.6, 2.8 and Lemmas 3.7, 3.8 and 3.9 it directly follows that every finite simple $\mathbf{Df}_2$-algebra is a subalgebra of a finite rectangular algebra. Hence, by Lemma 3.5, it is a subalgebra of a finite square algebra, thus is finitely representable.

Now we are in a position to prove the main result of this section:

Theorem 3.10. *$\mathcal{B} \in \mathbf{Df}_2$ is finitely representable iff it is finitely approximable.*

Proof. If $\mathcal{B}$ is finitely representable, then there exists a family $\{\mathcal{B}_i\}_{i\in I}$ of finite square algebras such that $\mathcal{B} \hookrightarrow \prod_{i\in I} \mathcal{B}_i$. Let $\pi_i \colon \prod_{i\in I} \mathcal{B}_i \to \mathcal{B}_i$ denote the ith projection. Then it is obvious that $\mathcal{B} \hookrightarrow \prod_{i\in I} \pi_i(\mathcal{B})$, and that $\mathcal{B}$ is a subdirect product of the family $\{\pi_i(\mathcal{B})\}_{i\in I}$. Now since every $\pi_i(\mathcal{B})$ is a subalgebra of $\mathcal{B}_i$, every $\pi_i(\mathcal{B})$ is finite, and hence $\mathcal{B}$ is finitely approximable.

Conversely, suppose $\mathcal{B}$ is finitely approximable. Then $\mathcal{B}$ is a subdirect product of its finite simple homomorphic images, that is $\mathcal{B} \in \mathbf{P_S}\{\mathcal{B}_i\}_{i\in I}$, where $\mathcal{B}_i$ are finite simple homomorphic images of $\mathcal{B}$. Now from Lemmas 3.7–3.9 it follows that every $\mathcal{B}_i$ is a subalgebra of a finite rectangular algebra (and from Lemma 3.5 even a subalgebra of a finite square algebra) $\hat{\mathcal{B}}_i$. But then $\mathcal{B} \in \mathbf{SP}\{\hat{\mathcal{B}}_i\}_{i\in I}$, and hence is finitely representable. $\square$

It is known from Malcev [14] that a variety $\mathbf{V}$ is finitely approximable iff every free $\mathbf{V}$-algebra is finitely approximable. It follows that every finite algebra, and by Theorem 2.12, also every free $\mathbf{Df}_2$-algebra, as well as subalgebras of free algebras are finitely representable. In particular, every projective algebra is finitely representable.

However, not every $\mathbf{Df}_2$-algebra is finitely representable. In fact, no infinite simple algebras are. Indeed, suppose $\mathcal{B}$ is an infinite simple algebra. Then the only homomorphic images of $\mathcal{B}$ are the trivial algebra and $\mathcal{B}$ itself. Hence $\mathcal{B}$ is not finitely approximable. By Theorem 3.10 neither is it finitely representable.

Finally, it should be noted that though there exist finitely non-representable $\mathbf{Df}_2$-algebras, that is $\mathbf{Df}_2 \neq \mathbf{SP}(\mathrm{Fin}\mathbf{Rect}) = \mathbf{SP}(\mathrm{Fin}\mathbf{Sq})$, from Theorem 2.12 and Lemmas 3.5, 3.7–3.9 it directly follows that $\mathbf{Df}_2$ is generated by finite rectangular (square) algebras, that is $\mathbf{Df}_2 = \mathbf{HSP}(\mathrm{Fin}\mathbf{Rect}) = \mathbf{HSP}(\mathrm{Fin}\mathbf{Sq})$.

3.2. Projective and injective $\mathbf{Df}_2$-algebras. As was mentioned above, none of the infinite simple $\mathbf{Df}_2$-algebras is projective. We can actually strengthen this result and prove that, although finite simple algebras are finitely representable, none of them is projective except the two-element $\mathbf{Df}_2$-algebra $\mathbf{2}$. Actually we have even a stronger result, which is similar to that of Kagan and Quackenbush [12] for $\mathbf{Df}_1$. For elementary notions of universal algebra we refer to [7].

Theorem 3.11. (1) *Every projective $\mathbf{Df}_2$-algebra contains $\mathbf{2}$ as a homomorphic image.*
(2) *A finite $\mathbf{Df}_2$-algebra $\mathcal{B}$ is projective iff $\mathcal{B}$ is isomorphic to $\mathcal{A} \times \mathbf{2}$ for a finite $\mathbf{Df}_2$-algebra $\mathcal{A}$.*
(3) *There are no non-trivial injective algebras in $\mathbf{Df}_2$.*
(4) *There are no non-trivial absolute retracts in $\mathbf{Df}_2$.*

Proof. (1) Let $\mathcal{B}$ be a projective $\mathbf{Df}_2$-algebra. Since $\mathcal{B}$ is projective and there is a homomorphism from $\mathcal{B} \times \mathbf{2}$ onto $\mathcal{B}$, there is a homomorphism $\alpha\colon \mathcal{B} \to \mathcal{B} \times \mathbf{2}$. Now since there is an onto homomorphism $\beta\colon \mathcal{B} \times \mathbf{2} \to \mathbf{2}$, the composition $\beta \circ \alpha\colon \mathcal{B} \to \mathbf{2}$ will be an onto homomorphism. Hence, $\mathbf{2}$ is a homomorphic image of $\mathcal{B}$.

(2) If $\mathcal{B}$ is a finite projective $\mathbf{Df}_2$-algebra, then as follows from (1), $\mathcal{B}$ contains $\mathbf{2}$ as a homomorphic image. Let $\mathcal{X}_\mathcal{B}$ be the dual space of $\mathcal{B}$. Then a singleton set $\{x\}$ is a E_i-saturated subset of $\mathcal{X}_\mathcal{B}$ for $i = 1, 2$. So $\mathcal{X}_\mathcal{B} = (\mathcal{X}_\mathcal{B} \setminus \{x\}) \cup \{x\}$, where both $\mathcal{X}_\mathcal{B} \setminus \{x\}$ and $\{x\}$ are E_i-saturated for $i = 1, 2$. This together with finiteness of $\mathcal{X}_\mathcal{B}$ imply that $\mathcal{B}$ is isomorphic to $\mathcal{A} \times \mathbf{2}$, where $\mathcal{A}$ is the $\mathbf{Df}_2$-algebra whose dual space is isomorphic to $\mathcal{X}_\mathcal{B} \setminus \{x\}$.

Conversely, suppose $\mathcal{B}$ is a finite $\mathbf{Df}_2$-algebra isomorphic to $\mathcal{A} \times \mathbf{2}$. Then there exists a finitely generated free $\mathbf{Df}_2$-algebra $F(X)$ and a surjective homomorphism $\alpha\colon F(X) \to \mathcal{A} \times \mathbf{2}$. It is sufficient to show that $\mathcal{B}$ is a retract of $F(X)$. Since $\mathcal{B}$ is a homomorphic image of $F(X)$, there exists a $\mathbf{Df}_2$-ideal I such that $\mathcal{B}$ is isomorphic to $F(X)/I$. Since $F(X)$ is finitely generated and $\mathcal{B}$ is finite, I can be chosen to be a principal ideal. Suppose $\mathcal{X}_{F(X)}$ and $\mathcal{X}_\mathcal{B}$ denote the dual spaces of $F(X)$ and $\mathcal{B}$, respectively. Then $\mathcal{X}_\mathcal{B}$ is a clopen E_i-saturated subset of $\mathcal{X}_{F(X)}$ and there is a point $x \in \mathcal{X}_\mathcal{B}$ such that $\{x\}$ is a clopen E_i-saturated subset of $\mathcal{X}_\mathcal{B}$ ($i = 1, 2$). Let ι be the dual of α. Then ι is an embedding of $\mathcal{X}_\mathcal{B}$ into $\mathcal{X}_{F(X)}$. Define $g\colon \mathcal{X}_{F(X)} \to \mathcal{X}_\mathcal{B}$ by putting

$$g(y) = \begin{cases} \iota^{-1}(y) & \text{for } y \in \iota(\mathcal{X}_\mathcal{B}), \\ x & \text{otherwise.} \end{cases}$$

It is easy to check that g is a $\mathbf{DS}$-morphism such that $g \circ \iota = \mathrm{id}_{\mathcal{X}_{\mathcal{B}}}$. Therefore, if γ denotes the dual of g, then γ is a $\mathbf{Df}_2$-homomorphism from $\mathcal{B}$ into $F(X)$ such that $\alpha \circ \gamma = \mathrm{id}_{\mathcal{B}}$. Hence, $\mathcal{B}$ is a retract of $F(X)$, and so is a projective algebra.

(3) Let $\mathcal{B}$ be a non-trivial injective $\mathbf{Df}_2$-algebra of the cardinality κ. Consider the square algebra $P(\kappa \times \kappa)$. Obviously $P(\kappa \times \kappa)$ is simple, $\mathbf{2}$ is a subalgebra of $P(\kappa \times \kappa)$ and there exists a homomorphism from $\mathbf{2}$ into $\mathcal{B}$. Since $\mathcal{B}$ is injective, there should exist a homomorphism h from $P(\kappa \times \kappa)$ to $\mathcal{B}$. Since $P(\kappa \times \kappa)$ is simple, $h[P(\kappa \times \kappa)]$ should be isomorphic to $P(\kappa \times \kappa)$. But the cardinality of $P(\kappa \times \kappa)$ is strictly bigger than κ. The obtained contradiction proves that there is no non-trivial injective $\mathbf{Df}_2$-algebra.

(4) Let $\mathcal{B}$ be a non-trivial absolute retract in $\mathbf{Df}_2$. Denote by $\mathcal{X}$ the dual space of $\mathcal{B}$. Consider the disjoint union $X \sqcup X$. As usual, we can view $X \sqcup X$ as $X \times \{0, 1\}$. Define E_1 and E_2 on $X \sqcup X$ by putting

$$(x, p)E_1(y, q) \text{ iff } xE_1y,$$
$$(x, p)E_2(y, q) \text{ iff } xE_2y \text{ and } p = q,$$

for any $x, y \in X$ and $p, q \in \{0, 1\}$. It is an easy exercise to check that $(X \sqcup X, E_1, E_2)$ is a $\mathbf{Df}_2$-space, and that the map $f\colon X \sqcup X \to X$ sending (x, p) to x is a $\mathbf{DS}$-morphism. Hence $\mathcal{B}$ is a subalgebra of $\mathcal{B}_{X \sqcup X}$. Denote by ι the embedding $\iota : \mathcal{B} \hookrightarrow \mathcal{B}_{X \sqcup X}$. Since $\mathcal{B}$ is an absolute retract, there exists a homomorphism h from $\mathcal{B}_{X \sqcup X}$ onto $\mathcal{B}$ such that $h(\iota(a)) = a$, for any $a \in \mathcal{B}$. Therefore, there is a $\mathbf{DS}$-morphism $g\colon X \to X \sqcup X$ such that $fg(x) = x$, for any $x \in X$. $fg(x) = x$ iff $g(x) = (x, 0)$ or $g(x) = (x, 1)$, for any $x \in X$. But then $(x, 0), (x, 1) \in E_1g(x)$, and in the former case $(x, 1) \notin gE_1(x)$ and in the latter case $(x, 0) \notin gE_1(x)$. Hence, g is not a $\mathbf{DS}$-morphism. The obtained contradiction proves that there are no non-trivial absolute retracts in $\mathbf{Df}_2$. □

4. Locally finite subvarieties of $\mathbf{Df}_2$

It was Tarski who first noticed that $\mathbf{Df}_2$ is not locally finite. Recall once again that:

Definition 4.1. A variety $\mathbf{V}$ is called *locally finite* if every finitely generated $\mathbf{V}$-algebra is finite.

We will sketch below his example. It can also be found in any of these references: Henkin, Monk and Tarski [9], Halmos [8], Erdös, Faber and Larson [5]. Consider the infinite square $\omega \times \omega$. Let $g = \{(n, m) : n \leq m\}$. Then the $\mathbf{Df}_2$-algebra $\mathcal{G} \subset P(\omega \times \omega)$ generated by g is infinite. Indeed, let $g_1 = (\omega \times \omega) \setminus E_2((\omega \times \omega) \setminus g)$ and $g_2 = (\omega \times \omega) \setminus E_1(g \setminus g_1)$. Then $g_1 \cap g_2 = \{(0, 0)\}$. Now define g_1' and g_2' for the infinite square $(\omega \times \omega) \setminus (g_1 \cup g_2)$ as we defined g_1 and g_2 for $\omega \times \omega$. Then $g_1' \cap g_2' =$

$\{(1,1)\}$. Continuing this process we obtain that every element of the diagonal $\Delta = \{(n,n)\}_{n\in\omega}$ is an element of $\mathcal{G}$. Hence $\mathcal{G}$ is infinite. Moreover, actually every singleton $\{(n,m)\}$ of $\omega\times\omega$ also belongs to $\mathcal{G}$, since $(n,m) = E_2(n,n)\cap E_1(m,m)$ (see Figure 5).

In contrast to this we will prove that every proper subvariety of $\mathbf{Df_2}$ is indeed locally finite. Let a simple $\mathbf{Df}_2$-algebra $\mathcal{B}$ and its dual $\mathcal{X}$ be given, $i = 1, 2$ and $n > 0$. $\mathcal{X}$ is said to be of *E_i-depth n* if the number of E_i-clusters of $\mathcal{X}$ is exactly n. $\mathcal{X}$ is said to have an *infinite E_i-depth* if the number of E_i-clusters of $\mathcal{X}$ is infinite. $\mathcal{B}$ is said to be of *E_i-depth $n<\omega$* if the E_i-depth of $\mathcal{X}$ is n. $\mathcal{B}$ is said to be of an *infinite E_i-depth* if $\mathcal{X}$ is of an infinite E_i-depth. $\mathbf{V}\subseteq\mathbf{Df}_2$ is said to be of *E_i-depth $n<\omega$* if there is a simple $\mathcal{B}\in\mathbf{V}$ of E_i-depth n, and the E_i-depth of every other simple member of $\mathbf{V}$ is less than or equal to n. $\mathbf{V}$ is said to be of *E_i-depth ω* if the E_i-depth of simple members of $\mathbf{V}$ is not bounded. For any simple $\mathbf{Df}_2$-algebra $\mathcal{B}$ and its dual $\mathcal{X}$ let $d_i(\mathcal{B})$ and $d_i(\mathcal{X})$ denote the E_i-depth of $\mathcal{B}$ and $\mathcal{X}$, respectively. Similarly, $d_i(\mathbf{V})$ will denote the E_i-depth of a variety $\mathbf{V}\subseteq\mathbf{Df}_2$.

Consider the following two lists of formulas, where $0<n<\omega$:

$$E_1^1 : \exists_2\exists_1 p_1 \le \exists_1 p_1;$$
$$E_1^2 : \exists_2\exists_1 p_1 \wedge \exists_2\exists_1 p_2 \le \exists_1 p_1 \vee \exists_1 p_2 \vee \exists_2(\exists_1 p_1 \wedge \exists_1 p_2);$$
$$E_1^n : \bigwedge_{k=1}^n \exists_2\exists_1 p_k \le \Big(\bigvee_{k=1}^n \exists_1 p_k \vee \exists_2 \bigvee_{k\neq l,\, 1\le k,\, l\le n}(\exists_1 p_k \wedge \exists_1 p_l)\Big);$$

$$E_2^1 : \exists_2\exists_1 p_1 \le \exists_2 p_1;$$
$$E_2^2 : \exists_2\exists_1 p_1 \wedge \exists_2\exists_1 p_2 \le \exists_2 p_1 \vee \exists_2 p_2 \vee \exists_1(\exists_2 p_1 \wedge \exists_2 p_2);$$
$$E_2^n : \bigwedge_{k=1}^n \exists_2\exists_1 p_k \le \Big(\bigvee_{k=1}^n \exists_2 p_k \vee \exists_1 \bigvee_{k\neq l,\, 1\le k,\, l\le n}(\exists_2 p_k \wedge \exists_2 p_l)\Big)$$

We have the following characterization of varieties of E_i-depth n, where $i = 1, 2$ and $0<n<\omega$:

Theorem 4.2 (M. Marx and N. Bezhanishvili)**.**

(1) *E_i^n is valid in a simple $\mathcal{B}$ iff the E_i-depth of $\mathcal{B}$ is less than or equal to n.*
(2) *$\mathbf{V}\in\Lambda(\mathbf{Df}_2)$ is of E_i-depth n iff $\mathbf{V}\subseteq\mathbf{Df}_2+E_i^n$ and $\mathbf{V}\not\subseteq\mathbf{Df}_2+E_i^{n-1}$.*

Proof. (1) Suppose $\mathcal{B}$ is a simple $\mathbf{Df}_2$-algebra and $\mathcal{X}$ is its dual component. Without loss of generality consider the case of E_1^n. Denote the E_1-clusters of $\mathcal{X}$ by $C_1,\dots,C_m$.

First suppose the E_1-depth of $\mathcal{X}$ is $m\le n$. Let $A_1,\dots,A_n$ be any clopen subsets of $\mathcal{X}$. Either $A_k=\emptyset$ for some $1\le k\le n$, or $A_k\neq\emptyset$ for every $1\le k\le n$. In the former case $\bigcap_{k=1}^n E_2E_1A_k=\emptyset$ and E_1^n is valid in $\mathcal{B}$. And in the latter case either there are $1\le k\neq l\le n$ such that $E_1A_k\cap E_1A_l\neq\emptyset$, or $E_1A_k\cap E_1A_l=\emptyset$ for all different k,l. In the first case $\bigcup_{k\neq l,\,1\le k,l\le n}(E_1A_k\cap E_1A_l))$ is a non-empty E_1-saturated set. Hence, $E_2\bigcup_{k\neq l,\,1\le k,l\le n}(E_1A_k\cap E_1A_l))=X$ and E_1^n is valid in $\mathcal{B}$.

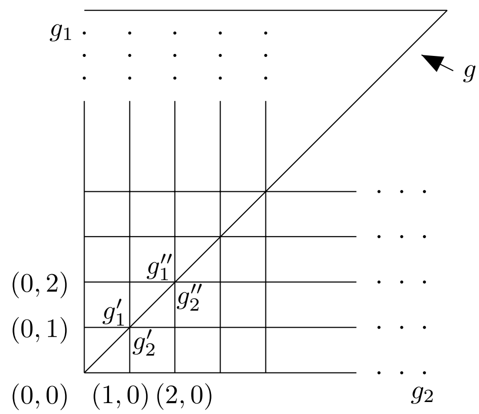

FIGURE 5

In the final case, for any $1 \leq j \leq m$, there exists $1 \leq k \leq n$ such that $A_k \cap C_j \neq \emptyset$. Hence, $\bigcup_{k=1}^{n} E_1 A_k = X$ and E_1^n is valid in $\mathcal{B}$.

Now suppose the E_1-depth of $\mathcal{X}$ is $m > n$. We find non-empty clopens $A_1, \dots, A_n$ such that $E_1(A_k) \cap E_1(A_l) = \emptyset$, for $k \neq l$, and $\bigcup_{k=1}^{n} E_1(A_k) \neq X$. Indeed, if m is a natural number, then every C_i is a clopen ($1 \leq i \leq m$), since, every C_i is closed, and the complement of every C_i is $\bigcup_{j \neq i} C_j$, which is closed too (as a finite union of closed sets). Hence, by putting $A_i = C_i$, $1 \leq i \leq n$, we obtain n different clopens satisfying the above condition. If $m \geq \omega$, then it follows from Claim 4.7 below that such clopens always exist.

Then obviously $\bigcap_{k=1}^{n} E_2 E_1 A_k = X$, but

$$E_2 \bigcup_{k \neq l,\, 1 \leq k,l \leq n} (E_1 A_k \cap E_1 A_l)) = \emptyset \text{ and } \bigcup_{k=1}^{n} E_1(A_k) \neq X.$$

Hence, $\bigcap_{k=1}^{n} E_2 E_1 A_k = X \not\subseteq \bigcup_{k=1}^{n} E_1 A_k \cup \ E_2 \bigcup_{k \neq l,\, 1 \leq k,l \leq n} (E_1 A_k \cap E_1 A_l))$ and E_1^n is not valid in $\mathcal{B}$.

(2) directly follows from (1). $\square$

Definition 4.3. For a variety $\mathbf{V}$, denote by $\mathbf{V}_{SI}$ and $\mathbf{V}_S$ the classes of all subdirectly irreducible and simple $\mathbf{V}$-algebras, respectively. Let also $Fin\mathbf{V}_{SI}$ and $Fin\mathbf{V}_S$ denote the class of all finite subdirectly irreducible and simple $\mathbf{V}$-algebras, respectively.

Now we are in a position to prove the following:

Lemma 4.4. $\mathbf{Df}_2 + E_i^m$ *is locally finite for any* $0 < m < \omega$ *and* $i = 1, 2$.

Proof. It follows from G. Bezhanishvili [2] that a variety $\mathbf{V}$ of a finite signature is locally finite iff the class $\mathbf{V}_{SI}$ is *uniformly locally finite*, that is for each finite n, there exists a natural number $M(n)$ such that for every n-generated $\mathcal{A} \in \mathbf{V}_{SI}$ we have $|A| \leq M(n)$. Now since the signature of $\mathbf{Df}_2$ is finite and $\mathbf{V}_{SI} = \mathbf{V}_S$ for any $\mathbf{V} \in \Lambda(\mathbf{Df}_2)$, it is left to be shown that $(\mathbf{Df}_2 + E_i^m)_S$ is uniformly locally finite.

We will prove that $(\mathbf{Df}_2 + E_1^m)_S$ is uniformly locally finite. The case of $(\mathbf{Df}_2 + E_2^m)_S$ is completely analogous. Suppose $\mathcal{B} = (B[g_1, \ldots, g_n], \exists_1, \exists_2)$ is an n-generated simple algebra from $\mathbf{Df}_2 + E_1^m$, where $g_1, \ldots, g_n$ denote the generators of $\mathcal{B}$, and $\mathcal{X} = (X, E_1, E_2)$ is the dual of $\mathcal{B}$. Then for any $a \in B[g_1, \ldots, g_n]$, there is a polynomial $P(g_1, \ldots, g_n)$, including Boolean operators as well as $\exists_1$ and $\exists_2$, such that $a = P(g_1, \ldots, g_n)$. Since there exist no more than m E_1-clusters of $\mathcal{X}$, there exist no more than 2^m E_1-saturated clopen subsets of $\mathcal{X}$. Hence $B_1 = \{\exists_1 a : a \in B\}$ is finite. Suppose $B_1 = \{a_1, \ldots, a_k\},\ k \leq 2^m$. Then any subformula of $P(g_1, \ldots, g_n)$ which begins with $\exists_1$ can be replaced by some $a_j, 1 \leq j \leq k$. So, we obtain that $a = P'(g_1, \ldots, g_n, a_1, \ldots, a_k)$, where P' is a new $\exists_1$-free polynomial. Hence $B[g_1, \ldots, g_n]$ is generated by $g_1, \ldots, g_n, a_1, \ldots, a_k$ as a $\mathbf{Df}_1$-algebra, and since $\mathbf{Df}_1$ is locally finite, there exists $M(n)$ such that $|B[g_1, \ldots, g_n]| \leq M(n)$. Therefore, $(\mathbf{Df}_2 + E_i^m)_S$ is uniformly locally finite. □

We proceed by showing that the join of two locally finite varieties is locally finite.

Lemma 4.5. *The join of two locally finite varieties is locally finite.*

Proof. Suppose $\mathbf{V} = \mathbf{V}_1 \vee \mathbf{V}_2$, where $\mathbf{V}_1$ and $\mathbf{V}_2$ are locally finite varieties and let $A \in \mathbf{V} = \mathbf{HSP}(\mathbf{V}_1 \cup \mathbf{V}_2)$ be a finitely generated infinite algebra. $A \in \mathbf{V}$ implies there exists a family $\{A_i\}_{i \in I}$ with $A_i \in \mathbf{V}_1 \cup \mathbf{V}_2$ such that $A \in \mathbf{HS}(\prod_{i \in I} A_i)$. For each $i \in I$ we have $A_i \in \mathbf{V}_1$ or $A_i \in \mathbf{V}_2$. Let $I_1 = \{\, i \in I \mid A_i \in \mathbf{V}_1 \,\}$ and $I_2 = \{\, i \in I \mid A_i \in \mathbf{V}_2 \setminus \mathbf{V}_1 \,\}$. Obviously $\prod_{i \in I} A_i$ is isomorphic to $\prod_{i \in I_1} A_i \times \prod_{i \in I_2} A_i$. Since $\mathbf{V}_1$ and $\mathbf{V}_2$ are varieties, $\prod_{i \in I_1} A_i \in \mathbf{V}_1$ and $\prod_{i \in I_2} A_i \in \mathbf{V}_2$. Hence there exist algebras $A_1 = \prod_{i \in I_1} A_i$ in $\mathbf{V}_1$ and $A_2 = \prod_{i \in I_2} A_i$ in $\mathbf{V}_2$ such that $A \in \mathbf{HS}(A_1 \times A_2)$. Therefore there is an algebra $A' \in \mathbf{V}$ such that A is a homomorphic image of A' and there is an embedding ι of A' into $A_1 \times A_2$. Without loss of generality assume that A' is finitely generated. Since A is infinite, A' is infinite as well. Let π_i be the natural projection of $A_1 \times A_2$ onto A_i $i = 1, 2$. Then A' is (isomorphic to) a subalgebra of $\pi_1 \iota(A') \times \pi_2 \iota(A')$. Therefore at least one of $\pi_i \iota(A')$ is infinite. On the other hand, the latter two algebras, being homomorphic images of A', are finitely generated. Hence at least one of $\mathbf{V}_i$ is not locally finite. A contradiction. □

Now we are in a position to prove that every proper subvariety of $\mathbf{Df}_2$ is locally finite.

Lemma 4.6. *If* $\mathbf{V}$ *is not locally finite, then* $\mathbf{V} = \mathbf{Df}_2$.

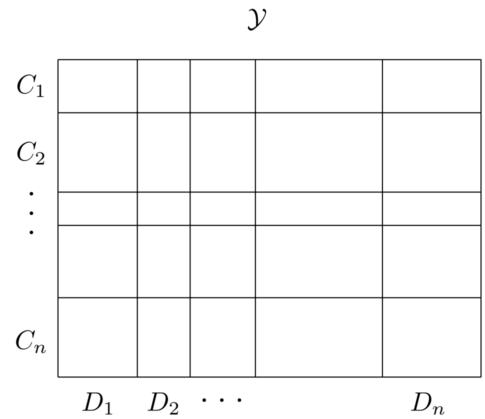

FIGURE 6

Proof. Suppose $\mathbf{V}$ is not locally finite. Then there exists a finitely generated infinite $\mathbf{V}$-algebra. Denote it by $\mathcal{B}$, and its dual space by $\mathcal{X}$. Then either there exists an infinite component of $\mathcal{X}$, or $\mathcal{X}$ consists of infinitely many finite components.

First suppose that $\mathcal{X}$ contains an infinite component $\mathcal{X}_0$. Consider $\mathcal{X}_0/E_0$ and denote it by $\mathcal{Y}$. By Lemma 4.4, $\mathcal{Y}$ is an infinite rectangle of infinite E_1- and E_2-depths. We will show now that $P(n \times n)$ is a subalgebra of $CP(\mathcal{Y})$ for any $n < \omega$.

Claim 4.7. There exists a correct partition R of $\mathcal{Y}$ such that $\mathcal{Y}/R$ is an $n \times n$ square.

Proof. Pick up $n-1$ points $x_1, \ldots, x_{n-1} \in Y$ such that $\neg(x_p E_i x_q)$, $p \neq q$, $1 \leq p, q \leq n-1$ and $i = 1, 2$. Obviously $\bigcup_{k=1}^{n-1} E_1(x_k)$ is a closed E_1-saturated set and $U_1 = Y \setminus \bigcup_{k=1}^{n-1} E_1(x_k)$ is an open E_1-saturated set. Hence, there exists a non-empty E_1-saturated clopen $C_1 \subseteq U_1$. Therefore, $Y = C_1 \cup (Y \setminus C_1)$. Now consider $U_2 = Y \setminus (C_1 \cup \bigcup_{k=2}^{n-1} E_1(x_k))$. Since $x_1 \in U_2$, U_2 is non-empty and obviously is a E_1-saturated open set. Hence, there exists a E_1-saturated clopen $C_2 \subseteq U_2$. Therefore, $(Y \setminus C_1) = C_2 \cup ((Y \setminus C_1) \setminus C_2)$. Now let $U_3 = Y \setminus (C_1 \cup C_2 \cup \bigcup_{k=3}^{n-1} E_1(x_k))$. Since $x_2 \in U_3$, U_3 is a non-empty E_1-saturated open set, and there exists a E_1-saturated clopen $C_3 \subseteq U_3$. Therefore, $(Y \setminus (C_1 \cup C_2) = C_3 \cup (((Y \setminus C_1) \setminus C_2) \setminus C_3)$. Continue this process $(n-1)$-times. At each stage U_k is non-empty, since $x_{k-1} \in U_k$. As a result we get the partition of Y into n E_1-saturated clopens $C_1, C_2, \ldots, C_{n-1}, C_n = Y \setminus \bigcup_{j=1}^{n-1} C_j$. Now do the same for E_2. This will give us the partition of Y into n E_2-saturated clopens $D_1, D_2, \ldots, D_{n-1}, D_n = Y \setminus \bigcup_{j=1}^{n-1} D_j$. Consider the partition $R = \{C_j \cap D_k\}_{1 \leq j,k \leq n}$ (see Figure 6).

Obviously R is a partition of Y into n^2 clopens, i.e. every R-equivalence class is a clopen. Hence, R is a separated partition. Now let us show that $E_iR(x) = RE_i(x)$ for any $x \in Y$ and $i = 1, 2$.

If $y \in RE_1(x)$, then there exists $z \in E_1(x)$ such that yRz. Also suppose that $x \in C_j \cap D_k$. Then $z, y \in C_j \cap D_l$ for some l. Obviously $E_1(y) \cap (C_j \cap D_k) \neq \emptyset$. Hence, $u \in R(x)$ for any $u \in E_1(y) \cap (C_j \cap D_k)$, and $y \in E_1R(x)$. Thus, $RE_1(x) \subseteq E_1R(x)$, which implies that $RE_1(x) = E_1R(x)$.

We can prove analogously that $RE_2(x) = E_2R(x)$ for any $x \in Y$. Hence, R is a correct partition of $\mathcal{Y}$. Clearly $\mathcal{Y}/R$ is isomorphic to $n \times n$. $\square$

Therefore, $P(n \times n)$ is a subalgebra of $CP(\mathcal{Y})$ for any $n < \omega$. Since $\mathbf{Df}_2$ is generated by finite square algebras, $\mathbf{V} = \mathbf{Df}_2$.

Now suppose that $\mathcal{X}$ consists of infinitely many finite components which we denote by $\{\mathcal{X}_j\}_{j \in J}$. Then from Lemma 4.4 it follows that both the E_1 and the E_2-depths of the members of $\{\mathcal{X}_j\}_{j \in J}$ are not bounded by any integer. But then two cases are possible: (1) either $\mathcal{X}$ consists of two families $\{\mathcal{X}'_j\}_{j \in J'}$ and $\{\mathcal{X}''_j\}_{j \in J''}$ such that the E_2-depth of the members of the first family is bounded by some natural n, but the E_1-depth of them is not bounded by any integer; and conversely, the E_1-depth of the members of the second family is bounded by some natural m, but the E_2-depth of them is unbounded; or (2) both the E_1- and the E_2-depths of $\mathcal{X}_j$ are not bounded by any integer.

(1) In the former case, consider the varieties $\mathbf{V}_1$ and $\mathbf{V}_2$, where $\mathbf{V}_1$ denotes the variety generated by the algebras corresponding to the members of the first family, while $\mathbf{V}_2$ denotes the variety generated by the algebras corresponding to the members of the second family. Observe, that $\mathcal{B} \in \mathbf{V}_1 \vee \mathbf{V}_2 = \mathbf{HSP}(\mathbf{V}_1 \cup \mathbf{V}_2)$. Now from Lemma 4.4 it follows that both $\mathbf{V}_1$ and $\mathbf{V}_2$ are locally finite. From Lemma 4.5 it follows that $\mathbf{V}_1 \vee \mathbf{V}_2$ is locally finite as well. Hence $\mathcal{B}$ must be finite, which contradicts our assumption.

(2) In the latter case, consider the quotient of every $\mathcal{X}_j$ by the equivalence E_0. As was mentioned above, $\mathcal{X}_j/E_0$ is a rectangle. Now since both the E_1- and the E_2-depths of $\{\mathcal{X}_j/E_0\}_{j \in J}$ are not bounded by any integer, for each rectangle $n \times m$ there exists a rectangle $\mathcal{X}_j/E_0 = k \times l$ such that $n \leq k$ and $m \leq l$. Hence, $P(n \times m)$ is a subalgebra of $P(\mathcal{X}_j/E_0)$, and $\mathbf{V}$ coincides with $\mathbf{Df}_2$.

Thus, if $\mathbf{V}$ is not locally finite, then $\mathbf{V} = \mathbf{Df}_2$, which completes the proof of our lemma. $\square$

Definition 4.8. A variety $\mathbf{V}$ is called *hereditarily finitely approximable* if every subvariety of $\mathbf{V}$ is finitely approximable.

From Lemma 4.6 and Theorem 2.12 we directly obtain the following:

Corollary 4.9. (1) $\mathbf{V} \in \Lambda(\mathbf{Df}_2)$ *is locally finite iff* $\mathbf{V}$ *is a proper subvariety of* $\mathbf{Df}_2$.

(2) $\mathbf{Df}_2$ *is hereditarily finitely approximable.* □

From Corollary 4.9 we can obtain a rough picture of the structure of the lattice of subvarieties of $\mathbf{Df}_2$.

Suppose $\mathbf{V}$ is a proper subvariety of $\mathbf{Df}_2$. From Corollary 4.9 it follows that $\mathbf{V}$ is generated by $\mathrm{Fin}\mathbf{V}_S$. Consider the sets

$$\Gamma_1 = \{n \in \omega : (\forall m \in \omega)(\exists \mathcal{B} \in \mathrm{Fin}\mathbf{V}_S)(d_1(\mathcal{B}) = n \ \&\ d_2(\mathcal{B}) > m)\}$$

and

$$\Gamma_2 = \{n \in \omega : (\forall m \in \omega)(\exists \mathcal{B} \in \mathrm{Fin}\mathbf{V}_S)(d_2(\mathcal{B}) = n \ \&\ d_1(\mathcal{B}) > m)\}.$$

For $i = 1, 2$ observe that if Γ_i is infinite, then every finite square algebra belongs to $\mathbf{V}$, hence $\mathbf{V} = \mathbf{Df}_2$, a contradiction. Therefore, Γ_i is finite. Let $n_i = \max \Gamma_i$. Also assume that $n_i = 0$ if $\Gamma_i = \emptyset$. Now consider the three subclasses of $\mathrm{Fin}\mathbf{V}_S$:

$\mathbf{K}_1 = \{\mathcal{B} \in \mathrm{Fin}\mathbf{V}_S : d_1(\mathcal{B}) \le n_1\}$,

$\mathbf{K}_2 = \{\mathcal{B} \in \mathrm{Fin}\mathbf{V}_S : d_2(\mathcal{B}) \le n_2\}$, and

$\mathbf{K}_3 = \mathrm{Fin}\mathbf{V}_S \setminus (\mathbf{K}_1 \cup \mathbf{K}_2)$.

Obviously $\mathrm{Fin}\mathbf{V}_S = \mathbf{K}_1 \cup \mathbf{K}_2 \cup \mathbf{K}_3$. Let $\mathbf{V}_r = \mathbf{Var}(\mathbf{K}_r)$ for $r = 1, 2, 3$. Since $\mathbf{V}$ is finitely approximable, $\mathbf{V} = \mathbf{V}_1 \vee \mathbf{V}_2 \vee \mathbf{V}_3$. Moreover, if E_2-depth of $\mathbf{V}$ is finite, then $n_1 = 0$, hence $\mathbf{K}_1 = \emptyset$ and $\mathbf{V}_1 = \emptyset$; if E_1-depth of $\mathbf{V}$ is finite, then $n_2 = 0$, hence $\mathbf{K}_2 = \emptyset$ and $\mathbf{V}_2 = \emptyset$; and if $\mathrm{Fin}\mathbf{V}_S = \mathbf{K}_1 \cup \mathbf{K}_2$, then $\mathbf{K}_3 = \emptyset$, and hence $\mathbf{V}_3 = \emptyset$.

Hence we obtain the following:

Theorem 4.10. *For every* $\mathbf{V} \in \Lambda(\mathbf{Df}_2)$, *either* $\mathbf{V} = \mathbf{Df}_2$, *or* $\mathbf{V} = \bigvee_{i \in S} \mathbf{V}_i$ *for some* $S \subseteq \{1, 2, 3\}$, *where* $d_1(\mathbf{V}_1)$ *is finite and* $d_2(\mathbf{V}_1) = \omega$; $d_1(\mathbf{V}_2) = \omega$ *and* $d_2(\mathbf{V}_2)$ *is finite; and both* $d_1(\mathbf{V}_3)$ *and* $d_2(\mathbf{V}_3)$ *are finite.*

5. Finitely generated and critical subvarieties of $\mathbf{Df}_2$

In this section we characterize finitely generated subvarieties of $\mathbf{Df}_2$ by showing that there exist exactly six critical varieties in $\Lambda(\mathbf{Df}_2)$.

Definition 5.1. A variety is said to be *finitely generated* if it is generated by a finite algebra. A variety is said to be *critical* if it is not finitely generated, but all its proper subvarieties are.

Let a finite component $\mathcal{X} = (X, E_1, E_2)$ be given.

Definition 5.2. (1) $\mathcal{X}$ is said to be E_i*-discrete* $(i = 1, 2)$ if $E_i(x) = \{x\}$ for any $x \in X$.

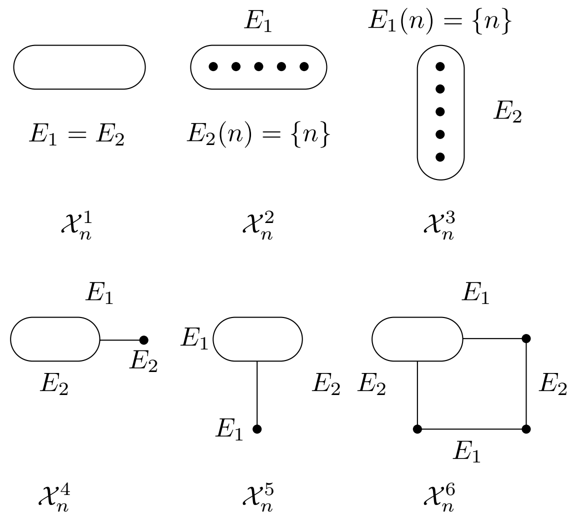

FIGURE 7

(2) $\mathcal{X}$ is said to be E_i-*quasi-bicluster* if $\mathcal{X}$ consists of two E_i-clusters, and one of them contains just one point.
(3) $\mathcal{X}$ is said to be a *quasi-rectangle* of type (n, m) if it is obtained from $n \times m$ by substituting a point of $n \times m$ by any finite E_0-cluster.
(4) $\mathcal{X}$ is said to be a *quasi-square* of type (n, n) if it is obtained from $n \times n$ by substituting a point of $n \times n$ by any finite E_0-cluster.

It should be clear that if $\mathcal{X}$ is either E_1-discrete or E_1-quasi-bicluster, then $E_2(x) = X$ for any $x \in X$, and vice versa.

We will use the following notation:

$\mathcal{X}_n^1 =$ a bicluster consisting of n points;
$\mathcal{X}_n^2 =$ a E_2-discrete $\mathcal{X}$ consisting of n points;
$\mathcal{X}_n^3 =$ a E_1-discrete $\mathcal{X}$ consisting of n points;
$\mathcal{X}_n^4 =$ a E_2-quasi-bicluster $\mathcal{X}$, whose non-singleton E_2-cluster consists of n points;
$\mathcal{X}_n^5 =$ a E_1-quasi-bicluster $\mathcal{X}$, whose non-singleton E_1-cluster consists of n points;
$\mathcal{X}_n^6 =$ a quasi-square $\mathcal{X}$ of type $(2, 2)$, whose the non-singleton E_0-cluster consists of n points (see Figure 7).

For $\mathcal{X}_n^i$ $(i = 1, \ldots, 6)$ denote by $\mathcal{B}_n^i$ the dual algebra of $\mathcal{X}_n^i$. Let also $\mathbf{V}_i$ $(i = 1, \ldots, 6)$ denote the variety $\mathbf{Var}(\{\mathcal{B}_n^i\}_{n=1}^{\infty})$.

Observe that every $\mathbf{V}_i$ is not a finitely generated variety. We will prove that $\mathbf{V}_1, \ldots, \mathbf{V}_6$ are the only critical varieties in $\Lambda(\mathbf{Df}_2)$. For this we need to show that every non-finitely generated variety contains one of the six varieties described above.

For a finite component $\mathcal{X}$ and $x \in X$, *girth* of x is the number of elements of $E_0(x)$. *Girth* of $\mathcal{X}$ is the maximum of girths of all $x \in X$. Let $\mathcal{B}$ be the dual of $\mathcal{X}$. *Girth* of $\mathcal{B}$ is girth of $\mathcal{X}$. From Corollary 4.9 we have that for any $\mathbf{V} \subseteq \mathbf{Df}_2$, $\mathbf{V} = \mathbf{Var}(\mathrm{Fin}\mathbf{V}_S)$. We say that *girth* of $\mathbf{V}$ is $n > 0$ if there is $\mathcal{B} \in \mathrm{Fin}\mathbf{V}_S$ whose girth is n, and girths of all the other members of $\mathrm{Fin}\mathbf{V}_S$ are less than or equal to n. Girth of $\mathbf{V}$ is said to be ω if girths of the members of $\mathrm{Fin}\mathbf{V}_S$ are not bounded by any finite n.

Lemma 5.3. *If E_1-depth, E_2-depth and girth of $\mathbf{V} \in \Lambda(\mathbf{Df}_2)$ are all bounded by some n, then $\mathbf{V}$ is a finitely generated variety.*

Proof. There exist only finitely many finite non-isomorphic components whose E_1-depth, E_2-depth and girth are all bounded by n. Hence the corresponding variety is finitely generated. □

Therefore we obtain that if $\mathbf{V}$ is not finitely generated, then either E_1-depth, E_2-depth or girth of $\mathbf{V}$ is not bounded. We will show now that in these cases one of the six varieties described above is contained in $\mathbf{V}$.

Theorem 5.4. *$\mathbf{V}_1, \dots, \mathbf{V}_6$ are the only critical varieties in $\Lambda(\mathbf{Df}_2)$.*

Proof. Suppose either E_1-depth, E_2-depth or girth of $\mathbf{V}$ is not bounded.

(a) If E_1-depth of $\mathbf{V}$ is not bounded, then for any n, there is a finite simple $\mathcal{B} \in \mathbf{V}$ whose E_1-depth is n. Suppose $\mathcal{X}$ is the dual of $\mathcal{B}$. Then E_1-depth of $\mathcal{X}$ is n. As was already mentioned, E_1 is a correct partition of $\mathcal{X}$. Consider the quotient of $\mathcal{X}$ by E_1. It should be clear that $\mathcal{X}/E_1$ is isomorphic to $\mathcal{X}_n^3$. Hence, $\mathcal{B}_n^3$ is a subalgebra of $\mathcal{B}$ and $\mathcal{B}_n^3 \in \mathbf{V}$ for any n. Thus, $\mathbf{V}_3 \subseteq \mathbf{V}$.

(b) If E_2-depth of $\mathbf{V}$ is not bounded, then similarly to (a) we can prove that $\mathbf{V}_2 \subseteq \mathbf{V}$.

(c) If girth of $\mathbf{V}$ is not bounded, then for any n there is a finite simple $\mathcal{B} \in \mathbf{V}$ whose girth is n. But then, at least one of (1)–(3) below holds for infinitely many $n \in \omega$:

(1) In the dual $\mathcal{X}$ of $\mathcal{B}$ both E_1 and E_2 are trivial, that is xE_iy for any $x, y \in X$ and $i = 1, 2$. In this case $\mathcal{B} = \mathcal{B}_n^1$ and $\mathbf{V}_1 \subseteq \mathbf{V}$.

(2) In the dual $\mathcal{X}$ of $\mathcal{B}$ either E_1 or E_2 is trivial. First suppose E_1 is trivial. Denote by C the E_0-cluster of $\mathcal{X}$ containing n points. Define an equivalence relation R on $\mathcal{X}$ by putting

xRy, for any $x, y \in X \setminus C$
xRy iff $x = y$, for any $x, y \in C$.

It is routine to check that R is a correct partition of $\mathcal{X}$, and that the quotient of $\mathcal{X}$ by R is isomorphic to $\mathcal{X}_n^4$. Hence, $\mathbf{V}_4 \subseteq \mathbf{V}$.

Now suppose E_2 is trivial. Then using analogous arguments we obtain that $\mathbf{V}_5 \subseteq \mathbf{V}$.

(3) In the dual $\mathcal{X}$ of $\mathcal{B}$ neither E_1 nor E_2 is trivial. As in (2), denote by C the E_0-cluster containing n points. Define R on $\mathcal{X}$ by putting

xRy iff xE_0y, for any $x, y \in X \setminus C$
xRy iff $x = y$, for any $x, y \in C$.

Obviously $\mathcal{Y} = \mathcal{X}/R$ is isomorphic to a quasi-rectangle with just one non-singleton E_0-cluster C. Now define the correct partition R' on $\mathcal{Y}$ by putting

$xR'y$, for any $x, y \in Y \setminus (E_1(C) \cup E_2(C))$
$xR'y$, for any $x, y \in E_2(C) \setminus C$
$xR'y$, for any $x, y \in E_1(C) \setminus C$
$xR'y$ iff $x = y$, for any $x, y \in C$.

Then obviously $\mathcal{Y}/R'$ is isomorphic to a quasi-square of type $(2, 2)$ with the non-singleton E_0-cluster C. Hence, $\mathbf{V}_6 \subseteq \mathbf{V}$.

Therefore, if $\mathbf{V}$ is non-finitely generated, it contains one of the six varieties $\mathbf{V}_1, \ldots, \mathbf{V}_6$. Since all of them are non-comparable to each other, they are the only critical varieties in $\Lambda(\mathbf{Df}_2)$. □

Now it is easy to recognize whether or not a variety $\mathbf{V} \subseteq \mathbf{Df}_2$ is finitely generated: If $\mathbf{V}$ contains one of the six critical varieties, then it is not. Otherwise it is. It follows that $\mathbf{V}$ is finitely generated iff $\mathbf{V}$ has only finitely many subvarieties iff $\mathbf{V}$ contains only finitely many simple algebras (and all of them are finite as well). Moreover, every finitely generated variety has only finitely many covers, and any cover of a finitely generated variety is finitely generated itself.

6. Representable subvarieties of $\mathbf{Df}_2$

Now we are in a position to investigate square and rectangularly representable subvarieties of $\mathbf{Df}_2$.

Definition 6.1. For a variety $\mathbf{V} \subseteq \mathbf{Df}_2$, denote by $\mathbf{Rect}_{\mathbf{V}}$ and $\mathbf{Sq}_{\mathbf{V}}$ the classes of rectangular and square $\mathbf{V}$-algebras, respectively. We also denote by $\mathit{Fin}\mathbf{Rect}_{\mathbf{V}}$ and $\mathit{Fin}\mathbf{Sq}_{\mathbf{V}}$ the classes of finite rectangular and finite square $\mathbf{V}$-algebras, respectively.

Now we define the main concept of this section:

Definition 6.2. A variety $\mathbf{V} \subseteq \mathbf{Df}_2$ is called *representable by (algebras from class)* $\mathbf{K}$ if $\mathbf{V} = \mathbf{SP}(\mathbf{K} \cap \mathbf{V})$.

Therefore, $\mathbf{V} \subseteq \mathbf{Df}_2$ is *rectangularly representable* if $\mathbf{V} = \mathbf{SP}(\mathbf{Rect_V})$, $\mathbf{V}$ is *square representable* if $\mathbf{V} = \mathbf{SP}(\mathbf{Sq_V})$, $\mathbf{V}$ is *finitely rectangularly representable* if $\mathbf{V} = \mathbf{SP}(\mathrm{Fin}\mathbf{Rect_V})$, and $\mathbf{V}$ is *finitely square representable* if $\mathbf{V} = \mathbf{SP}(\mathrm{Fin}\mathbf{Sq_V})$).

It is obvious that every square representable variety is also rectangularly representable, and that every finitely square representable variety is also finitely rectangularly representable. However, as we will see below, in contrast to $\mathbf{Df}_2$, there exist varieties $\mathbf{V} \subset \mathbf{Df}_2$ which are (finitely) rectangularly representable, but are not (finitely) square representable, that if a variety is finitely rectangularly (square) representable, then it is a finitely generated variety, and that for every proper subvariety $\mathbf{V}$ of $\mathbf{Df}_2$, $\mathbf{V}$ is square representable iff $\mathbf{V}$ is finitely square representable. A description of rectangularly representable varieties will also be given.

We start with the following useful:

Lemma 6.3. *If $\mathcal{B}$ is a simple $\mathbf{Df}_2$-algebra and $\{\mathcal{B}_i\}_{i\in I}$ is a family of non-trivial $\mathbf{Df}_2$-algebras such that $\mathcal{B} \in \mathbf{S}(\prod_{i\in I}\mathcal{B}_i)$, then $\mathcal{B}$ is a subalgebra of some $\mathcal{B}_i$.*

Proof. Let ι be an embedding of $\mathcal{B}$ into $\prod_{i\in I}\mathcal{B}_i$, and π_i be the projections of $\prod_{i\in I}\mathcal{B}_i$ onto $\mathcal{B}_i$. Observe that $\pi_i\iota(\mathcal{B})$ is either empty or a subalgebra of $\mathcal{B}_i$, which is a homomorphic image of $\mathcal{B}$. Since $\mathcal{B}$ is simple and $\pi_i\iota(\mathcal{B})$ is non-trivial, $\pi_i\iota(\mathcal{B})$ is isomorphic to $\mathcal{B}$. Hence, $\mathcal{B}$ is a subalgebra of $\mathcal{B}_i$ for some i. □

Theorem 6.4. (1) *$\mathbf{V} \subset \mathbf{Df}_2$ is square representable iff $\mathbf{V}$ is finitely square representable.*

(2) *$\mathbf{V}$ is finitely square representable iff there exists n such that $\mathbf{V} = \mathbf{Var}(P(n \times n))$.*

(3) *If $\mathbf{V} \subset \mathbf{Df}_2$ is square representable, then $\mathbf{V}$ is finitely generated.*

Proof. (1) It follows from the proof of Lemma 4.6 that if an infinite square algebra $\mathcal{B}$ belongs to $\mathbf{V}$, then $\mathbf{V}$ does coincide with $\mathbf{Df}_2$. Hence, if $\mathbf{V} \subset \mathbf{Df}_2$, $\mathbf{V}$ does not contain infinite square algebras. Therefore, $\mathbf{V}$ is square representable iff $\mathbf{V}$ is finitely square representable.

(2) From Jónsson's lemma [10] it directly follows that if $\mathbf{V} = \mathbf{Var}(P(n \times n))$ for some $n \in \omega$, then every simple $\mathbf{V}$-algebra is a subalgebra of $P(n \times n)$. Hence, by Birkhoff's theorem, $\mathbf{V}$ is finitely square representable.

Conversely, suppose $\mathbf{V}$ is finitely square representable and let $F = \mathbf{V} \cap \mathrm{Fin}\mathbf{Sq}$. Then $\mathbf{V} = \mathbf{SP}(F)$. Since $\mathbf{V}$ is a proper subvariety of $\mathbf{Df}_2$, F contains only finitely many non-isomorphic algebras. Let $P(n \times n)$ be the greatest among them. Then $F \subseteq \mathbf{S}(P(n \times n))$ and $\mathbf{Var}(P(n \times n)) = \mathbf{V}$.

(3) directly follows from (1) and (2). □

Now let us concentrate on rectangularly representable and finitely rectangularly representable subvarieties of $\mathbf{Df}_2$. For an infinite cardinal κ and positive integers n_1 and n_2 consider the varieties $\mathbf{V}_{(\kappa,n_1)}$ and $\mathbf{V}_{(n_2,\kappa)}$ generated by the algebras

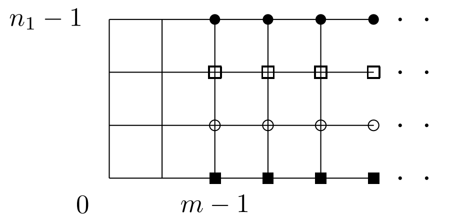

FIGURE 8

$P(\kappa \times n_1)$ and $P(n_2 \times \kappa)$, respectively. So, $\mathbf{V}_{(\kappa,n_1)} = \mathbf{Var}(P(\kappa \times n_1))$ and $\mathbf{V}_{(n_2,\kappa)} = \mathbf{Var}(P(n_2 \times \kappa))$.

The following proposition will be used below. We omit the proof which is standard.

Proposition 6.5. (1) *If a partition R of (X, E_1, E_2) satisfies the condition $RE_i(x) \subseteq E_i R(x)$ for every $x \in X$ and $i = 1, 2$, then $(P(X/R), (E_1)_R, (E_2)_R)$ is a subalgebra of $(P(X), E_1, E_2)$.*

(2) *If $\mathcal{B}$ is a finite subalgebra of $(P(X), E_1, E_2)$, then there exists a partition R of (X, E_1, E_2) such that $RE_i(x) \subseteq E_i R(x)$ for every $x \in X$ and $i = 1, 2$, and $\mathcal{B}$ is isomorphic to $(P(X/R), (E_1)_R, (E_2)_R)$.*

Lemma 6.6. (a) $\mathbf{V}_{(\kappa,n_1)} = \mathbf{V}_{(\omega,n_1)} = \mathbf{HSP}(\{P(m \times n_1)\}_{m\in\omega})$;
(b) $\mathbf{V}_{(n_2,\kappa)} = \mathbf{V}_{(n_2,\omega)} = \mathbf{HSP}(\{P(n_2 \times m)\}_{m\in\omega})$.

Proof. (a) First let us prove that $\mathbf{HSP}(\{P(m \times n_1)\}_{m\in\omega}) \subseteq \mathbf{V}_{(\kappa,n_1)}$. For every $m \in \omega$ define a partition R on $\kappa \times n_1$ by identifying the points $(m-1, k)$ and $(m + j, k)$, $j \in \kappa$ and $k \in n_1$ (see Figure 8).

It is then routine to check that R satisfies the condition $RE_i(x) \subseteq E_i R(x)$ for any $x \in \kappa \times n_1$ and $i = 1, 2$, and that the quotient of $\kappa \times n_1$ by R is isomorphic to $m \times n_1$. Hence, by Proposition 6.5 every rectangular algebra $P(m \times n_1)$ is a subalgebra of $P(\kappa \times n_1)$. Therefore, $\mathbf{HSP}(\{P(m \times n_1)\}_{m\in\omega}) \subseteq \mathbf{V}_{(\kappa,n_1)}$.

Conversely, let us show that for every finite $\mathcal{B} \in \mathbf{S}(P(\kappa \times n_1))$, there exists a rectangular algebra $P(m \times n_1)$ such that $\mathcal{B} \in \mathbf{S}(P(m \times n_1))$. Since $\mathcal{B}$ is a finite subalgebra of $P(\kappa \times n_1)$, by Proposition 6.5 there is a partition R of $\kappa \times n_1$ such that $RE_i(x) \subseteq E_i R(x)$ for any $x \in \kappa \times n_1$ and $i = 1, 2$, and $(\kappa \times n_1)/R$ is isomorphic to the dual of $\mathcal{B}$. Let $\{C_i\}_{i=1}^{n_1}$ and $\{B_k\}_{k=1}^{m}$ denote the sets of E_1 and R-clusters of $\kappa \times n_1$, respectively (see Figure 9).

Obviously, at least one R-cluster contains infinitely many points. Consider the sets $C_i \cap B_k$, $i = 1, \ldots, n_1$, $k = 1, \ldots, m$. From every non-empty $C_i \cap B_k$ choose a point x_i^k and consider a finite rectangle $\mathcal{Y} = \bigcup_{i=1}^{n_1} \bigcup_{k=1}^{m} E_2(x_i^k)$. Let R_Y denote the

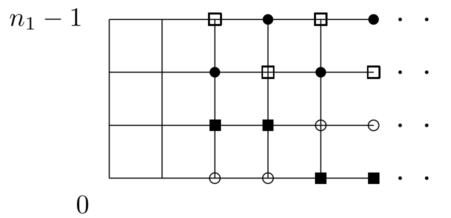

FIGURE 9

restriction of R to $\mathcal{Y}$ and show that $\kappa \times n_1/R$ is isomorphic to $\mathcal{Y}/R_Y$. Define a map $f\colon \mathcal{Y}/R_Y \to (\kappa \times n_1)/R$ by putting $f(R_Y(x)) = R(x)$ for any $x \in \mathcal{Y}$. Since R_Y is a restriction of R, f is an injection. From the construction of $\mathcal{Y}$ it follows that for any $R(z) \in (\kappa \times n_1)/R$ there exists $x \in \mathcal{Y}$ such that xRz. Hence $f(R_Y(x)) = R(z)$ and f is a surjection. Again since R_Y is a restriction of R, from $R_Y(x)E_iR_Y(y)$ it follows that $R(x)E_iR(y)$, for $i = 1, 2$. Conversely, let $R(x)E_1R(y)$. Then there exists $x' \in R(x)$ and $y' \in R(y)$ such that $x'E_1y'$. Hence $x', y' \in C_s$ for some $1 \leq s \leq n_1$. But then there exist $z, u \in \mathcal{Y}$ such that $z, u \in C_s$, $z \in R(x)$ and $u \in R(y)$. Obviously, $f(R_Y(z)) = R(x)$, $f(R_Y(u)) = R(y)$ and $R_Y(z)E_1R_Y(u)$. Finally, let $R(x)E_2R(y)$. Then there exists $x' \in R(x)$ and $y' \in R(y)$ such that $x'E_2y'$ and from the construction of $\mathcal{Y}$ it follows that there exists $z \in \mathcal{Y}$ such that $x'Rz$. It implies that $z \in RE_2(y')$ and since R is a correct partition, there exists $u \in \kappa \times n_1$ such that $y'Ru$ that is $u \in R(y)$ and uE_2z. Thus $u \in \mathcal{Y}$, $f(R_Y(z)) = R(x)$, $f(R_Y(u)) = R(y)$ and $R_Y(z)E_2R_Y(u)$. Therefore $(\kappa \times n_1)/R$ is isomorphic to $\mathcal{Y}/R_Y$, which implies that $\mathcal{B}$ is a subalgebra of $P(\mathcal{Y})$. But then $\mathcal{B}$ is a subalgebra of $P((n_1m) \times n_1)$ too. Hence we obtained that every finite subalgebra of $P(\kappa \times n_1)$ belongs to the variety $\mathbf{HSP}(\{P(m \times n_1)\}_{m\in\omega})$. Now since $\mathbf{V}_{(\kappa,n_1)}$ is a proper subvariety of $\mathbf{Df}_2$, Corollary 4.9 implies that it is locally finite. Hence $P(\kappa \times n_1)$ is the direct limit of its finite subalgebras. But every finite subalgebra of $P(\kappa \times n_1)$ belongs to $\mathbf{HSP}(\{P(m \times n_1)\}_{m\in\omega})$, hence so does $P(\kappa \times n_1)$. Therefore $\mathbf{V}_{(\kappa,n_1)} \subseteq \mathbf{HSP}(\{P(m \times n_1)\}_{m\in\omega})$. Since κ is any infinite cardinal, we get that $\mathbf{V}_{(\kappa,n_1)} = \mathbf{V}_{(\omega,n_1)}$ as well.

(b) is proved similarly to (a). □

We are in a position now to describe rectangularly representable and finitely rectangularly representable subvarieties of $\mathbf{Df}_2$.

Theorem 6.7. *For any* $\mathbf{V} \subset \mathbf{Df}_2$:

(1) $\mathbf{V}$ *is rectangularly representable iff every simple* $\mathbf{V}$*-algebra is a subalgebra of a rectangular* $\mathbf{V}$*-algebra.*
(2) *The following conditions are equivalent:*

(a) $\mathbf{V}$ *is finitely rectangularly representable.*
(b) $\mathbf{V}$ *is finitely generated and every finite simple* $\mathbf{V}$*-algebra is a subalgebra of a finite rectangular* $\mathbf{V}$*-algebra.*
(c) $\mathbf{V} = \mathbf{Var}(P(n_1 \times m_1)) \vee \cdots \vee \mathbf{Var}(P(n_k \times m_k))$*, for some non-negative* n_i, m_i *and positive* k*.*

Proof. (1) From Birkhoff's theorem it follows that if every simple $\mathbf{V}$-algebra is a subalgebra of some rectangular $\mathbf{V}$-algebra, then $\mathbf{V}$ is rectangularly representable. Conversely, if $\mathbf{V}$ is rectangularly representable and $\mathcal{B}$ is a simple $\mathbf{V}$-algebra, then there exists a family $\{P(\iota_i \times \kappa_i)\}_{i \in I} \subset \mathbf{V}$ of rectangular algebras such that $\mathcal{B} \in \mathbf{SP}\{P(\iota_i \times \kappa_i)\}_{i \in I}$. But then from Lemma 6.3 it follows that $\mathcal{B}$ is a subalgebra of $P(\iota_i \times \kappa_i)$ for some $i \in I$.

(2) (a) $\Rightarrow$ (b). If $\mathbf{V}$ is finitely rectangularly representable, then it does not contain an infinite rectangular algebra. For, by Theorem 3.10, any infinite rectangular algebra $P(\iota \times \kappa) \in \mathbf{V}$ would be finitely approximable, which is impossible since $P(\iota \times \kappa)$ is infinite and simple, and so has no finite homomorphic images. Moreover, if $\mathbf{V}$ contains an infinite number of finite rectangular algebras, then by Lemma 6.6 it also contains an infinite rectangular algebra. Hence $\mathbf{V}$ contains only finitely many finite rectangular algebras, thus is finitely generated. Now it directly follows from (1) that every finite simple $\mathbf{V}$-algebra is a subalgebra of the corresponding finite rectangular algebra.

(b) $\Rightarrow$ (c). Since $\mathbf{V}$ is finitely generated, $\mathbf{V} = \mathbf{HSP}(\mathcal{A})$ for some $\mathcal{A} \in \mathbf{V}$. $\mathcal{A}$ is a subdirect product of simple algebras $\mathcal{B}_i$ $(i \in I)$ in $\mathbf{V}$. Since $\mathcal{A}$ is finite, we can assume that I is finite. Each of $\mathcal{B}_i$ is a homomorphic image of $\mathcal{A}$, hence is finite as well. By (b), for each $i \in I$ there is a finite rectangular algebra $P(n_i \times m_i) \in \mathbf{V}$ with $\mathcal{B}_i \subseteq P(n_i \times m_i)$. But then, $\mathbf{V} = \bigvee_{i \in I} \mathbf{Var}(P(n_i \times m_i))$, proving (c).

(c) $\Rightarrow$ (a). Assume that $\mathbf{V} = \bigvee_{i=1}^{k} \mathbf{Var}(P(n_i \times m_i))$ for some finite rectangular algebras $P(n_i \times m_i)$ and positive k. Let $\mathcal{B} \in \mathbf{V}$ be a simple algebra. From Jónsson's lemma $\mathcal{B} \in \mathbf{Var}(P(n_i \times m_i))$ for some $1 \leq i \leq k$. Once again using Jónsson's lemma we get that $\mathcal{B}$ is a subalgebra of $P(n_i \times m_i)$. Therefore $\mathbf{V} = \mathbf{SPS}(\{P(n_i \times m_i)\}_{i=1}^{k}) = \mathbf{SP}(\{P(n_i \times m_i)\}_{i=1}^{k})$. Hence $\mathbf{V}$ is finitely rectangularly representable. $\square$

From Theorems 6.4 and 6.7 it follows that no non-finitely generated subvariety of $\mathbf{Df}_2$ is finitely rectangular (square) representable. On the other hand, it was shown in §3 that $\mathbf{Df}_2$ is generated by its finite rectangular (square) algebras, that is $\mathbf{Df}_2 = \mathbf{HSP}(\mathrm{Fin}\mathbf{Rect}) = \mathbf{HSP}(\mathrm{Fin}\mathbf{Sq})$. We will conclude the paper by showing that this property of being generated by its finite rectangular (square) algebras actually characterizes all rectangular (square) representable varieties. For this we need a number of additional lemmas.

Lemma 6.8. (1) *For any* $\mathbf{V}_1, \mathbf{V}_2 \subseteq \mathbf{Df}_2$, *if* $\mathbf{V}_1$ *and* $\mathbf{V}_2$ *are rectangularly representable, then so is* $\mathbf{V}_1 \vee \mathbf{V}_2$.

(2) *For any* $\mathbf{V}_1, \dots, \mathbf{V}_n \subseteq \mathbf{Df}_2$, *if each* $\mathbf{V}_i$ *is rectangularly representable, then so is* $\mathbf{V}_1 \vee \cdots \vee \mathbf{V}_n$.

Proof. (1) It is a consequence of Jónsson's lemma that for any two semi-simple and congruence distributive varieties $\mathbf{V}_1$ and $\mathbf{V}_2$, $(\mathbf{V}_1 \vee \mathbf{V}_2)_S = (\mathbf{V}_1)_S \cup (\mathbf{V}_2)_S$. Now let $\mathcal{B} \in (\mathbf{V}_1 \vee \mathbf{V}_2)_S$. Without loss of generality assume that $\mathcal{B} \in (\mathbf{V}_1)_S$. From Theorem 6.7 it follows that there exists a rectangular algebra $P(\kappa \times \iota) \in \mathbf{V}_1$ such that $\mathcal{B}$ is a subalgebra of $P(\kappa \times \iota)$. Obviously $P(\kappa \times \iota) \in \mathbf{V}_1 \vee \mathbf{V}_2$. So, applying Theorem 6.7 once again we get that $\mathbf{V}_1 \vee \mathbf{V}_2$ is rectangularly representable.

(2) is proved similarly to (1). □

Lemma 6.9. $\mathbf{V} \subset \mathbf{Df}_2$ *is generated by its finite rectangular algebras iff* $\mathbf{V} = \mathbf{V}_{(\omega,n_1)} \vee \mathbf{V}_{(n_2,\omega)} \vee \bigvee_{i=1}^{p} \mathbf{Var}(P(k_i \times r_i))$ *for some* $n_1, n_2, k_i, r_i \geq 0$, $p > 0$ *and* $1 \leq i \leq p$.

Proof. If $\mathbf{V} = \mathbf{V}_{(\omega,n_1)} \vee \mathbf{V}_{(n_2,\omega)} \vee \bigvee_{i=1}^{p} \mathbf{Var}(P(k_i \times r_i))$ for some $n_1, n_2, k_i, r_i \geq 0$, $p > 0$ and $1 \leq i \leq p$, then from Lemma 6.6 it directly follows that $\mathbf{V}$ is generated by its finite rectangular algebras.

Conversely, suppose $\mathbf{V}$ is generated by its finite rectangular algebras. Similarly to Theorem 4.10 we represent $\mathrm{Fin}\mathbf{Rect}_{\mathbf{V}}$ as the union of the three classes $\mathbf{R}_1, \mathbf{R}_2$ and $\mathbf{R}_3$ as follows. For $1 \leq i \neq j \leq 2$ let

$$n_i = \max\{n \in \omega : (\forall m \in \omega)(\exists \mathcal{B} \in \mathrm{Fin}\mathbf{Rect}_{\mathbf{V}})(d_i(\mathcal{B}) = n \;\&\; d_j(\mathcal{B}) > m)\},$$

and consider the three subclasses of $\mathrm{Fin}\mathbf{Rect}_{\mathbf{V}}$: $\mathbf{R}_1 = \{P(m \times k)\}_{m\in\omega, k\leq n_1}$, $\mathbf{R}_2 = \{P(k \times m)\}_{m\in\omega, k\leq n_2}$ and $\mathbf{R}_3 = \mathrm{Fin}\mathbf{Rect}_{\mathbf{V}} \setminus (\mathbf{R}_1 \cup \mathbf{R}_2)$. Obviously, $\mathbf{R}_3 = \{P(k_1 \times r_1)), \dots, P(k_p \times r_p)\}$, for some $k_i, r_i \geq 0$, $p > 0$ and $1 \leq i \leq p$. Since $\mathbf{V}$ is generated by its finite rectangular algebras, $\mathbf{V} = \mathbf{V}_1 \vee \mathbf{V}_2 \vee \mathbf{V}_3$, where $\mathbf{V}_1 = \mathbf{HSP}(\mathbf{R}_1), \mathbf{V}_2 = \mathbf{HSP}(\mathbf{R}_2)$ and $\mathbf{V}_3 = \mathbf{HSP}(\mathbf{R}_3)$. From Lemma 6.6 it follows that $\mathbf{V}_1 = \mathbf{V}_{(\omega,n_1)}$ and $\mathbf{V}_2 = \mathbf{V}_{(n_2,\omega)}$. Moreover, it is obvious that $\mathbf{V}_3 = \bigvee_{i=1}^{p} \mathbf{Var}(P(k_i \times r_i))$. □

Lemma 6.10. *If* $\mathcal{X}$ *is a finite rectangle and* R *is a correct partition of* $\mathcal{X}$ *such that* $\mathcal{X}/R$ *is an* n*-element bicluster, then both the* E_1*- and the* E_2*-depths of* $\mathcal{X}$ *are* $\geq n$.

Proof. Suppose $x \in X$ and C is a E_i-cluster of $\mathcal{X}$ ($i = 1, 2$). Consider $y \in C$. Since $\mathcal{X}/R$ is a bicluster, $R(x)(E_i)_R R(y)$. Hence, there exist $z, u \in X$ such that zRy, uRx and zE_iu. Thus, $y \in RE_i(u)$, and since R is correct, $y \in E_iR(u) = E_iR(x)$. Hence, $C \cap R(x) \neq \emptyset$.

Now let $d_1(\mathcal{X}) = k$. Since $\mathcal{X}$ is a rectangle, every E_2-cluster C of $\mathcal{X}$ contains precisely k points. Since there are exactly n R-equivalence classes, and each of

R-equivalence classes intersects C, we obtain $k \geq n$. A similar argument shows that the E_2-depth of $\mathcal{X}$ is also $\geq n$. $\square$

Lemma 6.11. *Suppose $n \times m$ is a finite rectangle, $m < n$, and R is a correct partition of $n \times m$. Then the number of elements of every E_0-cluster of $(n \times m)/R$ is less than or equal to m.*

Proof. Let $f_R\colon n \times m \to (n \times m)/R$ be the map sending every $x \in n \times m$ to $R(x)$. Suppose C is an E_0-cluster of $(n \times m)/R$, and $|C| = k$. Since R is correct, $f_R^{-1}(C)$ is a rectangle. By Lemma 6.10, $d_1(f_R^{-1}(C)), d_2(f_R^{-1}(C)) \geq k$. On the other hand, $f_R^{-1}(C) \subseteq n \times m$. Hence, $d_1(f_R^{-1}(C)) \leq m$ and $d_2(f_R^{-1}(C)) \leq n$. Thus, $k \leq m$. $\square$

Lemma 6.12. *Let $n \times m$ and R be as above. Also let C_i denote E_1-clusters and D denote E_0-clusters of $(n \times m)/R$, $k_i = \max\{|D| : D \subseteq C_i\}$, and $k = \sum_i k_i$. Then $k \leq m$.*

Proof. Let $f_R\colon n \times m \to (n \times m)/R$ be as above. For an E_1-cluster $C_i \subseteq (n \times m)/R$, choose an E_0-cluster $D_i \subseteq C_i$ containing k_i points, and consider a rectangle $f_R^{-1}(D_i)$. By Lemma 6.10, $d_1(E_1(f_R^{-1}(D_i))) \geq k_i$. Since R is correct, $E_1(f_R^{-1}(D_i)) \cap E_1(f_R^{-1}(D_j)) = \emptyset$ for $i \neq j$. Hence, $m = \sum_i d_1(E_1(f_R^{-1}(D_i))) \geq \sum_i k_i = k$. $\square$

Lemma 6.13. (1) $\mathbf{V}_{(\omega,n_1)}$ *is rectangularly representable for any $n_1 \in \omega$.*
(2) $\mathbf{V}_{(n_2,\omega)}$ *is rectangularly representable for any $n_2 \in \omega$.*

Proof. (1) Suppose $\mathcal{B} \in \mathbf{V}_{(\omega,n_1)}$ is a simple algebra and $\mathcal{X}$ its dual space. Also let $\{\mathcal{B}_i\}_{i \in I}$ be the family of all finite subalgebras of $\mathcal{B}$ and $\{\mathcal{X}_i\}_{i \in I}$ the family of their dual spaces.

Claim 6.14. The cardinality of each E_0-cluster of $\mathcal{X}$ is less than or equal to n_1.

Proof. Suppose there exists an E_0-cluster C of $\mathcal{X}$ containing k points and let $k > n_1$. Since $\mathbf{V}_{(\omega,n_1)}$ is locally finite, $\mathcal{B}$ is the direct limit of $\{\mathcal{B}_i\}_{i \in I}$. Then $\mathcal{X}$ is the inverse limit of $\{\mathcal{X}_i\}_{i \in I}$. Now from the definition of an inverse limit it follows that there exists $j \in I$ such that $\mathcal{X}_j$ contains an E_0-cluster consisting of $l \in \omega$ points, for $n_1 < l \leq k$. By Lemma 6.6, $\mathcal{B}_j \in \mathbf{Var}(\{P(m \times n_1)\}_{m \in \omega})$.

Using the standard splitting technique (see e.g. Kracht [13]) one can show that every finite simple $\mathbf{Df}_2$-algebra $\mathcal{A}$ is a *splitting algebra*, that is $\mathbf{Var}(\mathcal{A})$ is a strictly join prime element of $\Lambda(\mathbf{Df}_2)$.

Since $\mathcal{B}$ is a simple algebra, $\mathcal{B}_j$ is a subalgebra of $\mathcal{B}$, and $\mathbf{Df}_2$ has the congruence extension property, $\mathcal{B}_j$ is also simple, hence a splitting algebra. From $\mathcal{B}_j \in \mathbf{Var}(\{P(m \times n_1)\}_{m \in \omega})$ it follows that $\mathbf{Var}(\mathcal{B}_j) \leq \bigvee_{m \in \omega} \mathbf{Var}(P(m \times n_1))$. Since $\mathbf{Var}(\mathcal{B}_j)$ is strictly join prime, there exists $m_j \in \omega$ such that $\mathbf{Var}(\mathcal{B}_j) \leq \mathbf{Var}(P(m_j \times n_1))$. By Jonsson's lemma [10], $\mathcal{B}_j$ is a subalgebra of $P(m_j \times n_1)$.

Lemma 6.11 now shows that no E_0-cluster of $\mathcal{X}_j$ could have more than n_1 points – a contradiction, hence $k \leq n_1$. $\square$

Let $d_1(\mathcal{X}) = m < n_1$. For a E_1-cluster C_i and a E_0-cluster $D \subseteq C_i$, let

$$k_i = \max\{|D| : D \subseteq C_i\}.$$

Also let

$$k = \sum_{i=1}^{m} k_i.$$

Claim 6.15. $k \leq n_1$.

Proof. Let $D_i \subseteq C_i$ denote an E_0-cluster of $\mathcal{X}$ consisting of k_i points. Since $\mathcal{X}$ is an inverse limit of $\{\mathcal{X}_i\}_{i \in I}$, there exists $j \in I$ such that $\mathcal{X}_j$ contains a subset isomorphic to $\bigcup_{i=1}^{m} E_2(D_i)$. Similar arguments as in Claim 6.14 shows that $\mathcal{B}_j$ is a subalgebra of some finite rectangular algebra $P(m \times n_1)$. Hence, by Lemma 6.12, $k \leq n_1$. $\square$

Now we are in a position to prove Lemma 6.13. The proof is similar to the finite case, see Lemmas 3.7–3.9. So, we only give a sketch. The reader can easily reconstruct it from §3.

Let $\{C^j\}_{j \in J}$ be the class of E_2-clusters of $\mathcal{X}$ and $|J| = \kappa$. Consider a rectangle $(k\kappa) \times k$. Let $\Delta_i = ((k\kappa) \times \sum_{s=1}^{i} k_i) \setminus ((k\kappa) \times \sum_{s=1}^{i-1} k_i)$ for any $1 \leq i \leq m$, and $\Delta^\iota = (((k\iota) + k) \times k) \setminus ((k\iota) \times k)$ for any $\iota \in \kappa$. Note that Δ_i is isomorphic to a rectangle $(k\kappa) \times k_i$ and Δ^ι is isomorphic to a square $k \times k$. Also let $\Delta_i^\iota = \Delta_i \cap \Delta^\iota$. Obviously, Δ_i^ι is isomorphic to a rectangle $k \times k_i$. Then the same arguments as in Lemmas 3.5, 3.7–3.9 will show that there exists a partition R on $(k\kappa) \times k$ satisfying $RE_i(x) \subseteq E_i R(x)$ for any $x \in \kappa \times n_1$ and $i = 1, 2$ such that $((k\kappa) \times k)/R$ is isomorphic to $\mathcal{X}$. Hence, by Proposition 6.5, $P(\mathcal{X})$ is a subalgebra of $P((k\kappa) \times k)$. Since $\mathcal{B}$ is a subalgebra of $P(\mathcal{X})$, $\mathcal{B}$ is a subalgebra of $P((k\kappa) \times k)$ as well. By Lemma 6.6, $P((k\kappa) \times k)$ belongs to $\mathbf{V}_{(\omega, n_1)}$. Therefore, by Theorem 6.7, $\mathbf{V}_{(\omega, n_1)}$ is rectangularly representable.

(2) is proved analogously. $\square$

Remark 6.16. We remark here that in Lemma 6.13 we actually proved that $\mathbf{V}_{(\omega, n_1)}$ and $\mathbf{V}_{(n_2, \omega)}$ are canonical varieties, for any $n_1 \in \omega$ and $n_2 \in \omega$. We conjecture that every subvariety of $\mathbf{Df}_2$ is in fact a canonical variety.

Theorem 6.17. *Let* $\mathbf{V}$ *be a subvariety of* $\mathbf{Df}_2$*. Then:*

(1) $\mathbf{V}$ *is square representable iff* $\mathbf{V}$ *is generated by its finite square algebras.*
(2) $\mathbf{V}$ *is rectangularly representable iff* $\mathbf{V}$ *is generated by its finite rectangular algebras.*

Proof. (1) If $\mathbf{V}$ is square representable, then either $\mathbf{V} = \mathbf{Df}_2$ or $\mathbf{V}$ is finitely square representable. In both cases, it is obvious that $\mathbf{V}$ is generated by its finite square algebras. Conversely, if $\mathbf{V}$ is generated by its finite square algebras, then either all finite square algebras belong to $\mathbf{V}$, hence $\mathbf{V} = \mathbf{Df}_2$, or there is only a finite number of non-isomorphic finite square $\mathbf{V}$-algebras. In the latter case, the same arguments as in the proof of Theorem 6.4 ensure that $\mathbf{V} = \mathbf{Var}(P(n \times n))$ for some natural n and once again using Theorem 6.4 we get that $\mathbf{V}$ is square representable.

(2) Suppose $\mathbf{V}$ is rectangularly representable. If $\mathbf{V} = \mathbf{Df}_2$, then obviously $\mathbf{V}$ is generated by its finite rectangular algebras. Suppose $\mathbf{V}$ is a proper subvariety of $\mathbf{Df}_2$. Theorem 6.7 implies that for any $\mathcal{B} \in \mathbf{V}_S$ there exists a rectangular algebra $P(\kappa\times\iota) \in \mathbf{V}_S$ such that $\mathcal{B}$ is a subalgebra of $P(\kappa\times\iota)$. From $\mathbf{V} \subset \mathbf{Df}_2$ it follows that either κ or ι is a natural number. Hence, by Lemma 6.6, $\mathcal{B} \in \mathbf{HSP}(\mathrm{Fin}\mathbf{Rect}_{\mathbf{V}})$, $\mathbf{V}_S \subseteq \mathbf{HSP}(\mathrm{Fin}\mathbf{Rect}_{\mathbf{V}})$, and $\mathbf{V}$ is generated by its finite rectangular algebras.

Conversely, if $\mathbf{V} \subset \mathbf{Df}_2$ is generated by its finite rectangular algebras, then from Lemma 6.9 it follows that $\mathbf{V} = \mathbf{V}_{(\omega,n_1)} \vee \mathbf{V}_{(n_2,\omega)} \vee \bigvee_{i=1}^{p} \mathbf{Var}(P(k_i \times r_i))$ for some non-negative k_i, r_i, positive p, and $1 \leq i \leq p$. From Lemma 6.13 and Theorem 6.7 it follows that $\mathbf{V}_{(\omega,n_1)}$, $\mathbf{V}_{(n_2,\omega)}$ and each of $\mathbf{Var}(P(k_i \times r_i))$ is rectangularly representable. Hence, by Lemma 6.8, so is $\mathbf{V}$. $\square$

Corollary 6.18. *A variety $\mathbf{V} \in \Lambda(\mathbf{Df}_2)$ is rectangularly representable iff either $\mathbf{V} = \mathbf{Df}_2$, or $\mathbf{V} = \mathbf{V}_{(\omega,n_1)} \vee \mathbf{V}_{(n_2,\omega)} \vee \mathbf{V}'$, where $\mathbf{V}' = \bigvee_{i=1}^{p} \mathbf{Var}(P(k_i \times r_i))$ for some non-negative integers k_i, r_i, positive p, and $1 \leq i \leq p$.* $\square$

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