TARSKI'S THEOREM ON INTUITIONISTIC LOGIC, FOR POLYHEDRA

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Abstract. In 1938, Tarski proved that a formula is not intuitionistically valid if, and only if, it has a counter-model in the Heyting algebra of open sets of some topological space. In fact, Tarski showed that any Euclidean space $\mathbb{R}^n$ with $n \geqslant 1$ suffices, as does e.g. the Cantor space. In particular, intuitionistic logic cannot detect topological dimension in the Heyting algebra of all open sets of a Euclidean space. By contrast, we consider the lattice of open subpolyhedra of a given compact polyhedron $P \subseteq \mathbb{R}^n$, prove that it is a locally finite Heyting subalgebra of the (non-locally-finite) algebra of all open sets of P, and show that intuitionistic logic is able to capture the topological dimension of P through the bounded-depth axiom schemata. Further, we show that intuitionistic logic is precisely the logic of formulæ valid in all Heyting algebras arising from polyhedra in this manner. Thus, our main theorem reconciles through polyhedral geometry two classical results: topological completeness in the style of Tarski, and Jaśkowski's theorem that intuitionistic logic enjoys the finite model property. Several questions of interest remain open. E.g., what is the intermediate logic of all closed triangulable manifolds?

1. Introduction

If X is any topological space, the collection $\mathscr{O}(X)$ of its open subsets is a (complete) Heyting algebra whose underlying order is given by set-theoretic inclusion. One can then interpret formulæ of intuitionistic logic into $\mathscr{O}(X)$ by assigning open sets to propositional atoms, and then extending the assignment to formulæ using the operations of the Heyting algebra $\mathscr{O}(X)$. A formula is true under such an interpretation just when it evaluates to X. In 1938, Tarski ([35], English translation in [36]) proved that intuitionistic logic is complete with respect to this semantics. Moreover, Tarski showed that one can considerably restrict the class C of spaces under consideration without impairing completeness. In particular, one can take $\mathsf{C} := \{X \mid X \text{ is metrisable}\}$, and even $\mathsf{C} := \{\mathbb{R}\}$ or $\mathsf{C} := \{2^{\mathbb{N}}\}$, where $2^{\mathbb{N}}$ denotes the Cantor space. Tarski's result opened up a research area that continues to prosper to this day. Immediate descendants of [35] are the three seminal papers [24, 25, 26] by McKinsey and Tarski; [25, §3] offers a different proof of the main result of [35] in the dual language of closed sets and co-Heyting algebras. For an exposition of the different themes in spatial logic we refer to [2].

Intuitionistic logic has the finite model property. In 1936 Jaśkowski sketched a proof of this fact [19]; the first detailed exposition of the result[1] seems to be [31, Theorem 5.4] (see also [11, Theorem 2.57]). Algebraically, the finite model property

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[1]Though not exactly of the proof sketched by Jaśkowski: cf. [31, Lemma 5.3 and footnote (16)].

may be rephrased into the statement that there exists a set of finite Heyting algebras that generates the equational class (or *variety*) of all Heyting algebras. An algebraic proof of this result was first obtained by McKinsey and Tarski [23, 24] (see [5] for a discussion of this proof and a comparison with the model-theoretic method of filtration). Jaśkowski's proof shows that, in fact, there is a countable, recursively enumerable[2] such set.

Recall that an algebraic structure is *locally finite* if its finitely generated substructures are finite. The Heyting algebras $\mathscr{O}(\mathbb{R})$ and $\mathscr{O}(2^{\mathbb{N}})$ are very far from being locally finite. For example, [25, Theorem 3.33] shows that any Heyting algebra freely generated by a finite set embeds into both $\mathscr{O}(\mathbb{R})$ and $\mathscr{O}(2^{\mathbb{N}})$, and already the Heyting algebra freely generated by one element (the Rieger-Nishimura lattice [30, 28]) is infinite. Thus, while counter-models to formulæ that are not intuitionistically provable always exist in $\mathscr{O}(\mathbb{R})$, they are not automatically finite: one has to pick the open sets to be assigned to atomic formulæ with extra care in order to exhibit a finite counter-model such as the ones guaranteed by the finite model property, see e.g. [6].

Our main result provides a theorem in the style of Tarski that has the advantage of using locally finite Heyting algebras of open sets only, and hence affords at the same time the advantages of Jaśkowski's theorem. Our result exposes and exploits, we believe for the first time, the connection between intuitionistic logic and the classical PL (=piecewise linear) category of compact polyhedra in Euclidean spaces [32, 22]. The needed background is recalled in Section 2, to which the reader is referred for all unexplained notions in the rest of this Introduction. To state our results we prepare some notation.

For each $n \in \mathbb{N} := \{0, 1, 2, \ldots\}$ and each (always compact) polyhedron $P \subseteq \mathbb{R}^n$, we write $\mathrm{Sub_c}\, P$ for the collection of subpolyhedra of P — i.e., polyhedra in $\mathbb{R}^n$ contained in P. We set

$$\mathrm{Sub_o}\, P := \{O \subseteq P \mid P \setminus O \in \mathrm{Sub_c}\, P\},$$

where $\setminus$ is set-theoretic difference. Members of $\mathrm{Sub_o}\, P$ are called *open (sub)polyhedra* (of P) throughout this paper. (This choice of terminology requires some clarifications which we offer in Remark 2.11, when we discuss background on PL topology.)

It is a standard fact that $\mathrm{Sub_o}\, P$ is a distributive lattice under set-theoretic intersections and unions, and hence a sublattice of[3] $\mathscr{O}(P)$. In Section 3 we prove that $\mathrm{Sub_o}\, P$ is, in fact, a Heyting subalgebra of $\mathscr{O}(P)$. In the same section we prove that, unlike $\mathscr{O}(P)$, $\mathrm{Sub_o}\, P$ is always locally finite. The proof provides one of the key insights of the present paper: local finiteness essentially amounts to the Triangulation Lemma of PL topology, and thus reflects algebraically a crucial tameness property of polyhedra as opposed to general compact subsets of $\mathbb{R}^n$.

Further tameness properties of polyhedra emerge from their dimension theory, which is far simpler than the dimension theory of general metric spaces. All standard topological dimension theories agree on polyhedra [18, 29]. In fact, an elementary notion of dimension is available for every nonempty polyhedron $\emptyset \neq P$, in that $\dim P \leqslant d$ holds if, and only if, any $d + 2$ distinct points of P are affinely dependent. In Section 4, we establish a fundamental connection between the topological dimension of P and the structure of $\mathrm{Sub_o}\, P$: the latter lies in the variety of Heyting algebras of bounded depth d if, and only if, $\dim P \leqslant d$. Let us recall how

[2] Each finite Heyting algebra being presented, e.g., by the finite multiplication tables for its operations. Jaśkowski's theorem yields at once the decidability of intuitionistic logic. More is known: the problem of deciding whether a formula is intuitionistically provable is PSPACE-complete [34].

[3] Here and throughout, P is always equipped with the subspace topology inherited from the Euclidean topology of $\mathbb{R}^n$.

these varieties are defined. Consider the following inductively defined terms (over the variables, say, $X_0, X_1, \ldots$) in the similarity type of Heyting algebras:

$$\text{BD}_d := \begin{cases} (\, X_0 \vee \neg X_0 \,) & \text{if } d = 0, \text{ and} \\ (\, X_d \vee (X_d \to \text{BD}_{d-1} \,) & \text{if } d \geqslant 1. \end{cases}$$

Writing $\top$ for the constant in the type interpreted as the top element of Heyting algebras, to each $d \in \mathbb{N}$ we have a corresponding *bounded-depth equation*

$$\text{BD}_d = \top\,. \tag{1}$$

The variety of *Heyting algebras of bounded depth d* is the class of those Heyting algebras H that satisfy equation (1); in other words, H must have the property that each evaluation of the term BD_d in H results in the top element of H. For example, Heyting algebras of bounded depth 0 coincide with Boolean algebras. Under the standard one-one correspondence between varieties of Heyting algebras and intermediate logics, the variety of Heyting algebras of bounded depth d corresponds to the *intermediate logic of bounded depth d* (see, e.g., [11, Section 2.5]).

If now $\mathscr{P}$ is any family of polyhedra, we write $\mathsf{Log}\,\mathscr{P}$ for the extension of intuitionistic logic determined by $\mathscr{P}$, namely, the unique intermediate logic corresponding to the variety of Heyting algebras generated by the collection of Heyting algebras $\{\text{Sub}_o\, P \mid P \in \mathscr{P}\}$.

Given $d \in \mathbb{N}$, let us denote by P_d the set of all polyhedra of dimension less than or equal to d. Consider any finite poset A of depth $d \in \mathbb{N}$. (That is, suppose the cardinality of the longest chain in A is $d+1$; please see Subsection 2.1 for a formal definition.) In Section 5, using Alexandrov's notion of nerve [3], we construct a polyhedron P of dimension d such that the Heyting algebra of upper sets of A embeds into the Heyting algebra $\text{Sub}_o\, P$. This leads to our main result:

Theorem 1.1. *For each $d \in \mathbb{N}$, $\mathsf{Log}\,\mathsf{P}_d$ is the intermediate logic of bounded depth d. Therefore, the logic $\mathsf{Log}\,\bigcup_{d\in\mathbb{N}} \mathsf{P}_d$ of all polyhedra is intuitionistic logic.*

We prove the theorem in Section 6. Our proof is self-contained to within the standard facts from PL topology and Heyting algebras recalled in Section 2.

Returning to Tarski's theorem, let us consider Euclidean spaces $\mathbb{R}^N$ and $\mathbb{R}^n$ with $N > n \in \mathbb{N}$. In line with the compact setting of the present paper, let us in fact confine attention to their unit cubes $[0,1]^N$ and $[0,1]^n$. Then Tarski's results show, *inter alia*, that the Heyting algebras $\mathscr{O}\,([0,1]^N)$ and $\mathscr{O}\,([0,1]^n)$ satisfy precisely the same equations — i.e., in both cases the corresponding logic is intuitionistic logic — regardless of the fact that one cube has strictly larger topological dimension than the other. However, if we consider the smaller Heyting algebras of open subpolyhedra of the two cubes, then BD_n is valid in $[0,1]^n$ and is refuted in $[0,1]^N$. Restriction to a class of tame, geometric subsets of Euclidean space such as the polyhedra of our paper thus allows us to express the dimension of Euclidean spaces by means of intuitionistic logic and Heyting algebras.

2. Preliminaries

We assume familiarity with intuitionistic logic and Heyting algebras. A few standard references are [4, 20, 21, 11]. In this section we recall what we need. On the other hand, we assume rather less about PL topology. All needed definitions and results are recalled in detail in this section. A few standard references are [33, 17, 16, 22, 32].

'Distributive lattice' means 'bounded distributive lattice'; homomorphisms are to preserve both the maximum ($\top$) and the minimum ($\bot$) element. We write $\wedge$ and $\vee$ for meets and joins, and write $\to$ and $\neg$ for Heyting implication and negation.

2.1. **Posets and p-morphisms.** We denote the partial order relation on any poset by $\leqslant$, unless otherwise specified. Given any poset A and any $a \in A$, we set

$$\begin{aligned} \uparrow a &:= \{x \in A \mid a \leqslant x\}\,, \\ \downarrow a &:= \{x \in A \mid x \leqslant a\}\,. \end{aligned}$$

An *upper set* in A is a subset $U \subseteq A$ closed under $\uparrow$: if $a \in A$ satisfies $a \in U$, then $\uparrow a \subseteq U$. Similarly, a *lower set* in A is a subset closed under $\downarrow$. A *chain* is a totally ordered set. A *chain in* A is a subset $C \subseteq A$ that is a chain when equipped with the order inherited from A. We define the *depth* of A to be

$$\operatorname{dep} A := \sup\{|C| - 1 \mid C \subseteq A \text{ is a chain in } A\} \in \mathbb{N} \cup \{\infty\}\,.$$

If A and B are posets, a *p-morphism* from A to B is an order-preserving function $f\colon A \to B$ that commutes with $\uparrow$: for each $a \in A$,

$$f[\uparrow a] = \uparrow f(a).$$

Here and throughout, $f[\cdot]$ denotes direct image under the function f. Similarly, $f^{-1}[\cdot]$ will denote inverse image under the function f.

A poset is *rooted* if it has a minimum. Any poset A gives rise to a Heyting algebra. First, set

$$\operatorname{Up} A := \{U \subseteq A \mid U \text{ is an upper set in } A\}\,.$$

Under the inclusion order, $\operatorname{Up} A$ is a complete distributive lattice; arbitrary meets and joins are provided by set-theoretic unions and intersections. Hence the meet operation has an adjoint, the uniquely determined implication of $\operatorname{Up} A$ that makes it into a Heyting algebra.[4] For later use in the paper, we also prepare the dual notation

$$\operatorname{Lo} A := \{L \subseteq A \mid L \text{ is lower set in } A\}\,.$$

As for $\operatorname{Up} A$, we will always regard $\operatorname{Lo} A$ as a complete distributive lattice under the inclusion order. $\operatorname{Lo} A$ has a uniquely determined co-Heyting algebraic structure.[5]

Conversely, we can associate a poset to any Heyting algebra H. Set

$$\operatorname{Spec} H := \{F \subseteq H \mid F \text{ is a prime filter of the distributive lattice } H\}\,,$$

where the notation 'Spec' is for 'Spectrum'. Equipping $\operatorname{Spec} H$ with the inclusion order, we obtain a poset.

The Heyting algebras of the form $\operatorname{Up} A$, as A ranges over all finite posets, are precisely the finite Heyting algebras. To see this, given a Heyting algebra H, we consider the *Stone map*:

$$\begin{aligned} \widehat{\cdot}\colon H &\longrightarrow \operatorname{Up}\operatorname{Spec} H \\ h \in H &\longmapsto \widehat{h} := \{\mathfrak{p} \in \operatorname{Spec} H \mid h \in \mathfrak{p}\}\,. \end{aligned} \tag{2}$$

The following goes back to [9].

Lemma 2.1. *For any finite Heyting algebra H, the Stone map* (2) *is an isomorphism of Heyting algebras.*

Proof. For detailed proofs see [11, Sec. 8.4] and [27]. □

With the above in place, a modern statement of a part of Jaśkowski's result cited in the Introduction is:

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[4]See Subsection 2.4 for the generalisation of this construction to all topological spaces.

[5]As well as a Heyting one that will not be used in this paper.

Lemma 2.2 (The finite model property)**.** *The equational class of Heyting algebras is generated by the finite Heyting algebras: any Heyting algebra is a homomorphic image of a subalgebra of a product of Heyting algebras of the form* $\mathrm{Up}\, A$*, as* A *ranges over all finite posets.*

Proof. See e.g. [11, Thms. 2.57 and 7.21]. □

Remark 2.3. One can restrict the class of finite posets featuring in Lemma 2.2 in various ways. Thus, Jaśkowski exhibited a specific recursive sequence of posets. It is also known, for instance, that the class of all finite trees (=rooted posets T such that $\downarrow t$ is a chain for each $t \in T$) suffices, see e.g., [11, Cor. 2.33 and Ex. 2.17]. In this paper we only need the general form of the result as stated in Lemma 2.2. □

2.2. **Finite Esakia duality.** Lemma 2.1 can be lifted to a contravariant equivalence of categories between Heyting algebras and Esakia spaces [12]. These are ordered topological spaces with specific properties; for detailed definitions we refer the reader to [7]. In the finite case of interest here, topology can and will be dispensed with, because a finite Esakia space can be identified with a finite poset. Given a homomorphism of finite Heyting algebras $h\colon H \to K$, set

$$\begin{aligned} \mathrm{Spec}\, h\colon\ \mathrm{Spec}\, K &\longrightarrow \mathrm{Spec}\, H \\ \mathfrak{p} \in \mathrm{Spec}\, K &\longmapsto h^{-1}[\mathfrak{p}] \in \mathrm{Spec}\, H. \end{aligned}$$

Dually, given a p-morphism of posets $f\colon A \to B$, set

$$\begin{aligned} \mathrm{Up}\, f\colon\ \mathrm{Up}\, B &\longrightarrow \mathrm{Up}\, A \\ U \in \mathrm{Up}\, B &\longmapsto f^{-1}[U] \in \mathrm{Up}\, A. \end{aligned}$$

Let now $\mathsf{HA_f}$ and $\mathsf{Pos_f}$ denote the categories of finite Heyting algebras and their homomorphisms, and of finite posets and p-morphisms, respectively. Then the above defines functors

$$\begin{aligned} \mathrm{Spec}\colon \mathsf{HA_f} &\longrightarrow \mathsf{Pos_f^{op}}, \\ \mathrm{Up}\colon \mathsf{Pos_f} &\longrightarrow \mathsf{HA_f^{op}}. \end{aligned}$$

(We are indicating by C^{op} the category opposite to the category C, as is standard.)

Lemma 2.4 (Esakia duality, finite case)**.** *The functors* Spec *and* Up *are an equivalence of categories.*

Proof. See [11, Exs. 7.5, 7.6 and Sec. 8.5] and [27]. □

Remark 2.5. As with all duality results, Lemma 2.4 provides a dictionary between notions in $\mathsf{HA_f}$ and notions in $\mathsf{Pos_f}$. For example, one shows that a surjective p-morphism of finite posets dualises to an injective homomorphism of finite Heyting algebras, i.e. to a Heyting subalgebra, and conversely. We do not dwell on the details of such translations, and use them whenever needed in the sequel. □

2.3. **Bounded depth.** Through the bounded-depth equations BD_d of the Introduction, one can equationally express the analogue for Heyting algebras of the Krull dimension of commutative rings.[6]

Lemma 2.6. *For any non-trivial Heyting algebra* H *and each* $d \in \mathbb{N}$*, the following are equivalent.*

(i) *The longest chain of prime filters in* H *has cardinality* $d + 1$*.*
(ii) $\mathrm{dep}\, \mathrm{Spec}\, H = d$*.*

[6] For recent related literature see [8], where a Krull dimension is defined for any topological space and is used in obtaining fine-grained topological completeness results for modal and intermediate logics.

(iii) *H satisfies the equation* $\mathrm{BD}_d = \top$*, and fails each equation* $\mathrm{BD}_{d'} = \top$ *with* $0 \leqslant d' < d$.

Proof. See [11, Prop. 2.38 and Table 9.7] and [8]. □

2.4. **Heyting and co-Heyting algebras of open and closed sets.** The open (closed) sets of a topological space provide important examples of (co-)Heyting algebras. For background on co-Heyting algebras we refer to [25, §1 and *passim*], where these structures were first axiomatised equationally, and systematically investigated under the name of 'Brouwerian algebras'. We write $\lnot$ to denote co-Heyting negation, and $\leftarrow$ to denote co-Heyting implication.[7]

If X is any topological space, we write $\mathscr{O}(X)$ for its collection of open sets. Then $\mathscr{O}(X)$ is a complete distributive lattice, bounded above by $\top := X$ and below by $\bot := \emptyset$, with joins given by set-theoretic unions and meets given by

$$\bigwedge F := \operatorname{int} \bigcap F$$

for any family F of open subsets of X, where int denotes the interior operator of the given topology on X. Therefore $\mathscr{O}(X)$ has exactly one structure of Heyting algebra compatible with its distributive-lattice structure; namely, for any $U, V \in \mathscr{O}(X)$ the Heyting implication is given by

$$U \to V := \bigcup \{O \in \mathscr{O}(X) \mid U \cap O \subseteq V\} = \operatorname{int}\left((X \setminus U) \cup V \right). \tag{3}$$

In particular, the Heyting negation is given by

$$\neg U := U \to \bot = \operatorname{int}(X \setminus U).$$

Dually, the family $\mathscr{C}(X)$ of closed sets of X is a complete distributive lattice, bounded above by $\top := X$ and below by $\bot := \emptyset$, with meets given by set-theoretic intersections and joins given by

$$\bigvee F := \operatorname{cl} \bigcup F$$

for any family F of closed subsets of X, where cl denotes the closure operator of the given topology on X. Therefore $\mathscr{C}(X)$ has exactly one structure of co-Heyting algebra compatible with its distributive-lattice structure; namely, for any $C, D \in \mathscr{C}(X)$ the co-Heyting implication is given by

$$C \leftarrow D := \bigcap \{K \in \mathscr{C}(X) \mid C \subseteq D \cup K\} = \operatorname{cl}(C \setminus D). \tag{4}$$

In particular, the co-Heyting negation is given by

$$\llcorner D := \top \leftarrow D = \operatorname{cl}(X \setminus D).$$

Remark 2.7. All our results in this paper have versions for Heyting and co-Heyting algebras. We stressed the Heyting version in the Introduction, as this relates most directly to intuitionistic logic. However, we will see below that it is at times convenient in proofs to establish the co-Heyting version of the results first, because it is traditional in simplicial topology to work with closed simplices and polyhedra. Proofs for the corresponding Heyting versions are obtained through dual arguments, which we sometimes omit. □

[7] McKinsey's and Tarski's original notations were $\neg$ and $\dot{-}$, respectively.

2.5. **Polyhedra: basic notions.** An *affine combination* of $x_0, \ldots, x_d \in \mathbb{R}^n$ is an element $\sum_{i=0}^{d} r_i x_i \in \mathbb{R}^n$, where $r_i \in \mathbb{R}$ and $\sum_{i=0}^{d} r_i = 1$. If, additionally, $r_i \geqslant 0$ for each $i \in \{0, \ldots, d\}$, $\sum_{i=0}^{d} r_i x_i$ is a *convex combination.* Given any subset $S \subseteq \mathbb{R}^n$, the *convex hull* of S, written $\operatorname{conv} S$, is the collection of all convex combinations of finite subsets of S. Then S is *convex* if $S = \operatorname{conv} S$, and a *polytope* if $S = \operatorname{conv} V$ for a finite set $V \subseteq \mathbb{R}^n$. A *polyhedron* in $\mathbb{R}^n$ is any subset that can be written as a finite union of polytopes. The union over an empty index set is allowed, so that $\emptyset$ is a polyhedron. Any polyhedron is closed and bounded, hence compact. If $P \subseteq \mathbb{R}^n$ is a polyhedron, then by an *open* (*sub*)*polyhedron* in P we mean the complement (within P) of a polyhedron which is included in P. The points $x_0, \ldots, x_d \in \mathbb{R}^n$ are *affinely independent* if the vectors $x_1 - x_0, x_2 - x_0, \ldots, x_d - x_0$ are linearly independent, a condition which is invariant under permutations of the index set $\{0, \ldots, d\}$. A *simplex* in $\mathbb{R}^n$ is a *non-empty*[8] subset of the form $\sigma := \operatorname{conv} V$, where $V := \{x_0, \ldots, x_d\}$ is a set of affinely independent points. Then V is the uniquely determined such affinely independent set [22, Proposition 2.3.3], and σ is a *d-simplex* with *vertices* $x_0, \ldots, x_d$. A *face* of the simplex σ is the convex hull of a non-empty subset of V, and thus is itself a d'-simplex for a uniquely determined $d' \in \{0, \ldots, d\}$. Hence the 0-faces of σ are precisely its vertices.

We write

$$\sigma = x_0 \cdots x_d, \ \ \sigma \preccurlyeq \tau, \ \text{ and } \sigma \prec \tau$$

to indicate that σ is the d-simplex whose vertices are $x_0, \ldots, x_d$, that σ is a face of τ, and that σ is a proper (i.e. $\neq \tau$) face of τ, respectively. If $\sigma = x_0 \cdots x_d \in \mathbb{R}^n$, the *relative interior* of σ, denoted $\operatorname{relint} \sigma$, is the topological interior of σ in the affine subspace of $\mathbb{R}^n$ spanned by[9] σ. (Thus, the relative interior of a 0-dimensional simplex — a point — is the point itself.) To rephrase through coordinates, note that by the affine independence of the vertices of σ, for each $x \in \sigma$ there exists a unique choice of $r_i \in \mathbb{R}$ with $x = \sum_{i=0}^{d} r_i x_i$ and $r_i \geqslant 0$, $\sum_{i=1}^{d} r_i = 1$. The r_i's are traditionally called the *barycentric coordinates* of x. Then $\operatorname{relint} \sigma$ coincides with the subset of σ of those points $x \in \sigma$ whose barycentric coordinates are strictly positive. Note that $\operatorname{cl} \operatorname{relint} \sigma = \sigma$, the closure being taken in the ambient Euclidean space $\mathbb{R}^n$. In the rest of this paper, for any set $S \subseteq \mathbb{R}^n$ we use the notation

$$\operatorname{cl} S \tag{5}$$

to denote the closure of S in the ambient Euclidean space $\mathbb{R}^n$. Observe that if $P \subseteq \mathbb{R}^n$ is a polyhedron and $S \subseteq P$, then the closure of S in the subspace P of $\mathbb{R}^n$ agrees with $\operatorname{cl} S$, because P is closed in $\mathbb{R}^n$.

2.6. **Polyhedra: the Triangulation Lemma.**

Definition 2.8 (Triangulation)**.** A *triangulation*[10] is a finite set Σ of simplices in $\mathbb{R}^n$ satisfying the following conditions.

(1) If $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$.
(2) If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is either empty, or a common face of σ and τ.

The *support*, or *underlying polyhedron*, of the triangulation Σ is

$$|\Sigma| := \bigcup \Sigma \ \subseteq \mathbb{R}^n.$$

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[8] It is expedient in this paper not to regard $\emptyset$ as a simplex.

[9] Recall that the affine subspace spanned by a subset $S \subseteq \mathbb{R}^n$ is the collection of all affine combinations of finite subsets of S, or equivalently, the intersection of all affine subspaces of $\mathbb{R}^n$ containing S.

[10] Also known as (*geometric*) *simplicial complex.* Note that the empty triangulation $\emptyset$ is allowed.

One also says that Σ *triangulates* the subset $|\Sigma|$ of $\mathbb{R}^n$. A *subtriangulation* of the triangulation Σ is any subset $\Delta \subseteq \Sigma$ that is itself a triangulation. This is equivalent to the condition that Δ be closed under taking faces — i.e. satisfies just (1) in Definition 2.8 — for then (2) follows [22, Proposition 2.3.6]. By the *vertices* of Σ we mean the vertices of the simplices in Σ.

Observe that a subtriangulation of Σ is precisely the same thing as a lower set of Σ, the latter being regarded as a poset under inclusion. This fact will be heavily exploited below, cf. in particular Section 4. The following standard fact makes precise the idea that a triangulation Σ provides a finitary description of the triangulated space $|\Sigma|$.

Lemma 2.9. *If Σ is a triangulation, for each $x \in |\Sigma|$ there is exactly one simplex $\sigma^x \in \Sigma$ such that $x \in \operatorname{relint} \sigma$.*

Proof. See [22, Proposition 2.3.6]. □

In light of Lemma 2.9, in the sequel we adopt the notation σ^x without further comment; the simplex σ^x is called the *carrier* of x (in Σ).

Any subset of $\mathbb{R}^n$ that admits a triangulation, being a finite union of simplices, is evidently a polyhedron. The rather less trivial converse is true, too, in the following strong sense.

Lemma 2.10 (Triangulation Lemma)**.** *Given finitely many polyhedra $P, P_1, \ldots, P_m$ in $\mathbb{R}^n$ with $P_i \subseteq P$ for each $i \in \{1, \ldots, m\}$, there exists a triangulation Σ of P such that, for each $i \in \{1, \ldots, m\}$, the collection*

$$\Sigma_i \coloneqq \{\sigma \in \Sigma \mid \sigma \subseteq P_i\}$$

is a triangulation of P_i, i.e. $|\Sigma_i| = P_i$.

Proof. [32, Theorem 2.11 and Addendum 2.12]. □

Remark 2.11. Recall from the Introduction that $\mathrm{Sub_c}\, P$ and $\mathrm{Sub_o}\, P$ denote the collections of polyhedra and open polyhedra in $P \subseteq \mathbb{R}^n$, respectively. As an anonymous referee appropriately pointed out to us, a remark on our terminology is in order. Let us first recall that in the literature on PL topology a polyhedron is a subset $Q \subseteq \mathbb{R}^n$ such that $Q \coloneqq \bigcup \Sigma$ for some locally finite, not necessarily finite, triangulation Σ. In turn, Σ is a *locally finite triangulation* if it is a (possibly infinite) set of simplices in $\mathbb{R}^n$ meeting conditions (1) and (2) in Definition 2.8, and such that each simplex in Σ has a neighbourhood in $\mathbb{R}^n$ that intersects finitely many members of Σ. Cf. e.g. [13, pag. 97]. For the purposes of this remark, let us call a subset $Q \subseteq \mathbb{R}^n$ satisfying these conditions a *polyhedral set*, to distinguish it from our polyhedra. A polyhedral set is in general neither open nor closed in the subspace topology it inherits from $\mathbb{R}^n$. And *closed* and *open polyhedral sets* in $\mathbb{R}^n$ have the expected meaning — they are just the polyhedral sets in $\mathbb{R}^n$ that are closed and open subsets of $\mathbb{R}^n$, respectively. It follows that a polyhedral set is a polyhedron (in the sense of this paper) if, and only if, it is compact if, and only if, it admits a finite triangulation. This last equivalence is based on the Triangulation Lemma above. Our usage of "polyhedron" *tout court* as a shorthand for "compact polyhedron" is frequent in the literature. Now, if $P \subseteq \mathbb{R}^n$ is a polyhedron, we defined an open (sub)polyhedron O in P to be the set-theoretic complement in P of a polyhedron $Q \subseteq P$. Such a subset $O \subseteq P$ is then open in the topology of P, and can be shown to be also a polyhedral set in $\mathbb{R}^n$. However, the notion of "polyhedral set in $\mathbb{R}^n$ that is a subset of P and happens to be open in the topology of P" is much more general than our notion of "open polyhedron in P". Indeed, it is a non-trivial fact (see e.g. [13, Corollary 3.2.22]) that the former sentence just describes the collection of *all* subsets of P that are open in the topology of P — whether their complement in P is a polyhedron or not! The usual duality between open and closed sets thus breaks down: in $\mathbb{R}^n$, on standard PL terminology, the posets of open and closed polyhedral sets are not order-dual. Our more restrictive notion of open subpolyhedron, though not standard in PL topology, reinstates that duality. □

The Triangulation Lemma is the fundamental tool in this paper. Here is a first consequence[11] of Lemma 2.10.

Corollary 2.12. *For any polyhedron $P \subseteq \mathbb{R}^n$, both $\mathrm{Sub_c}\, P$ and $\mathrm{Sub_o}\, P$ are distributive lattices (under set-theoretic intersections and unions) bounded above by P and below by $\emptyset$.* □

Proof. Given polyhedra $A, B \subseteq P$, by Lemma 2.10 there is a triangulation Σ of P along with two subtriangulations Σ_A, Σ_B with $A = |\Sigma_A|$ and $B = |\Sigma_B|$. Then the triangulation $\Delta \coloneqq \Sigma_A \cap \Sigma_B$ triangulates $A \cap B$. Indeed, $|\Delta| \subseteq A \cap B$. Conversely, if $x \in A \cap B$ then there are $\sigma_A \in \Sigma_A, \sigma_B \in \Sigma_B$ with $x \in \sigma_A$ and $x \in \sigma_B$. Setting $\tau \coloneqq \sigma_A \cap \sigma_B$, we have $x \in \tau \neq \emptyset$ and $\tau \in \Sigma$. Since τ is a face of $\sigma_A \in \Sigma_A$, $\tau \in \Sigma_A$. Similarly, $\tau \in \Sigma_B$. Hence $\tau \in \Delta$, and $A \cap B \subseteq |\Delta|$. Similarly, it is elementary that the triangulation $\nabla \coloneqq \Sigma_A \cup \Sigma_B$ triangulates $A \cup B$. It is obvious that P and $\emptyset$ are the upper and lower bounds of $\mathrm{Sub_c}\, P$. The statements about $\mathrm{Sub_o}\, P$ follow at once by taking complements. □

In Subsection 3.1 we shall strengthen Corollary 2.12 to the effect that $\mathrm{Sub_o}\, P$ is a Heyting subalgebra of the Heyting algebra $\mathscr{O}(P)$.

2.7. **Polyhedra: dimension theory.** The (*affine*) *dimension* of a d-simplex $\sigma = x_0 \cdots x_d$ in $\mathbb{R}^n$ is the linear-space dimension of the affine subspace of $\mathbb{R}^n$ spanned by σ, and that dimension is precisely d because of the affine independence of the vertices of σ. The (*affine*) *dimension* of a nonempty polyhedron P in $\mathbb{R}^n$ is the maximum of the dimensions of all simplices contained in P; if $P = \emptyset$, its dimension is -1. We write $\dim P$ for the dimension of P. Given a triangulation Σ in $\mathbb{R}^n$, the (*combinatorial*) *dimension* of Σ is

$$\dim \Sigma \coloneqq \max \{d \in \mathbb{N} \mid \text{ there exists } \sigma \in \Sigma \text{ such that } \sigma \text{ is a } d\text{-simplex}\}.$$

Again, the dimension of an empty triangulation is -1. The facts stated in the following lemma are standard.

Lemma 2.13. *For any polyhedron $\emptyset \neq P \subseteq \mathbb{R}^n$ and every $d \in \mathbb{N}$, the following are equivalent.*

(i) $\dim P = d$.
(ii) *There exists a triangulation Σ of P such that* $\dim \Sigma = d$.
(iii) *All triangulations Σ of P satisfy* $\dim \Sigma = d$.

Proof. With the Triangulation Lemma 2.10 available, the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follow from linear algebra. □

Remark 2.14. Items (i)–(iii) in Lemma 2.13 refer to the affine structure of the Euclidean space $\mathbb{R}^n$. We point out that the dimension of P could also be defined in purely topological terms as the Lebesgue covering dimension [29, Definition 3.1.1] of the topological space P. The equivalence of this latter definition with item (i) in Lemma 2.13 is, in essence, the content of the Lebesgue Covering Theorem [18, Theorem IV 2]. □

[11] Cf. [22, Proposition 2.3.6(d)].

3. The locally finite Heyting algebra of a polyhedron

Throughout this section we fix $n \in \mathbb{N}$ along with a polyhedron $P \subseteq \mathbb{R}^n$. We shall study the distributive lattice $\mathrm{Sub_o}\, P$ (Corollary 2.12). We begin by proving that $\mathrm{Sub_o}\, P$ is in fact a Heyting algebra. We then prove that $\mathrm{Sub_o}\, P$ is always locally finite.

3.1. The Heyting algebra of open subpolyhedra.

Let us record a well-known, elementary observation on relative interiors for which we know no convenient reference.

Lemma 3.1. *Let Σ be a triangulation in $\mathbb{R}^n$, let $\tau = x_0 \cdots x_d$ be a simplex of Σ, and let $x \in \mathrm{relint}\, \tau$. Then no proper face $\sigma \prec \tau$ contains x. Hence, in particular, the carrier σ^x of x in Σ is the inclusion-smallest simplex of Σ containing x.*

Proof. There are $r_0, \ldots, r_d \in (0, 1]$ such that $x = \sum_{i=0}^{d} r_i x_i$ and $\sum_{i=0}^{d} r_i = 1$. Let $\rho_i := x_0 \cdots x_{i-1} x_{i+1} \cdots x_d$. Clearly $\rho_i \prec \tau$ for each $i \in \{0, \ldots, d\}$, and for each $\sigma \prec \tau$ there exists $i \in \{0, \ldots, d\}$ such that $\sigma \preccurlyeq \rho_i$. Hence, if we assume by way of contradiction that $x \in \sigma \prec \tau$, then $x \in \rho_i$ for some $i \in \{0, \ldots, d\}$; say $x \in \rho_0$. Then $x = \sum_{i=1}^{d} s_i x_i$, for some $s_1, \ldots, s_d \in [0, 1]$ such that $\sum_{i=1}^{d} s_i = 1$. It follows that $r_0 = \sum_{i=1}^{d} (s_i - r_i)$, and so

$$0 = x - x = \sum_{i=1}^{d} s_i x_i - \sum_{i=0}^{d} r_i x_i = \sum_{i=1}^{d} (s_i - r_i) x_i - r_0 x_0 = \sum_{i=1}^{d} (s_i - r_i)(x_i - x_0).$$

Since $r_0 > 0$, there must be $i \in \{1, \ldots, d\}$ such that $s_i - r_i \neq 0$, contradicting the affine independence of $x_0, \ldots, x_d$. $\square$

The next lemma is the key fact of this subsection.[12]

Lemma 3.2. *Let P and Q be polyhedra in $\mathbb{R}^n$ with $Q \subseteq P$, and suppose Σ is a triangulation of P such that*

$$\Sigma_Q := \{\sigma \in \Sigma \mid \sigma \subseteq Q\}$$

triangulates Q. Define

- $C := \mathrm{cl}\,(P \setminus Q)$,
- $\Sigma_C := \{\sigma \in \Sigma \mid \sigma \subseteq C\}$, *and*
- $\Sigma^* := \{\sigma \in \Sigma \mid$ *There exists* $\tau \in \Sigma \setminus \Sigma_Q$ *such that* $\sigma \preccurlyeq \tau\}$.

Then

(1) $\Sigma_C = \Sigma^*$, *and*
(2) $|\Sigma_C| = |\Sigma^*| = C$.

In particular, C is a polyhedron.

Proof. We first show that Σ^* triangulates C, that is:

$$|\Sigma^*| := \bigcup \Sigma^* = C. \tag{*}$$

To show $|\Sigma^*| \subseteq C$, let $\sigma \in \Sigma^*$, and pick $\tau \in \Sigma \setminus \Sigma_Q$ such that $\sigma \preccurlyeq \tau$. We prove that $\mathrm{relint}\, \tau \subseteq P \setminus Q$. For, if $x \in \mathrm{relint}\, \tau$, by Lemma 3.1 there are no simplices $\sigma \in \Sigma$ such that $x \in \sigma \prec \tau$. Then, by definition of triangulation, for any simplex $\rho \in \Sigma$, $x \in \rho$ entails $\tau \preccurlyeq \rho$. Hence no simplex of Σ_Q contains x, or equivalently, $x \notin Q$ and therefore $\mathrm{relint}\, \tau \subseteq P \setminus Q$.

Now, it is clear that any simplex τ satisfies $\tau = \mathrm{cl}\, \mathrm{relint}\, \tau$. It follows that $\sigma \subseteq \tau = \mathrm{cl}\, \mathrm{relint}\, \tau \subseteq \mathrm{cl}\,(P \setminus Q)$, and thus $|\Sigma^*| \subseteq C$ as was to be shown.

[12] Cf. [22, Proposition 2.3.7].

Conversely, to show $C \subseteq |\Sigma^*|$, let $x \in C$. Since C is the closure of $P \setminus Q$ in $\mathbb{R}^n$, there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subseteq P \setminus Q$ that converges to x. Clearly the carrier σ^{x_i} of x_i in Σ lies in $\Sigma \setminus \Sigma_Q$, for all $i \in \mathbb{N}$. Since $\Sigma \setminus \Sigma_Q$ is finite, there must exist a simplex $\tau \in \Sigma \setminus \Sigma_Q$ containing infinitely many elements of $\{x_i\}_{i\in\mathbb{N}}$. Then there exists a subsequence of $\{x_i\}_{i\in\mathbb{N}}$ that is contained in τ and converges to x. Since τ is closed, $x \in \tau$, and therefore $x \in |\Sigma^*|$ as was to be shown.

This establishes (*). It now suffices to prove (1). For the non-trivial inclusion $\Sigma_C \subseteq \Sigma^*$, let $\sigma \in \Sigma$ be such that $\sigma \subseteq C$, and pick $\beta \in \operatorname{relint}\sigma$. There is a sequence $\{x_i\}_{i\in\mathbb{N}} \subseteq P \setminus Q$ converging to $\beta \in \sigma$. Since each x_i is in some simplex of $\Sigma \setminus \Sigma_Q$ and Σ is finite, there must exist a simplex $\tau \in \Sigma \setminus \Sigma_Q$ containing a subsequence of $\{x_i\}_{i\in\mathbb{N}}$ that converges to β. Since τ is closed, $\beta \in \tau$. But by Lemma 3.1, $\sigma^\beta = \sigma$, so that $\sigma \subseteq \tau$ and $\sigma \in \Sigma^*$. □

Corollary 3.3. *Given polyhedra Q_1, Q_2 in $\mathbb{R}^n$, the set* $\operatorname{cl}(Q_2 \setminus Q_1)$ *is a polyhedron.*

Proof. Observe that $Q_2 \setminus Q_1 = Q_2 \setminus (Q_1 \cap Q_2)$ and apply Corollary 2.12 together with Lemma 3.2 to the set $P \coloneqq \operatorname{conv}(Q_1 \cup Q_2)$, which clearly is a polyhedron. □

Corollary 3.4. *The lattice* $\operatorname{Sub_c} P$ *is closed under the co-Heyting implication* (4) *of* $\mathscr{C}(P)$. *Dually, the lattice* $\operatorname{Sub_o} P$ *is closed under the Heyting implication* (3) *of* $\mathscr{O}(P)$.

Proof. The first statement is an immediate consequence of Corollary 3.3. The second statement follows by dualising. □

3.2. **Local finiteness through triangulations.** Having established that $\operatorname{Sub_o} P$ is a Heyting subalgebra of $\mathscr{O}(P)$, we infer an important structural property of $\operatorname{Sub_o} P$, local finiteness. For this, we first identify the class of subalgebras of $\operatorname{Sub_o} P$ that corresponds to triangulations of P. These algebras will have a central rôle in the sequel, too.

Definition 3.5 (Σ-definable polyhedra)**.** For any triangulation Σ in $\mathbb{R}^n$, we write $\mathsf{P_c}(\Sigma)$ for the sublattice of $\mathscr{C}(|\Sigma|)$ generated by Σ, and $\mathsf{P_o}(\Sigma)$ for the sublattice of $\mathscr{O}(|\Sigma|)$ generated by $\{|\Sigma| \setminus C \mid C \in \mathsf{P_c}(\Sigma)\}$. We call $\mathsf{P_c}(\Sigma)$ the set of Σ-*definable polyhedra*, and $\mathsf{P_o}(\Sigma)$ the set of Σ-*definable open polyhedra*.

Note that we have

$$\mathsf{P_c}(\Sigma) = \{C \subseteq \mathbb{R}^n \mid C \text{ is the union of some subset of } \Sigma\}.$$

Lemma 3.6. *For any triangulation Σ of P, $\mathsf{P_c}(\Sigma)$ is a co-Heyting subalgebra of* $\operatorname{Sub_c} P$. *Dually, $\mathsf{P_o}(\Sigma)$ is a Heyting subalgebra of* $\operatorname{Sub_o} P$.

Proof. For any $\emptyset \neq C, D \in \mathsf{P_c}(\Sigma)$, it follows immediately by the assumptions that C and D are triangulated by the collection of simplices of Σ contained in C and D, respectively. Hence $C \leftarrow D \coloneqq \operatorname{cl}(C \setminus D) = |\Sigma^*| = \bigcup \Sigma^*$ by Corollary 3.3 and Lemma 3.2, where Σ^* is the appropriate subset of Σ as per Lemma 3.2. Thus $C \leftarrow D \in \mathsf{P_c}(\Sigma)$. □

Corollary 3.7. *Let H be the co-Heyting subalgebra of* $\operatorname{Sub_c} P$ *generated by finitely many polyhedra $P_1, \ldots, P_m \subseteq P$. Let further Σ be any triangulation of P that triangulates each P_i, $i \in \{1, \ldots, m\}$. Then H is a co-Heyting subalgebra of $\mathsf{P_c}(\Sigma)$. In particular, H is finite. Dually for the Heyting subalgebra of* $\operatorname{Sub_o} P$ *generated by $P \setminus P_i$, $i \in \{1, \ldots, m\}$.*

Proof. Each P_i is the union of those simplices of Σ that are contained in P_i, by assumption. It follows that the distributive lattice L generated in $\operatorname{Sub_c} P$ by $\{P_1, \ldots, P_m\}$ is entirely contained in $\mathsf{P_c}(\Sigma)$. Now, if $C, D \in L$, $C \leftarrow D \coloneqq$

$\mathrm{cl}\,(C \setminus D) = |\Sigma^*| = \bigcup \Sigma^*$ by Corollary 3.3 and Lemma 3.2, where Σ^* is the appropriate subset of Σ as per Lemma 3.2. Hence $C \leftarrow D \in \mathsf{P}_{\mathrm{c}}\,(\Sigma)$, as was to be shown. □

Corollary 3.8. *The Heyting algebra* $\mathrm{Sub}_{\mathrm{o}}\, P$ *is locally finite, and so is the co-Heyting algebra* $\mathrm{Sub}_{\mathrm{c}}\, P$*.*

Proof. The second statement is Corollary 3.7 together with the Triangulation Lemma 2.10. The first statement follows by dualising. □

4. Topological dimension and bounded depth

The aim of this section is to prove:

Theorem 4.1. *For any polyhedron* $\emptyset \neq P \subseteq \mathbb{R}^n$ *and every* $d \in \mathbb{N}$*, the following are equivalent.*

(i) $\dim P = d$.
(ii) *The Heyting algebra* $\mathrm{Sub}_{\mathrm{o}}\, P$ *satisfies the equation* $\mathrm{BD}_d = \top$*, and fails each equation* $\mathrm{BD}_{d'} = \top$ *for each integer* $0 \leqslant d' < d$*.*

We deduce the theorem from a combinatorial counterpart of the result for triangulations, Lemma 4.5 below. In turn, this lemma will follow from the analysis of posets arising from triangulations that we carry out first.

4.1. Posets dual of algebras of definable polyhedra.

Consider a triangulation Σ, and the finite Heyting algebra $\mathsf{P}_{\mathrm{o}}\,(\Sigma)$. We shall henceforth regard Σ as a poset under the inclusion order, whenever convenient. Note that the inclusion order of Σ is the same thing as the "face order" $\sigma \preccurlyeq \tau$ we have been using above: since Σ is a triangulation (as opposed to a mere set of simplices), $\sigma \subseteq \tau$ implies $\sigma \preccurlyeq \tau$, and the converse implication is obvious. Indeed, the poset Σ is a much-studied object in combinatorics, where it is known as the *face poset* of a simplicial complex. We next show what rôle Σ plays for the Heyting algebra $\mathsf{P}_{\mathrm{o}}\,(\Sigma)$, by establishing an isomorphism of Heyting algebras $\mathrm{Up}\,\Sigma \cong \mathsf{P}_{\mathrm{o}}\,(\Sigma)$; equivalently, through Esakia duality (Lemma 2.4), the face poset Σ is isomorphic to the dual poset of the algebra $\mathsf{P}_{\mathrm{o}}\,(\Sigma)$. This result is technically important, because the prime filters of $\mathsf{P}_{\mathrm{o}}\,(\Sigma)$, or what amounts to the same, its join-irreducible elements, are somewhat harder to visualise than the simplices of Σ. There are corresponding results for the co-Heyting algebra $\mathsf{P}_{\mathrm{c}}\,(\Sigma)$ which we do not spell out as we do not need them for the proof of our main result.

We recall the notion of open star of a simplex, cf. e.g. [22, Definition 2.4.2].

Definition 4.2 (Open star)**.** For Σ a triangulation, the *open star* of $\sigma \in \Sigma$ is the subset of $|\Sigma|$ defined by

$$\mathrm{o}(\sigma) \coloneqq \bigcup_{\sigma \subseteq \tau \in \Sigma} \mathrm{relint}\, \tau.$$

Although not immediately obvious, it is classical (see e.g. [22, Proposition 2.4.3]) that the open star of any simplex is an open subpolyhedron, that is, for each $\sigma \in \Sigma$

$$\mathrm{o}(\sigma) \in \mathsf{P}_{\mathrm{o}}\,(\Sigma). \tag{6}$$

Indeed, set

$$K_\sigma \coloneqq \{\tau \in \Sigma \mid \sigma \not\subseteq \tau\}.$$

Then K_σ is clearly a subtriangulation of Σ, $|K_\sigma|$ is a subpolyhedron of $|\Sigma|$, and thus $O \coloneqq |\Sigma| \setminus |K_\sigma| \in \mathsf{P}_{\mathrm{o}}\,(|\Sigma|)$; but one can show using Lemma 2.9 that $O = \mathrm{o}(\sigma)$, so (6) holds.

We now define a function

$$\begin{aligned} \gamma^{\uparrow}\colon\ \mathrm{Up}\,\Sigma &\longrightarrow \mathsf{P}_{\mathrm{o}}\,(\Sigma) \\ U \in \mathrm{Up}\,\Sigma &\longmapsto \bigcup_{\sigma\in U} \operatorname{relint}\sigma. \end{aligned} \tag{7}$$

To see that $\gamma^{\uparrow}(U)$ indeed lies in $\mathsf{P}_{\mathrm{o}}\,(\Sigma)$ use the fact that Σ is a finite poset to list the minimal elements $\sigma_1,\dots,\sigma_u$ of the upper set U. Then

$$U = {\uparrow}\sigma_1 \cup \cdots \cup {\uparrow}\sigma_u,$$

so that

$$\begin{aligned} \gamma^{\uparrow}(U) &= \gamma^{\uparrow}({\uparrow}\sigma_1) \cup \cdots \cup \gamma^{\uparrow}({\uparrow}\sigma_u) \\ &= \left(\bigcup_{\sigma_1\subseteq\tau\in\Sigma} \operatorname{relint}\tau\right) \cup \cdots \cup \left(\bigcup_{\sigma_u\subseteq\tau\in\Sigma} \operatorname{relint}\tau\right) \\ &= \mathrm{o}(\sigma_1) \cup \cdots \cup \mathrm{o}(\sigma_u). \end{aligned}$$

Thus $\gamma^{\uparrow}(U)$ is a union of open stars and hence a member of $\mathsf{P}_{\mathrm{o}}\,(\Sigma)$.

Lemma 4.3. *The map* $\gamma^{\uparrow}$ *of* (7) *is an isomorphism of the finite Heyting algebras* $\mathrm{Up}\,\Sigma$ *and* $\mathsf{P}_{\mathrm{o}}\,(\Sigma)$.

Proof. It suffices to show that $\gamma^{\uparrow}$ is an isomorphism of distributive lattices. It is clear that $\gamma^{\uparrow}$ preserves the top and bottom elements, and that it preserves unions: if $U, V \in \mathrm{Up}\,\Sigma$ then

$$\gamma^{\uparrow}(U\cup V) = \bigcup_{\sigma\in U\cup V} \operatorname{relint}\sigma = \left(\bigcup_{\sigma\in U} \operatorname{relint}\sigma\right) \cup \left(\bigcup_{\sigma\in V} \operatorname{relint}\sigma\right) = \gamma^{\uparrow}(U) \cup \gamma^{\uparrow}(V).$$

Concerning intersections,

$$\begin{aligned} \gamma^{\uparrow}(U) \cap \gamma^{\uparrow}(V) &= \left(\bigcup_{\sigma\in U} \operatorname{relint}\sigma\right) \cap \left(\bigcup_{\tau\in V} \operatorname{relint}\tau\right) \\ &= \bigcup_{\sigma\in U} \left(\operatorname{relint}\sigma \cap \bigcup_{\tau\in V} \operatorname{relint}\tau\right) \\ &= \bigcup_{\sigma\in U} \bigcup_{\tau\in V} (\operatorname{relint}\sigma \cap \operatorname{relint}\tau) \\ &= \bigcup_{\sigma\in U,\, \tau\in V} (\operatorname{relint}\sigma \cap \operatorname{relint}\tau) \end{aligned} \tag{8}$$

By Lemma 2.9, for any two $\sigma, \tau \in \Sigma$ the intersection $\operatorname{relint}\sigma \cap \operatorname{relint}\tau$ is empty as soon as $\sigma \neq \tau$. Hence from (8) we deduce

$$\gamma^{\uparrow}(U) \cap \gamma^{\uparrow}(V) = \bigcup_{\delta\in U\cap V} \operatorname{relint}\delta = \gamma^{\uparrow}(U\cap V),$$

as was to be shown.

To prove $\gamma^{\uparrow}$ is surjective, let $O \in \mathsf{P}_{\mathrm{o}}\,(\Sigma)$ and set $P := |\Sigma| \setminus O \in \mathsf{P}_{\mathrm{c}}\,(\Sigma)$. Then, by definition of $\mathsf{P}_{\mathrm{c}}\,(\Sigma)$, there is exactly one subtriangulation Δ of Σ such that $P = |\Delta|$, and Δ is a lower set of (the poset) Σ. Set $U := \Sigma \setminus \Delta$, so that U is an upper set of Σ. We show:

$$O = \bigcup_{\sigma\in U} \operatorname{relint}\sigma. \tag{9}$$

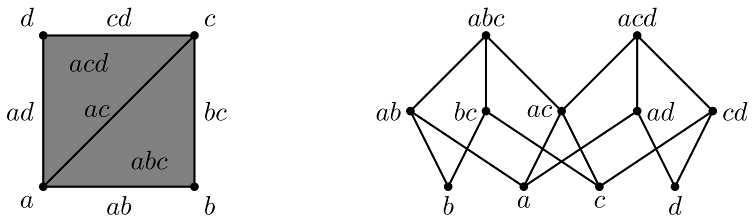

FIGURE 1. A triangulation Σ of $[0,1]^2$ and the corresponding poset that is (isomorphic to) the Esakia-dual of the Heyting algebra $\mathsf{P}_{\mathrm{o}}(\Sigma)$ of Σ-definable open polyhedra. Cf. Example 4.4.

To prove (9) we use the fact that, since P is a member of $\mathsf{P}_{\mathrm{c}}(\Sigma)$, for every $\sigma \in \Sigma$ we have

$$\operatorname{relint} \sigma \cap P \neq \emptyset \text{ if, and only if, } \sigma \subseteq P. \tag{10}$$

Only the left-to-right implication in (10) is non-trivial, and we prove the contrapositive. Assume $\sigma \not\subseteq P$. Then we have

$$\sigma \cap P = \sigma \cap |\Delta| = \sigma \cap \bigcup_{\delta \in \Delta} \delta = \bigcup_{\delta \in \Delta} \sigma \cap \delta.$$

Then $\sigma \cap \delta$ is either empty or else a proper face of σ, and hence $\sigma \cap P$ is a union of finitely many (possibly zero) proper faces of σ, which entails $\operatorname{relint} \sigma \cap (\sigma \cap P) = \emptyset$ and therefore $\operatorname{relint} \sigma \cap P = \emptyset$. This establishes (10).

Now, to show (9), if $x \in O$ then the carrier $\sigma^x \in \Sigma$ is such that $\operatorname{relint} \sigma^x \cap P = \emptyset$, so $\sigma^x \not\subseteq P$; equivalently, $\sigma^x \notin \Delta$. Then $\sigma^x \in U$ and hence $x \in \bigcup_{\sigma \in U} \operatorname{relint} \sigma$. Conversely, if $x \notin O$, then $x \in P$, so $\operatorname{relint} \sigma^x \cap P \neq \emptyset$ and thus $\sigma^x \subseteq P$; equivalently, $\sigma^x \in \Delta$. Then $\sigma^x \notin U$ and hence $x \notin \bigcup_{\sigma \in U} \operatorname{relint} \sigma$. This proves (9).

In light of (9) we now have $\gamma^{\uparrow}(U) = O$, so that $\gamma^{\uparrow}$ is surjective.

Finally, to prove injectivity, it suffices to recall that relative interiors of simplices in Σ are pairwise-disjoint, so the union in (7) is in fact a disjoint one, which makes the injectivity of $\gamma^{\uparrow}$ evident. □

Example 4.4. Consider the unit square $[0,1]^2$, and let Σ be its triangulation shown on the left of Fig. 1. The reader can verify that the set Σ ordered by inclusion — whose Hasse diagram is depicted on the right in Fig. 1 — is isomorphic to the Esakia-dual poset of the Heyting algebra $\mathsf{P}_{\mathrm{o}}(\Sigma)$ of Σ-definable open polyhedra. □

4.2. **Topological dimension through bounded depth.** We can now prove:

Lemma 4.5. *Let Σ be a triangulation in $\mathbb{R}^n$.*

(1) *The join-irreducible elements of $\mathsf{P}_{\mathrm{c}}(\Sigma)$ are the simplices of Σ.*
(2) *The join-irreducible elements of $\mathsf{P}_{\mathrm{o}}(\Sigma)$ are the open stars of simplices of Σ.*
(3) *In both $\mathsf{P}_{\mathrm{c}}(\Sigma)$ and $\mathsf{P}_{\mathrm{o}}(\Sigma)$ there is a chain of prime filters having cardinality $\dim \Sigma + 1$. In neither $\mathsf{P}_{\mathrm{c}}(\Sigma)$ nor $\mathsf{P}_{\mathrm{o}}(\Sigma)$ is there a chain of prime filters having strictly larger cardinality.*

Proof. Item (1) follows from direct inspection of the definitions. Item (2) is an immediate consequence of Lemma 4.3 along with Esakia duality (Subsection 2.2). To prove (3), set $d := \dim \Sigma$ and note that by definition Σ contains at least one d-simplex $\sigma = x_0 \cdots x_d \in \Sigma$. By item (1) the chain of simplices $x_0 < x_0 x_1 < \cdots < x_0 x_1 \cdots x_d = \sigma$ is a chain of join-irreducible elements of $\mathsf{P}_{\mathrm{c}}(\Sigma)$, and the principal filters generated by these elements yields a chain of prime filters of $\mathsf{P}_{\mathrm{c}}(\Sigma)$ of cardinality $d+1$. On the other hand, any chain of prime filters of $\mathsf{P}_{\mathrm{c}}(\Sigma)$ must be finite because $\mathsf{P}_c(\Sigma)$ is. If $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{p}_l$ is any such chain of prime filters, then each $\mathfrak{p}_i$ is principal — again because $\mathsf{P}_c(\Sigma)$ is finite — its unique generator p_i is join-irreducible, and we have $p_l < p_{l-1} < \cdots < p_2 < p_1$ in the order of the lattice $\mathsf{P}_c(\Sigma)$. Then $p_i \in \Sigma$, and clearly, since the simplex p_1 has $l-1$ proper faces of distinct dimensions, $\dim p_1 \geqslant l-1$. But $d \geqslant \dim p_1$ by definition of $d \coloneqq \dim \Sigma$, and therefore $d+1 \geqslant l$, as was to be shown. The proof for $\mathsf{P}_o(\Sigma)$ is analogous, using item (2). □

To finally relate the bounded-depth formulæ to topological dimension, we give a proof of Theorem 4.1.

Proof of Theorem 4.1. (i) ⇒ (ii) By Lemma 2.13, $\dim \Sigma = d$ for any triangulation Σ of P. By Lemmas 2.6, 3.6, and 4.5, the subalgebra $\mathsf{P}_o(\Sigma)$ of $\mathrm{Sub}_o P$ satisfies the equation $\mathrm{BD}_d = \top$, and fails each equation $\mathrm{BD}_{d'} = \top$ for each integer $0 \leqslant d' < d$. To complete the proof it thus suffices to show that any finitely generated subalgebra of $\mathrm{Sub}_o P$ is a subalgebra of $\mathsf{P}_o(\Sigma)$ for some triangulation Σ of P. But this is precisely the content of the Triangulation Lemma 2.10.

(ii) ⇒ (i) We prove the contrapositive. Suppose first $\dim P > d \geqslant 0$. Then, by (i) ⇒ (ii), $\mathrm{Sub}_o P$ fails the equation BD_d, so that (ii) does not hold. On the other hand, if $0 \leqslant d' \coloneqq \dim P < d$, by (i) ⇒ (ii) we know that $\mathrm{Sub}_o P$ satisfies the equation $\mathrm{BD}_{d'} = \top$, so again (ii) does not hold. □

5. Nerves of posets, and the geometric finite model property

In this section we use a classical construction in polyhedral geometry to realise finite posets geometrically. Our aim is to prove:

Theorem 5.1. *Let A be a finite, nonempty poset of cardinality $n \in \mathbb{N}$. There exists a triangulation Σ in $\mathbb{R}^n$ satisfying the following conditions.*

(1) $\mathrm{dep}\, A = \dim \Sigma$.
(2) *There is a surjective p-morphism $\Sigma \twoheadrightarrow A$, where Σ is equipped with the inclusion order.*

Construction. The *nerve* ([3, *passim*], [10, p. 1844]) of a finite poset A is the set

$$\mathscr{N}(A) \coloneqq \{\emptyset \neq C \subseteq A \mid C \text{ is totally ordered by the restriction of } \leqslant \text{ to } C \times C\}.$$

In other words, the nerve of A is the collection of all chains of A. We always regard the nerve $\mathscr{N}(A)$ as a poset under inclusion order.[13] Let us display the elements of A as $\{a_1, \ldots, a_n\}$. Let $e_1, \ldots, e_n$ denote the vectors in the standard basis of the linear space $\mathbb{R}^n$. The *triangulation induced by* the nerve $\mathscr{N}(A)$ is the set of simplices

$$\nabla(\mathscr{N}(A)) \coloneqq \{\mathrm{conv}\{e_{i_1}, \ldots, e_{i_l}\} \subseteq \mathbb{R}^n \mid \{a_{i_1}, \ldots, a_{i_l}\} \in \mathscr{N}(A)\}.$$

Then it is immediate that $\nabla(\mathscr{N}(A))$ indeed is a triangulation in $\mathbb{R}^n$, and its underlying polyhedron $|\nabla(\mathscr{N}(A))|$ is called the *geometric realisation of* the poset A. For the proof of Theorem 5.1, we set

$$\Sigma \coloneqq \nabla(\mathscr{N}(A)).$$

Using the fact that simplices are uniquely determined by their vertices (see Subsection 2.5), we see that the map

$$a_{i_1} < a_{i_2} < \cdots < a_{i_l} \in \mathscr{N}(A) \quad \longmapsto \quad \mathrm{conv}\{e_{i_1}, \ldots, e_{i_l}\} \in \Sigma$$

[13] In the literature on polyhedral geometry the nerve is most often regarded as an "abstract simplicial complex", or "vertex scheme". See e.g. [3]. We do not need to explicitly use this notion in this paper.

is an order-isomorphism between $\mathscr{N}(A)$ and Σ, the latter ordered by inclusion. Therefore,

$$\dim \Sigma = \text{cardinality of the longest chain in } A = \operatorname{dep} A,$$

so that (1) holds. To prove Theorem 5.1 it will therefore suffice to construct a p-morphism $\mathscr{N}(A) \twoheadrightarrow A$. To this end, let us define a function

$$f\colon \mathscr{N}(A) \longrightarrow A$$
$$C \in \mathscr{N}(A) \longmapsto \max C \in A,$$

where the maximum is computed in the poset A. □

Proof of Theorem 5.1. To show that f preserves order, just note that $C \subseteq D \in \mathscr{N}(A)$ entails $\max C \leqslant \max D$ in A. To show that f is a p-morphism, for each $C \in \mathscr{N}(A)$ we prove:

$$f[\uparrow C] = \{a_k \in A \mid a_k \geqslant \max C\} = \uparrow \max C = \uparrow f(C). \tag{11}$$

Only the first equality in (11) needs proof, and only the right-to-left inclusion is non-trivial. So let $a_k \in A$ be such that $a_k \geqslant \max C$. Then the set $D \coloneqq C \cup \{a_k\}$ is a chain in A, i.e. a member of $\mathscr{N}(A)$, and $D \in \uparrow C$ because $C \subseteq D$. Further, $\max D = a_k$, because $a_k \geqslant \max C$, so that $f(D) = a_k$. Hence $a_k \in f[\uparrow C]$, and the proof is complete. □

Remark 5.2. The reader may be interested in comparing the construction above of the Heyting algebra $\operatorname{Up}\Sigma$ from the finite distributive lattice $\operatorname{Up} A$ with the description of the prelinear Heyting algebra[14] freely generated by a finite distributive lattice in [1], along with that of the Heyting algebra freely generated by a finite distributive lattice in [14] (see also [15]). It is an interesting open question whether the construction given here using the nerve is the solution to a universal problem, too. □

6. Proof of Theorem 1.1

Proof of Theorem 1.1. By Theorem 4.1, $\mathsf{Log}\,\mathsf{P}_d$ contains the intermediate logic of bounded depth d. Conversely, suppose a formula α is not contained in the intermediate logic of bounded depth d. By Lemmas 2.1, 2.2 and 2.6, there exists a finite poset A satisfying $\operatorname{dep} A \leqslant d$ such that there is an evaluation into the poset A that provides a counter-model to α; equivalently, interpreting α as a term in the similarity type of Heyting algebras, the equation $\alpha = \top$ fails in the Heyting algebra $\operatorname{Up} A$. By Theorem 5.1 there exists a triangulation Σ in $\mathbb{R}^{|A|}$ such that $\operatorname{dep} A = \dim \Sigma \leqslant d$, along with a surjective p-morphism

$$p\colon \Sigma \longrightarrow\!\!\!\!\!\rightarrow A. \tag{12}$$

We set $P \coloneqq |\Sigma|$ and consider the Heyting algebra $\operatorname{Sub_o} P$ and its subalgebra $\mathsf{P_o}(\Sigma)$, per Corollary 3.4 and Lemma 3.6, respectively. Since $\dim P \leqslant d$, we have $\mathsf{Log}\,P \supseteq \mathsf{Log}\,\mathsf{P}_d$ by Theorem 4.1.

By Lemma 4.3 there is an isomorphism of (finite) Heyting algebras

$$\gamma^{\uparrow}\colon \operatorname{Up}\Sigma \longrightarrow \mathsf{P_o}(\Sigma)$$

defined as in (7). By finite Esakia duality (Lemma 2.4) we have isomorphisms of posets

$$\Sigma \cong \operatorname{Spec}\operatorname{Up}\Sigma \cong \operatorname{Spec}\mathsf{P_o}(\Sigma).$$

[14]A Heyting algebra is *prelinear* if it satisfies the equation $(X \to Y) \vee (Y \to X) = \top$.

The Esakia dual $\operatorname{Spec} p\colon \operatorname{Up} A \hookrightarrow \operatorname{Up} \Sigma$ of the surjective p-morphism (12) is an injective homomorphism. We thus have homomorphisms

$$\operatorname{Up} A \xhookrightarrow{\operatorname{Spec} p} \operatorname{Up} \Sigma \overset{\gamma^{\uparrow}}{\cong} \mathsf{P}_{\mathrm{o}}(\Sigma) \subseteq \operatorname{Sub}_{\mathrm{o}} P,$$

where the inclusion preserves the Heyting structure by Lemma 3.6. Since the equation $\alpha = \top$ fails in $\operatorname{Up} A$, it also fails in the larger algebra $\operatorname{Sub}_{\mathrm{o}} P$; equivalently, $\alpha \notin \mathsf{Log}\, P \supseteq \mathsf{Log}\, \mathsf{P}_d$, and the proof of the first statement is complete. The second statement follows easily from the first using Lemma 2.2. □

Remark 6.1. Intuitionistic logic is capable of expressing properties of polyhedra other than their dimension. To show this, let $\mathscr{P}$ consist of the class of all polyhedra that are, as topological spaces, closed (=without boundary) topological manifolds. Then $\mathsf{Log}\, \mathscr{P}$ contains intuitionistic logic properly. Indeed, it is a classical theorem that for any triangulation Σ of any d-dimensional manifold $M \in \mathscr{P}$, each $(d-1)$-simplex $\sigma \in \Sigma$ is a face of exactly two d-simplices of Σ. It follows from our results above that $\mathsf{Log}\, \mathscr{P}$ contains (all instances of) the well-known *bounded top-width axiom schema* of index 2, cf. [11, p. 112], which is refuted by intuitionistic logic. The problem of determining which intermediate logics are complete for classes of polyhedra is open; e.g., what is the logic of the class $\mathscr{P}$ of all closed triangulable manifolds? □

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