Free Heyting algebras: revisited

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Abstract. We use coalgebraic methods to describe finitely generated free Heyting algebras. Heyting algebras are axiomatized by rank 0-1 axioms. In the process of constructing free Heyting algebras we first apply existing methods to weak Heyting algebras—the rank 1 reducts of Heyting algebras—and then adjust them to the mixed rank 0-1 axioms. On the negative side, our work shows that one cannot use arbitrary axiomatizations in this approach. Also, the adjustments made for the mixed rank axioms are not just purely equational, but rely on properties of implication as a residual. On the other hand, the duality and coalgebra perspectives do allow us, in the case of Heyting algebras, to derive Ghilardi's (Ghilardi, 1992) powerful representation of finitely generated free Heyting algebras in a simple, transparent, and modular way using Birkhoff duality for finite distributive lattices.

1 Introduction

Coalgebraic methods and techniques are becoming increasingly important in investigating non-classical logics [19]. In particular, logics axiomatized by rank 1 axioms allow coalgebraic representation as coalgebras for a functor [14, 18]. We recall that an equation is of rank 1 for an operation f if each variable occurring in the equation is under the scope of exactly one occurrence of f. As a result the algebras for these logics become algebras for a functor. Consequently, free algebras in the corresponding varieties are initial algebras in the category of algebras for this functor. This correspondence immediately gives a constructive description of free algebras for rank 1 logics [11, 1, 5]. Examples of rank 1 logics are the basic modal logic **K**, basic positive modal logic, graded modal logic, probabilistic modal logic, coalition logic and so on [18]. For a coalgebraic approach to the complexity of rank 1 logics we refer to [18]. On the other hand, rank 1 axioms are too simple—very few well-known logics are axiomatized by rank 1 axioms. Therefore, one would, of course, want to extend the existing coalgebraic techniques to non-rank 1 logics. As follows from [15] algebras for these logics cannot be represented as algebras for a functor. Therefore, for these algebras we cannot use the standard construction of free algebras in a straightforward way.

In this paper, which is a facet of a larger joint project with Alexander Kurz [5], we try to take the first steps toward a coalgebraic treatment of modal logics beyond rank 1. We recall that an equation is of rank 0-1 for an operation f if each variable occurring in the equation is under the scope of at most one occurrence of f. With the ultimate

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goal of generalizing a method of constructing free algebras for varieties axiomatized by rank 1 axioms to the case of rank 0-1 axioms, we consider the case of Heyting algebras (intuitionistic logic, which is of rank 0-1 for $f = \rightarrow$). In particular, we construct free Heyting algebras. For an extension of coalgebraic techniques to deal with the finite model property of non-rank 1 logics we refer to [17].

Free Heyting algebras have been the subject of intensive investigation for decades. The one-generated free Heyting algebra was constructed by Rieger and Nishimura in the 50s. In the 70s Urquhart gave an algebraic characterization of finitely generated free Heyting algebras. A very detailed description of finitely generated free Heyting algebras in terms of their dual spaces was obtained in the 80s by Grigolia, Shehtman, Bellissima and Rybakov. This method is based on a description of the points of finite depth of the dual frame of the free Heyting algebra. For the details of this construction we refer to [9, Section 8.7] and [4, Section 3.2] and the references therein. Finally, Ghilardi [10] introduced a different method for describing free Heyting algebras. His technique builds the free Heyting algebra on a distributive lattice step by step by freely adding to the original lattice the implications of degree n, for each $n \in \omega$. Ghilardi [10] used this technique to show that every finitely generated free Heyting algebra is a bi-Heyting algebra. A more detailed account of Ghilardi's construction can be found in [7] and [12]. Ghilardi and Zawadowski [12], based on this method, derive a model-theoretic proof of Pitts' uniform interpolation theorem. In [3] a similar construction is used to describe free linear Heyting algebras over a finite distributive lattice and [16] uses the same method to construct high order cylindric Heyting algebras.

Our contribution is to derive Ghilardi's representation of finitely generated free Heyting algebras in a simple, transparent, and modular way using Birkhoff duality for finite distributive lattices. We split the process into two parts. We first apply the initial algebra construction to weak Heyting algebras—the rank 1 reducts of Heyting algebras. Then we adjust this method to the mixed rank 0-1 axioms. Finally, by using Birkhoff duality we obtain Ghilardi's [10] powerful representation of the finite approximants of the dual of finitely generated free Heyting algebra in a simple and systematic way. On the negative side, our work shows that one cannot use arbitrary axiomatizations in this approach. In particular, we give an example of a valid equation of Heyting algebras of rank 1 that cannot be derived, within the setting of distributive lattices, from other equations of rank 0-1 that are known to provide a full axiomatization of Heyting algebras. In addition, we use properties of Heyting algebras that are not directly equational, and thus our work does not yield a method that applies in general. Nevertheless, we expect that the approach, though it would have to be tailored, is likely to be successful in other instances as well.

The paper is organized as follows. In Section 2 we recall the so-called Birkhoff (discrete) duality for distributive lattices. We use this duality in Section 3 to build free weak Heyting algebras and in Section 4 to build free Heyting algebras. We conclude the paper by listing some future work.

2 Discrete duality for distributive lattices

We recall that a non-zero element a of a distributive lattice D is called *join-irreducible* if for every $b, c \in D$ we have that $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. For each distributive lattice (DL for short) D let $J(D)$ denote the set of all join-irreducible elements of D. Let also $\leq$ be the restriction of the order of D to $J(D)$. Then $(J(D), \leq)$ is a poset. Recall also that for every poset X a subset $U \subseteq X$ is called a *downset* if $x \in U$ and $y \leq x$ imply $y \in U$. For each poset X we denote by $\mathcal{O}(X)$ the distributive lattice $(\mathcal{O}(X), \cap, \cup, \emptyset, X)$ of all downsets of X. Then every finite distributive lattice D is isomorphic to the lattice of all downsets of $(J(D), \leq)$ and vice versa, every poset X is isomorphic to the poset of join-irreducible elements of $\mathcal{O}(X)$. We call $(J(D), \leq)$ the *dual poset* of D and we call $\mathcal{O}(X)$ the *dual lattice* of X.

This duality can be extended to the duality of the category $\mathbf{DL}_{fin}$ of finite bounded distributive lattices and bounded lattice morphisms and the category $\mathbf{Pos}_{fin}$ of finite posets and order-preserving maps. In fact, if $h : D \to D'$ is a bounded lattice morphism, then the restriction of $h^\flat$, the lower adjoint of h, to $J(D')$ is an order-preserving map between $(J(D'), \leq')$ and $(J(D), \leq)$, and if $f : X \to X'$ is an order-preserving map between two posets X and X', then $f^\downarrow : \mathcal{O}(X) \to \mathcal{O}(X'),\ S \mapsto \ {\downarrow} f(S)$ is $\bigvee$-preserving and its upper adjoint $(f^\downarrow)^\sharp = f^{-1} : \mathcal{O}(X') \to \mathcal{O}(X)$ is a bounded lattice morphism. Moreover, injective bounded lattice morphisms (i.e. embeddings or, equivalently, regular monomorphisms) correspond to surjective order-preserving maps, and surjective lattice morphisms (homomorphic images) correspond to order embeddings that are in one-to-one correspondence with subsets of the corresponding poset.

We also recall that an element a, $a \neq 1$, of a distributive lattice D is called *meet-irreducible* if for every $b, c \in D$ we have that $b \wedge c \leq a$ implies $b \leq a$ or $c \leq a$. We let $M(D)$ denote the set of all meet-irreducible elements of D.

Proposition 2.1. *Let D be a finite distributive lattice. Then for every $p \in J(D)$, there exists $\kappa(p) \in M(D)$ such that $p \nleq \kappa(p)$ and for every $a \in D$ we have*

$$p \leq a \ \text{ or } \ a \leq \kappa(p).$$

Proof. For $p \in J(D)$, let $\kappa(p) = \bigvee\{a \in D \mid p \nleq a\}$. Then it is clear that the condition involving all $a \in D$ holds. Note that if $p \leq \kappa(p) = \bigvee\{a \in D \mid p \nleq a\}$, then, applying the join-irreducibility of p, we get $a \in D$ with $p \nleq a$ but $p \leq a$, which is clearly a contradiction. So it is true that $p \nleq \kappa(p)$. Now we show that $\kappa(p)$ is meet irreducible. First note that since p is not below $\kappa(p)$, the latter cannot be equal to 1. Also, if $a, b \nleq \kappa(p)$ then $p \leq a, b$ and thus $p \leq a \wedge b$. Thus it follows that $a \wedge b \nleq \kappa(p)$. This concludes the proof of the proposition.

Proposition 2.2. *Let X be a finite set and $F_{DL}(X)$ the free distributive lattice over X. Then the poset $(J(F_{DL}(X)), \leq)$ of join-irreducible elements of $F_{DL}(X)$ is isomorphic to $(\mathcal{P}(X), \supseteq)$, where $\mathcal{P}(X)$ is the power set of X and each subset $S \subseteq X$ corresponds to the conjunction $\bigwedge S \in F_{DL}(X)$. Moreover, for $x \in X$ and $S \subseteq X$ we have*

$$\bigwedge S \leq x \text{ iff } x \in S.$$

Proof. This is equivalent to the disjunctive normal form representation for elements of $F_{DL}(X)$.

3 Weak Heyting algebras

3.1 Freely adding weak implications

Definition 3.1. *[8] A pair $(A, \to)$ is called a* weak Heyting algebra[3] *if A is a bounded distributive lattice and $\to: A^2 \to A$ a* weak implication, *that is, a binary operation satisfying the following axioms for all $a, b, c \in A$:*

(1) $a \to a = 1$,
(2) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.
(3) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$.
(4) $(a \to b) \wedge (b \to c) \leq a \to c$.

Let D and D' be distributive lattices. We let $\to (D \times D')$ denote the set $\{a \to b : a \in D \text{ and } b \in D'\}$. We stress that this is just a set bijective with $D \times D'$. The implication symbol is just a formal notation. For every distributive lattice D we also let $F_{DL}(\to(D \times D))$ denote the free distributive lattice over $\to(D \times D)$. Moreover, we let

$$H(D) = F_{DL}(\to(D \times D))/_{\approx}$$

where $\approx$ is the DL congruence generated by the axioms (1)–(4). We want to stress that we are not thinking of the axioms as a basis for an equational theory for a binary operation $\to$ here. The point of view is that of describing a bounded distributive lattice by generators and relations. That is, we want to find the quotient of the free bounded distributive lattice over the set $\to(D \times D)$ with respect to the lattice congruence generated by the pairs of elements of $F_{DL}(\to(D \times D))$ in (1)–(4) with a, b, c ranging over D. For an element $a \to b \in F_{DL}(\to(D \times D))$ we denote by $[a \to b]_{\approx}$ the $\approx$ equivalence class of $a \to b$.

The rest of the section will be devoted to showing that for each finite distributive lattice D the poset $(J(H(D)), \leq)$ is isomorphic to $(\mathcal{P}(J(D)), \subseteq)$. Below we give a dual proof of this fact. The dual proof, which relies on the fact that identifying two elements of an algebra simply corresponds to throwing out those points of the dual that are below one and not the other, is produced in a simple, modular, and systematic way that doesn't require any prior insight.

We start with a finite distributive lattice D and the free DL generated by the set

$$\to(D \times D) = \{a \to b \mid a, b \in D\}$$

of all formal arrows over D. As follows from Proposition 2.2, $J(F_{DL}(\to(D \times D)))$ is isomorphic to the power set of $\to(D \times D)$, ordered by reverse inclusion. Each subset of $\to(D \times D)$ corresponds to the conjunction of the elements in that subset; the empty set of course corresponds to 1. Now we want to take quotients of this free distributive lattice wrt various lattice congruences, namely the ones generated by the set of instances of the axioms of weak Heyting algebras.

The axiom $x \to x = 1$.

Here we want to take the quotient of $F_{DL}(\to(D \times D))$ with respect to the lattice congruence of $F_{DL}(\to(D \times D))$ generated by the set $\{(a \to a, 1) \mid a \in D\}$.

[3] In [8] weak Heyting algebras are called 'weakly Heyting algebras'.

By duality this quotient is given dually by the *subset*, call it P_1, of our initial poset $P_0 = J(F_{DL}(\rightarrow(D \times D)))$, consisting of those join-irreducibles of $F_{DL}(\rightarrow(D \times D))$ that do not violate this axiom. Thus, for $S \in J(F_{DL}(\rightarrow(D \times D)))$, S is admissible provided

$$\forall a \in D \qquad (\bigwedge S \leq 1 \quad \iff \quad \bigwedge S \leq a \to a).$$

Since all join-irreducibles are less than or equal to 1, it follows that the only join-irreducibles that are admissible are the ones that are below $a \to a$ for all $a \in D$. That is, viewed as subsets of $\rightarrow(D \times D)$, only the ones that contain $a \to a$ for each $a \in D$:

$$P_1 = \{S \in P_0 \mid a \to a \in S \text{ for each } a \in D\}.$$

The axiom $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

We now want to take a further quotient and thus we want to keep only those join-irreducibles from P_1 that do not violate this second axiom. That is, $S \in P_1$ is admissible provided

$$\forall a, b, c \qquad (\bigwedge S \leq a \to (b \wedge c) \quad \iff \quad \bigwedge S \leq a \to b \text{ and } \bigwedge S \leq a \to c).$$

which means

$$\forall a, b, c \qquad (a \to (b \wedge c) \in S \quad \iff \quad a \to b \in S \text{ and } a \to c \in S).$$

Proposition 3.2. *The poset P_2 of admissible join-irreducibles at this stage is order isomorphic to the set*

$$Q_2 = \{f : D \to D \mid \forall a \in D \quad f(a) \leq a\}$$

ordered pointwise.

Proof. An admissible S from
P_2 corresponds to the function $f_S : D \to D$ given by

$$f_S(a) = \bigwedge \{b \in D \mid a \to b \in S\}.$$

In the reverse direction a function in P_2 corresponds to the admissible set

$$S_f = \{a \to b \mid f(a) \leq b\}.$$

The proof that this establishes an order isomorphism is a straightforward verification.

The axiom $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

We want the subposet of P_2 consisting of those f's such that

$$\forall a, b, c \qquad \big((a \vee b) \to c \in S_f \quad \iff \quad a \to c \in S_f \text{ and } b \to c \in S_f\big).$$

To this end notice that

$$\begin{array}{lll} & \forall a,b,c & \big((a\vee b)\to c\in S_f \iff (a\to c\in S_f \text{ and } b\to c\in S_f)\big) \\ \iff & \forall a,b,c & \big(f(a\vee b)\le c \iff (f(a)\le c \text{ and } f(b)\le c)\big) \\ \iff & \forall a,b & f(a\vee b)=f(a)\vee f(b). \end{array}$$

That is, the poset, P_3, of admissible join-irreducibles left at this stage is isomorphic to the set

$$Q_3=\{f:D\to D \mid f \text{ is join preserving and } \forall a\in D \quad f(a)\le a\}.$$

The axiom $(x\to y)\wedge(y\to z)\le x\to z$.

It is not hard to see that this yields, in terms of join-preserving functions $f:D\to D$,

$$\begin{aligned} Q_4 &= \{f\in Q_3 \mid \forall a\in D\ f(a)\le f(f(a))\} \\ &= \{f:D\to D \mid f \text{ is join-preserving and } \forall a\in D\ f(a)\le f(f(a))\le f(a)\le a\} \\ &= \{f:D\to D \mid f \text{ is join-preserving and } \forall a\in D\ f(f(a))=f(a)\le a\}. \end{aligned}$$

We note that the elements of Q_4 are nuclei [13] on the order-dual lattice of D. Since the f's in Q_4 are join and 0 preserving, they are completely given by their action on $J(D)$. The additional property shows that these functions have lots of fixpoints. In fact, we can show that they are completely described by their join-irreducible fixpoints.

Lemma 3.3. *Let $f\in Q_4$, then for each $a\in D$ we have*

$$f(a)=\bigvee\{r\in J(D)\mid f(r)=r\le a\}.$$

Proof. Clearly $\bigvee\{r\in J(D)\mid f(r)=r\le a\}\le f(a)$. For the converse, let r be maximal in $J(D)$ wrt the property that $r\le f(a)$. Now it follows that

$$r\le f(a)=f(f(a))=\bigvee\{f(q)\mid J(D)\ni q\le f(a)\}.$$

Since r is join-irreducible, there is $q\in J(D)$ with $q\le f(a)$ and $r\le f(q)$. Thus $r\le f(q)\le q\le f(a)$ and by maximality of r we conclude that $q=r$. Now $r\le f(q)$ and $q=r$ yields $r\le f(r)$. However, $f(r)\le r$ as this holds for any element of D and thus $f(r)=r$. Since any element in a finite lattice is the join of the maximal join-irreducibles below it, we obtain

$$\begin{aligned} f(a) &= \bigvee\{r\in J(D)\mid r \text{ is maximal in } J(D) \text{ wrt } r\le f(a)\} \\ &\le \bigvee\{r\in J(D)\mid f(r)=r\le f(a)\} \ \le\ f(a). \end{aligned}$$

Finally, notice that if $f(r)=r\le f(a)$ then as $f(a)\le a$, we have $f(r)=r\le a$. Conversely, if $f(r)=r\le a$ then $r=f(r)=f(f(r))\le f(a)$ and we have proved the lemma.

Proposition 3.4. *The set of functions in Q_4, ordered pointwise, is order isomorphic to the powerset of $J(D)$ in the usual inclusion order.*

Proof. The order isomorphism is given by the following one-to-one correspondence

$$\begin{array}{rcl} Q_4 & \leftrightarrows & \mathcal{P}(J(D)) \\ f & \mapsto & \{p \in J(D) \mid f(p) = p\} \\ f_T & \leftarrow\!\shortmid & T \end{array}$$

where $f_T : D \to D$ is given by $f_T(a) = \bigvee\{p \in J(D) \mid T \ni p \le a\}$. Using the lemma, it is straightforward to see that these two assignments are inverse to each other. Checking that f_T is join preserving and satisfies $f^2 = f \le id_D$ is also straightforward. Finally, it is clear that $f_T \le f_S$ if and only if $T \subseteq S$.

Theorem 3.5. *Let D be a finite distributive lattice and $X = (J(D), \le)$ its dual poset. Then*

1. *The poset $(J(H(D)), \le)$ is isomorphic to the poset $(\mathcal{P}(X), \subseteq)$ of all subsets of X ordered by inclusion.*
2. *$J(H(D)) = \{[\bigwedge_{q \notin T}(q \to \kappa(q))]_\approx \mid T \subseteq J(D)\}$, (where $\kappa(q)$ is the element defined in Proposition 2.1).*

Proof. As shown above, the poset $J(H(D))$, obtained from $J(F_{DL}(\to(D \times D)))$ by removing the elements that violate the congruence schemes (1)–(4), is isomorphic to the poset Q_4, and Q_4 is in turn isomorphic to $\mathcal{P}(J(D))$ ordered by inclusion, see Proposition 3.4.

In order to prove the second statement, let $q \in J(D)$, and consider $q \to \kappa(q) \in F_{DL}(\to (D \times D))$. If we represent $H(D)$ as the lattice of downsets $\mathcal{O}(J(H(D)))$, then the action of the quotient map on this element is given by

$$\begin{array}{rcl} F_{DL}(\to (D \times D)) & \to & H(D) \\ q \to \kappa(q) & \mapsto & \{T' \in \mathcal{P}(J(D)) \mid q \to \kappa(q) \in S_{T'}\}. \end{array}$$

Now

$$\begin{aligned} q \to \kappa(q) \in S_{T'} & \iff f_{T'}(q) \le \kappa(q) \\ & \iff \bigvee(\downarrow\! q \cap T') \le \kappa(q) \\ & \iff q \notin T'. \end{aligned}$$

The last equivalence follows from the fact that $a \le \kappa(q)$ if and only if $q \not\le a$ and the only element of $\downarrow\! q$ that violates this is q itself. We now can see that for any $T \subseteq J(D)$ we have

$$\begin{array}{rcl} F_{DL}(\to (D \times D)) & \to & H(D) \\ [\bigwedge_{q \notin T}(q \to \kappa(q))]_\approx & \mapsto & \{T' \in \mathcal{P}(J(D)) \mid \forall q \quad (q \notin T \ \Rightarrow \ q \to \kappa(q) \in S_{T'}\} \\ & & = \{T' \in \mathcal{P}(J(D)) \mid \forall q \quad (q \notin T \ \Rightarrow \ q \notin T'\} \\ & & = \{T' \in \mathcal{P}(J(D)) \mid \forall q \quad (q \in T' \ \Rightarrow \ q \in T\} \\ & & = \{T' \in \mathcal{P}(J(D)) \mid T' \subseteq T\}. \end{array}$$

That is, under the quotient map $F_{DL}(\to (D \times D)) \to H(D)$, the elements $\bigwedge_{q \notin T}(q \to \kappa(q))$ are mapped to the principal downsets $\downarrow T$, for each $T \in \mathcal{P}(J(D)) = J(H(D))$. Since these principal downsets are exactly the join-irreducibles of $\mathcal{O}(J(H(D))) = H(D)$, we have that $\{ [\bigwedge_{q \notin T}(q \to \kappa(q))]_{\approx} \mid T \subseteq J(D) \} = J(H(D))$.

3.2 Free weak Heyting algebras

In the coalgebraic approach to generating the free algebra, it is a fact of central importance that H as described here is actually a functor. That is, for a DL homomorphism $h : D \to E$ one can define a DL homomorphism $H(h) : H(D) \to H(E)$ so that H becomes a functor on the category of DLs. To see this, we only need to note that H is defined by rank 1 axioms, which the given axioms (1)-(4) for weak Heyting algebras clearly are. Therefore, H gives rise to a functor $H : \mathbf{DL} \to \mathbf{DL}$ [2, 15]. Moreover, the category of weak Heyting algebras is isomorphic to the category $Alg(H)$ of the algebras for the functor H. For the details of such correspondences we refer to [2, 1, 11, 5, 15]. We would like to give a concrete description of how H applies to DL homomorphisms. We describe this in algebraic terms here and give the dual construction via Birkhoff duality.

Let $h : D \to E$ be a DL homomorphism. Recall that the dual map from $J(E)$ to $J(D)$ is just the lower adjoint $h^\flat$ with domain and codomain properly restricted. By abuse of notation we will just denote this map by $h^\flat$, leaving it to the reader to decide what the proper domain and codomain is. Now $H(D) = F_{DL}(\to (D \times D))/<Ax(D)>$, where $<Ax(D)>$ is the DL congruence generated by $Ax(D)$ and $Ax(D)$ is the set of all instances of the axioms (1)-(4) with $a, b, c \in D$. Also let q_D be the quotient map corresponding to mod'ing out by $<Ax(D)>$. The map $h : D \to E$ yields a map $h \times h : D \times D \longrightarrow E \times E$ and this of course yields a lattice homomorphism $F_{DL}(h \times h) : F_{DL}(\to (D \times D)) \longrightarrow F_{DL}(\to (E \times E))$. Now the point is that $F_{DL}(h \times h)$ carries elements of $Ax(D)$ to elements of $Ax(E)$ and thus in particular to elements of $<Ax(E)>$ (it is an easy verification and only requires h to be a homomorphism for axiom schemes (2) and (3)). This is equivalent to saying that $Ax(D) \subseteq Ker(q_E \circ F_{DL}(h \times h))$ and thus $<Ax(D)> \subseteq Ker(q_E \circ F_{DL}(h \times h))$, or equivalently that there is a unique map $H(h) : H(D) \to H(E)$ that makes the following diagram commute

$$\begin{array}{ccc} F_{DL}(\to (D \times D)) & \xrightarrow{F_{DL}(h \times h)} & F_{DL}(\to (E \times E)) \\ \downarrow{\scriptstyle q_D} & & \downarrow{\scriptstyle q_E} \\ H(D) & \overset{H(h)}{\dashrightarrow} & H(E). \end{array}$$

The dual diagram is

$$\begin{array}{ccc} \mathcal{P}(D \times D) & \xleftarrow{(h \times h)^{-1}} & \mathcal{P}(E \times E) \\ \uparrow{\scriptstyle e_D} & & \uparrow{\scriptstyle e_E} \\ \mathcal{P}(J(D)) & \overset{\mathcal{P}(h^\flat)}{\dashleftarrow} & \mathcal{P}(J(E)) \end{array}$$

The map $e_D : \mathcal{P}(D) \hookrightarrow \mathcal{P}(D \times D)$ is the embedding, via Q_4 and so on into P_0 as obtained above. That is, $e_D(T) = \{a \to b \mid \forall p \in T\ (p \leq a \Rightarrow p \leq b\}$. Now in this dual setting, the fact that there is a map $\mathcal{P}(h^\flat)$ is equivalent to the fact that $(h \times h)^{-1} \circ e_E$ maps into the image of the embedding e_D. This is easily verified:

$$
\begin{aligned}
(h \times h)^{-1}(e_E(T)) &= \{a \to b \mid \forall q \in T\ (q \leq h(a) \Rightarrow q \leq h(b)\} \\
&= \{a \to b \mid \forall q \in T\ (h^\flat(q) \leq a \Rightarrow h^\flat(q) \leq b\} \\
&= \{a \to b \mid \forall p \in h^\flat(T)\ (p \leq a \Rightarrow p \leq b\} \\
&= e_D(h^\flat(T)).
\end{aligned}
$$

Thus we can read off directly what the map $\mathcal{P}(h^\flat)$ is: it is just forward image under $h^\flat$. That is, if we call the dual of $h : D \to E$ by the name $f : J(E) \to J(D)$, then $\mathcal{P}(f) = f[\ ]$ where $f[\ ]$ is the lifted forward image mapping subsets of $J(E)$ to subsets of $J(D)$. Finally, we note that $\mathcal{P}$ satisfies $\mathcal{P}(f)$ is an embedding if and only if f is injective, and $\mathcal{P}(f)$ is surjective if and only if f is surjective.

Since weak Heyting algebras are the algebras for the functor H, we can make use of coalgebraic methods for constructing free weak Heyting algebras. Similarly to [5], where free modal algebras and free distributive modal algebras were constructed, we construct finitely generated free weak Heyting algebras as initial algebras of $Alg(H)$. That is, we have a sequence of bounded distributive lattices, each embedded in the next:

$$
\begin{array}{lcl}
n & \longrightarrow & F_{DL}(n), \text{ the free bounded distributive lattice on } n \text{ generators} \\
D_0 & = & F_{DL}(n) \\
D_{k+1} & = & D_0 + H(D_k), \text{ where } + \text{ is the coproduct in } \mathbf{DL} \\
i_0 & : & D_0 \to D_0 + H(D_0) = D_1 \text{ the embedding given by coproduct} \\
i_k & : & D_k \to D_{k+1} \text{ where } i_k = id_{D_0} + H(i_{k-1})
\end{array}
$$

For $a, b \in D_k$, we denote by $a \to_k b$ the equivalence class $[a \to b]_\approx \in H(D_k) \subseteq D_{k+1}$. Now, by applying the technique of [2], [1], [11], [5] to weak Heyting algebras, we arrive at the following theorem.

Theorem 3.6. *The direct limit $(D_\omega, (D_k \to D_\omega)_k)$ in $\mathbf{DL}$ of the system $(D_k, i_k : D_k \to D_{k+1})_k$ with the binary operation $\to_\omega$: $D_\omega \times D_\omega \to D_\omega$ defined by $a \to_\omega b = a \to_k b$, for $a, b \in D_k$ is the free n-generated weak Heyting algebra when we embed n in D_ω via $n \to D_0 \to D_\omega$.*

Now we will look at the dual of $(D_\omega, \to_\omega)$. Let $X_0 = \mathcal{P}(n)$ be the dual of D_0 and let

$$X_{k+1} = X_0 \times \mathcal{P}(X_k)$$

be the dual of D_{k+1}.

Theorem 3.7. *The sequence $(X_k)_{k<\omega}$ with maps $\pi_k : X_0 \times \mathcal{P}(X_k) \to X_k$ defined by*

$$\pi_k(x, A) = (x, \pi_{k-1}[A])$$

is dual to the sequence $(D_k)_{k<\omega}$ with maps $i_k : D_k \to D_{k+1}$. In particular, the π_k's are surjective.

Proof. The dual of D_0 is $X_0 = \mathcal{P}(n)$, and since $D_{k+1} = D_0 + H(D_k)$, it follows that $X_{k+1} = X_0 \times \mathcal{P}(X_k)$ as sums go to products and as H is dual to $\mathcal{P}$. For the maps, $\pi_0 : X_0 \times \mathcal{P}(X_0) \to X_0$ is just the projection onto the first coordinate since i_0 is the injection given by the sum construction. We note that π_0 is surjective. Now the dual $\pi_k : X_{k+1} = X_0 \times \mathcal{P}(X_k) \to X_k = X_0 \times \mathcal{P}(X_{k-1})$ of $i_k = id_{D_0} + H(i_{k-1})$ is $id_{X_0} \times \mathcal{P}(\pi_{k-1})$ which is exactly the map given in the statement of the theorem. Note that a map of the form $X \times Y \to X \times Z$ given by $(x, y) \mapsto (x, f(y))$ where $f : Y \to Z$ is surjective if and only the map f is. Also, as we saw above $\mathcal{P}(\pi_k)$ is surjective if and only if π_k is. Thus by induction, all the π_k's are surjective.

4 Heyting algebras

4.1 Freely adding Heyting implications

Definition 4.1. *[13] A weak Heyting algebra $(A, \to)$ is called a* Heyting algebra, HA *for short, if the following two axioms are satisfied for all $a, b \in A$:*

(5) $b \leq a \to b$,
(6) $a \wedge (a \to b) \leq b$.

Since both D and $H(D)$ are embedded in $D + H(D)$ (where $+$ is the coproduct in the category of distributive lattices) we will not distinguish between the elements of D and $H(D)$ and their images in $D + H(D)$. It is a well-known consequence of duality that the dual of the coproduct $D + H(D)$ is the product $J(D) \times J(H(D))$, where $(p, T) \leq a \in D$ if and only if $p \leq a$ and $(p, T) \leq \alpha \in H(D)$ if and only if $T \leq \alpha$. The latter implies in particular that $(p, T) \leq a \to b$ if and only if $a \to b \in S_T$ if and only if $f_T(a) \leq b$ if and only if, for each $q \in T$ we have $q \leq a$ implies $q \leq b$. Let $\equiv$ be a distributive lattice congruence of the lattice $D + H(D)$ generated by the axioms (5)–(6) viewed as congruence schemes. We denote $(D + H(D))/_{\equiv}$ by $V(D)$. For a poset P, call $T \subseteq P$ *rooted* provided there is a $p \in P$ with $p \in T \subseteq {\downarrow}p$, see [10]. Though a rooted subset T is completely determined just by T, we often write (p, T) to identify the root. We denote the set of all rooted subsets of P by P^r.

Theorem 4.2. *Let D be a distributive lattice and $X = (J(D), \leq)$ its dual poset. Then*

1. *The poset $(J(V(D)), \leq)$ is isomorphic to the poset $(X^r, \subseteq)$ of all rooted subsets of X ordered by inclusion.*
2. *$J(V(D)) = \{p \wedge \bigwedge_{q \notin T} q \to \kappa(q) : J(D) \supseteq T$ is rooted with root $p\}$.*

Proof. We start from the coproduct $D + H(D)$, or dually speaking from the poset $P = J(D) \times J(H(D)) = J(D) \times \mathcal{P}(J(D))$ and we impose the axiom scheme (5), which means dually that we obtain a subset $P_5 \subseteq P$ of all join-irreducible elements that are admissible wrt the axiom scheme (5). That is, $(p, T) \in P_5$ if and only if

$$\begin{aligned} & \forall a, b \in D \quad ((p, T) \leq b \quad \Rightarrow \quad (p, T) \leq a \to b) \\ \Longleftrightarrow \quad & \forall a, b \in D \quad (p \leq b \quad \Rightarrow \quad f_T(a) \leq b) \\ \Longleftrightarrow \quad & \forall a \in D \quad (f_T(a) \leq p) \\ \Longleftrightarrow \quad & \forall q \in T \quad (q \leq p). \end{aligned}$$

That is, the poset dual to the lattice obtained by mod'ing out by the axiom scheme (5) is

$$P_5 = \{(p, T) \mid T \subseteq {\downarrow}p\}.$$

Now further imposing the axiom scheme (6), we retain those elements of $(p, T) \in P_5$ satisfying

$$\begin{aligned}
& \forall a, b \in D \quad \big(((p,T) \leq a \text{ and } (p,T) \leq a \to b) \quad \Rightarrow \quad (p,T) \leq b\big) \\
\iff \quad & \forall a, b \in D \quad (p \leq a \text{ and } f_T(a) \leq b) \quad \Rightarrow \quad p \leq b) \\
\iff \quad & \forall b \quad (f_T(p) \leq b) \quad \Rightarrow \quad p \leq b) \\
\iff \quad & p \leq f_T(p) = \bigvee \{q \in T \mid q \leq p\} \\
\iff \quad & p \in T.
\end{aligned}$$

That is, $P_6 = \{(p, T) \mid p \in T \subseteq {\downarrow}p\}$, which corresponds exactly to the set of all rooted subsets of $J(D)$ ordered by inclusion. This proves the first statement. The second statement is now an easy consequence of this and Theorem 3.5.

Let D be a finite distributive lattice and X its dual poset. Then $D + H(D)$ is dual to $X \times \mathcal{P}(X)$. Consequently, the canonical embedding $i : D \hookrightarrow D + H(D)$ corresponds to the first projection $\pi^1 : X \times \mathcal{P}(X) \to X$ mapping a pair (x, T) for $x \in X$ and $T \subseteq X$ to x. Let $h : D + H(D) \twoheadrightarrow V(D)$ be the quotient map. Then it follows from Theorem 4.2 that h corresponds to an embedding $e : X^r \to X \times \mathcal{P}(X)$ mapping each rooted subset T to $(root(T), T)$. Now we define $j : D \to V(D)$ as the composition $j = h \circ i$. Then, by duality, the dual of j is the map $\pi : X^r \to X$ such that $\pi(T) = root(T)$, or denoting T by (x, T) we have $\pi(x, T) = x$. This implies that π is surjective and therefore, by duality, $j : D \to V(D)$ is an embedding.

4.2 Free Heyting algebras

In this section we relate our method to that of Ghilardi [10]. Consider the sequence

$$D_0 \xrightarrow{j_0} D_1 \xrightarrow{j_1} D_2 \ldots$$

where D_0 is the free distributive lattice on n generators, $D_{k+1} = V(D_k)$, and $j_k : D_k \to D_{k+1}$ is the embedding of D_k into $V(D_k)$ discussed in the previous section. Now let H be any n-generated Heyting algebra. Let $H_0 = <n>$, $H_{k+1} =< H_k \cup \{a \to_H b \mid a, b \in H_k >$ where all these are DL subalgebras of H generated by the given sets. Then we have a sequence

$$H_0 \xrightarrow{g_0} H_1 \xrightarrow{g_1} H_2 \ldots$$

as well as maps $\to_k$: $H_k \times H_k \to H_{k+1}$ given by $a \to_k b = a \to_H b$ whenever $a, b \in H_k$. By freeness of D_0, there is a quotient map $q_0 : D_0 \twoheadrightarrow H_0$, and since H_1 is generated by $H_0 \cup \to_H (H_0 \times H_0)$ and H satisfies (1)-(6), it follows that $ker(g_0 \circ q_0) \supseteq ker(j_0)$ and thus $g_0 \circ q_0$ factors through j_0. By induction on this argument we get a sequence of quotient maps q_k making a commutative diagram

$$\begin{array}{ccccccc}
D_0 & \overset{j_0}{\hookrightarrow} & D_1 & \overset{j_1}{\hookrightarrow} & D_2 & \overset{j_2}{\hookrightarrow} & \cdots \\
\downarrow{\scriptstyle q_0} & & \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_2} & & \\
H_0 & \overset{g_0}{\hookrightarrow} & H_1 & \overset{g_1}{\hookrightarrow} & H_2 & \overset{g_2}{\hookrightarrow} & \cdots
\end{array}$$

On the lower sequence we have that each map is a 'partial' homomorphism in the sense that for each $k \geq 0$ we have $\to_k : H_k \times H_k \to H_{k+1}$ and for $k \geq 1$ in the sequence $H_{k-1} \overset{g_{k-1}}{\longrightarrow} H_k \overset{g_k}{\longrightarrow} H_{k+1}$ we have $g_k(a \to_{k-1} b) = g_{k-1}(a) \to_k g_{k-1}(b)$. Now because this is a HA implication and each finite DL is a HA we have in addition that $a \to_k b = g_k(a) \to_{H_{k+1}} g_k(b)$ for each k. This is of course very special to HAs. As was applied in [10], this property is equivalent to saying that the dual sequence

$$Q_0 \overset{\pi_0}{\twoheadleftarrow} Q_1 \overset{\pi_1}{\twoheadleftarrow} Q_2 \ldots$$

of maps are 'partial p-morphisms', i.e., for each $k \geq 1$

$$\forall \tau \in Q_{k+1}\, \forall S \in Q_k\, (S \leq \pi_k(\tau) \Rightarrow \exists \tau' \in Q_{k+1}\, (\tau' \subseteq \tau \& \pi_{k-1}\pi_k(\tau') = \pi_{k-1}(S)).$$

Note that the commutative diagram between the D_k and the H_k sequences translates to a dual diagram

$$\begin{array}{ccccccc}
P_0 & \overset{r_0}{\twoheadleftarrow} & P_1 & \overset{r_1}{\twoheadleftarrow} & P_2 & \overset{r_2}{\twoheadleftarrow} & \cdots \\
\uparrow{\scriptstyle i_0} & & \uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle i_2} & & \\
Q_0 & \overset{\pi_0}{\twoheadleftarrow} & Q_1 & \overset{\pi_1}{\twoheadleftarrow} & Q_2 & \overset{\pi_2}{\twoheadleftarrow} & \cdots
\end{array}$$

which tells us that $Q_{k+1} \subseteq Q_k^r$, the set of rooted subsets of Q_k, and that the action of the π_k's is to take the root. Now, a second fact that is very special to HAs is that not only is $Q_{k-1} \twoheadleftarrow Q_k \twoheadleftarrow Q_{k+1}$ a partial p-morphism diagram, but so is $Q_{k-1} \twoheadleftarrow Q_k \hookleftarrow \tau$ for any $\tau \in Q_{k+1}$ viewed as a subset of Q_k (and thus as an embedding). The ensuing property on rooted subsets $\tau \in Q_k^r$ for them to be admissible in a sequence of Q_k's for a Heyting algebra H is easily derivable in the same manner as our earlier calculations. This was done by Ghilardi in [10] and results in

$$\forall T \in \tau\, \forall S \in Q_k\, (S \leq T \Rightarrow \exists T' \in \tau\, (T' \leq T \,\&\, root(T') = root(S)) \qquad \text{(G)}$$

The point is now that since, in each step and for each n-generated HA, H, the admissible rooted subsets can at most be those satisfying (G), if we start from the largest initial poset namely $R_0 = P_0 = J(F_{DL}(n))$ and proceed with $R_1 = R_0^r$, and $R_{k+1} = \{\tau \in R_k^r \mid \tau \text{ satisfies (G)}\}$ then, for any Heyting algebra with dual sequence $\{Q_k\}$ we have

$$\begin{array}{ccccccc}
P_0 & \overset{root}{\twoheadleftarrow} & P_1 & \overset{root}{\twoheadleftarrow} & P_2 & \overset{root}{\twoheadleftarrow} & \cdots \\
\uparrow{\scriptstyle i_0} & & \uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle i_2} & & \\
R_0 & \twoheadleftarrow & R_1 & \twoheadleftarrow & R_2 & \twoheadleftarrow & \cdots \\
\uparrow{\scriptstyle i_0} & & \uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle i_2} & & \\
Q_0 & \overset{\pi_0}{\twoheadleftarrow} & Q_1 & \overset{\pi_1}{\twoheadleftarrow} & Q_2 & \twoheadleftarrow & \cdots
\end{array}$$

and thus H factors through $\varinjlim \mathcal{O}(R_k)$ and this latter algebra thus has the universal mapping property for HA. By the same argument, the same is true for any algebra given by any sequence R'_k between the P_k's and the R_k's for which the local operations $\rightarrow_k$ glue together correctly. However there is of course no guarantee that any of these algebras are themselves Heyting algebras (at most one is, as it is then the free n-generated HA).

Theorem 4.3. *The limit $\varinjlim \mathcal{O}(R_k)$ of the sequence $\{\mathcal{O}(R_k)\}_{k\in\omega}$ in the category* **DL** *of distributive lattices is the free Heyting algebra on n generators.*

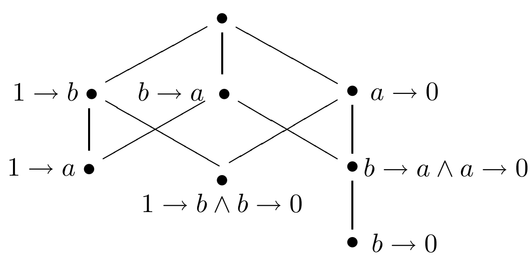
Proof. (Sketch) It follows from the discussion above that $\varinjlim \mathcal{O}(R_k)$ satisfies the required universal properties of the free algebra. Therefore, we only need to show that $\varinjlim \mathcal{O}(R_k)$ is a Heyting algebra, or dually that $\varprojlim R_k = R_\omega$, with the standard topology of the inverse limit, is an Esakia space (see e.g., [4, Section 2.3.3], for details of this duality). The crucial part is that $\uparrow U$ must be clopen for U clopen in $\varprojlim R_k$. For each $k \in \omega$ let $\pi^\omega_k : R_\omega \to R_k$ be the projection map. A subset U is clopen in the limit R_ω provided $U = (\pi^\omega_{k-1})^{-1}(V)$ for some $k-1$ and $V \subseteq R_{k-1}$ and then $U = (\pi^\omega_k)^{-1}(V_k)$ with $V_k = \pi_k^{-1}(V)$ is also true. Now, clearly $\uparrow U \subseteq (\pi^\omega_k)^{-1}(\uparrow V_k)$. The crux of this proof is that (G) implies that the reverse inclusion also holds. To see this, let $x \in R_\omega$ with $\pi^\omega_k(x) = x_k \geq y_k$ for some $y_k \in V_k$. Applying (G) with $\tau = \pi^\omega_{k+1}(x)$ and $T = root(\tau) = x_k$ and $S = y_k$, there is $T' \leq T, T' \in \tau$ with $root(S) = root(T')$. Now note that $root(T') = root(S) = root(y_k) \in \pi_{k-1}(V_k) = V$. Thus T' is in V_k. Also, we now take $y_{k+1} = \tau' = \tau \cap \downarrow T'$. As mentioned by Ghilardi this is easily seen to be admissible again, and since $root(\tau') = T' \in V_k$, then $y_{k+1} = \tau' \in (\pi_{k+1})^{-1}(V_k) = V_{k+1}$. Also $y_{k+1} = \tau' \leq \tau = x_{k+1}$ and in this way we build a sequence $y = (y_k) \in (\pi^\omega_{k-1})^{-1}(V)$ with $y \leq x$. This proves $x \in \uparrow U$ and we are done.

We conclude this section with a few points on where this leaves us in the quest for a systematic approach to the generative description of free finitely generated algebras in DL based varieties. First an example concerning the choice of axiomatization.

Example 4.4. Let D be a finite distributive lattice and let $H'(D)$ denote $F_{DL}(\rightarrow(D \times D))$ modulo axioms (1),(2), (3) of Definition 3.1. This means that the dual of $H'(D)$ is isomorphic to the set $Q_3 = \{f : D \to D \mid f$ is join-preserving and $\forall a \in D\; f(a) \leq a\}$. Since the f's are join preserving we may consider them as order preserving functions $f : J(D) \to \mathcal{O}(J(D)) \cong D$ as this restriction uniquely determines f. We also let $V'(D)$ denote $D + H'(D)$ modulo axioms (5),(6) of Definition 4.1. The dual of $D + H'(D)$ is isomorphic to $J(D) \times P_3$ and by imposing axiom schemes (5) and (6), we get

$$P' = \{(p, f) \mid f(q) \subseteq \downarrow q \cap \downarrow p, f(p) = \downarrow p\}.$$

Then we can show that in general, $V'(D)$ is not isomorphic to $V(D)$.



In fact, the inequality $(a \to b) \wedge (b \to c) \leq a \to c$ will not be valid on $V'(D)$ for all $a, b, c \in D$, whereas on $V(D)$ it is valid by definition. To see this consider the four element chain $0 < a < b < 1$. The poset P', where we write each S_f as the conjunction of the $q \to f(q)$ for which $f(q) < q$ is depicted above.

This poset is larger than $J(V(D))$ as the point $(b \to a) \wedge (a \to 0)$ is not in $J(V(D))$. We recall that axioms (1),(2) of Definition 3.1 and (5),(6) of Definition 4.1 are already sufficient to axiomatize Heyting algebras; see e.g., [13, Lemma 1.10] or [4, Theorem 2.2.6]. In logical terms the above observation means that the inequality $(a \to b) \wedge (b \to c) \leq (a \to c)$ is an example of a valid rank 1 inequality of the theory of Heyting algebras whose derivation is not a DL derivation on the basis of the axioms.

Finally a remark on mod'ing out to make the partial operations into operations.

Remark 4.5. Notice that our first sequence for the HA case with $D_{k+1} = V(D_k)$ is definitely 'too free' and we can make a first cut on general principles. For any $k \geq 1$ and for $a, b \in D_{k-1}$, we may take $j_k(a \to_{k-1} b)$ or $j_{k-1}(a) \to_k j_{k-1}(b)$, and if $\{H_k\}$ is any sequence obtained from an algebra it factors through the DL congruence sequence generated by

$$\forall a, b \in D'_{k-1} \quad j_k(a \to_{k-1} b) \approx j_{k-1}(a) \to_k j_{k-1}(b)$$

with $D'_0 = D_0$ and $D'_1 = D_1$. Let $\{P'_k\}$ be the corresponding dual sequence. In the case of HA one can prove the following

Claim: $\tau \in (P'_k)^r$ is admissible if and only if for all $p \in root(\tau)$ there is $T \in \tau$ with $root(T) = p$.

Mod'ing out this congruence ensures that the corresponding sequence of algebras yields an $\to$-algebra in the limit. However mod'ing out must have destroyed the truth of (1)-(4) as one can show that this is not quite Ghilardi's sequence. For example, for the four element chain as in the above example, with P its dual, $P_1 = P^r$, we have that $\tau = \{P, \{1, b\}, \{a, b\}, \{a\}\}$ satisfies the above admissibility condition but does not satisfy (G) as can be seen from taking $T = \{1, b\}$ and $P_1 \ni S = \{b\} \leq T$.

5 Conclusions and future work

In this paper we described finitely generated free (weak) Heyting algebras using an initial algebra-like construction. The main idea is to split the axiomatization of Heyting algebras into its rank 1 and non-rank 1 parts. The rank 1 reducts of Heyting algebras are weak Heyting algebras. For weak Heyting algebras we applied the standard initial algebra construction and then adjusted it for Heyting algebras. We used Birkhoff duality for finite distributive lattices and finite posets to obtain the dual characterization of the finite posets that approximate the duals of free algebras. As a result we obtained Ghilardi's representation of these posets in a more systematic and transparent way.

There are a few possible directions for further research. As we mentioned in the introduction, although we considered Heyting algebras (intuitionistic logic), this method could be applied to other non-classical logics. More precisely, the method is available if a signature of the algebras for this logic can be obtained by adding an extra operator to a locally finite variety. Thus, various non-rank 1 modal logics such as $\mathbf{S4}$, $\mathbf{K4}$ and other more complicated modal logics, as well as distributive modal logics, are the obvious candidates. On the other hand, one cannot always expect to have such a simple representation of free algebras. The algebras corresponding to other many-valued logics such as MV-algebras, l-groups, BCK-algebras and so on, are other examples where this method could lead to interesting representations. The recent work [6] that connects ontologies with free distributive algebras with operators shows that such representations of free algebras are not only interesting from a theoretical point of view, but could have very concrete applications.

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