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An Algebraic Approach to Canonical Formulas: Modal Case

Dedicated to Ryszard Wójcicki on his 80th birthday

Abstract. We introduce relativized modal algebra homomorphisms and show that the category of modal algebras and relativized modal algebra homomorphisms is dually equivalent to the category of modal spaces and partial continuous p-morphisms, thus extending the standard duality between the category of modal algebras and modal algebra homomorphisms and the category of modal spaces and continuous p-morphisms. In the transitive case, this yields an algebraic characterization of Zakharyaschev's subreductions, cofinal subreductions, dense subreductions, and the closed domain condition. As a consequence, we give an algebraic description of canonical, subframe, and cofinal subframe formulas, and provide a new algebraic proof of Zakharyaschev's theorem that each logic over **K4** is axiomatizable by canonical formulas.

Keywords: Modal logic, duality theory, relativization.

1. Introduction

Refutation patterns play an important role in developing axiomatic bases for modal logics. As was shown by Fine [8], each finite rooted **S4**-frame $\mathfrak{F}$ gives rise to the formula $\chi(\mathfrak{F})$ such that a general **S4**-frame $\mathfrak{G}$ refutes $\chi(\mathfrak{F})$ iff $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$. This yields an axiomatization of a large class of logics over **S4**. Fine's formulas are a frame-theoretic version of Jankov's formulas for intuitionistic logic, developed several years earlier by Jankov [10, 11] using algebraic techniques. (De Jongh [6] also developed a version of Jankov formulas for intuitinostic logic via frame-theoretic methods.) The results of Fine easily generalize to logics over **K4**. For an algebraic account of these formulas, which is a direct generalization of Jankov's approach, see Rautenberg [13]. Similar results were obtained by Fine [9] and Zakharyaschev [17] for subframe and cofinal subframe logics over **K4**. Namely Fine showed that each rooted **K4**-frame $\mathfrak{F}$ gives rise

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to the formula $\alpha_s(\mathfrak{F})$ such that a general **K4**-frame $\mathfrak{G}$ refutes $\alpha_s(\mathfrak{F})$ iff $\mathfrak{G}$ is subreducible to $\mathfrak{F}$, thus providing an axiomatization of subframe logics over **K4**. Zakharyaschev generalized Fine's results by showing that each rooted **K4**-frame $\mathfrak{F}$ gives rise to the formula $\alpha_{cs}(\mathfrak{F})$ such that a general **K4**-frame $\mathfrak{G}$ refutes $\alpha_{cs}(\mathfrak{F})$ iff $\mathfrak{G}$ is cofinally subreducible to $\mathfrak{F}$, thus providing an axiomatization of cofinal subframe logics over **K4**. These are large classes of logics over **K4**, but not every logic over **K4** is axiomatizable by these means.

The problem of axiomatizing every logic over **K4** was resolved by Zakharyaschev [16], who developed the theory of canonical formulas and showed that each logic over **K4** is axiomatizable by canonical formulas. Canonical formulas of Zakharyaschev are also built from finite rooted **K4**-frames $\mathfrak{F}$, but they have an additional parameter—a set of antichans $\mathfrak{D}$ in $\mathfrak{F}$. The two main ingredients of Zakharyaschev's proof are: (i) the refutation of a formula α in a general **K4**-frame $\mathfrak{G}$ can be coded by means of finitely many canonical formulas (which are constructed effectively from α) and (ii) if $\mathfrak{F}$ is a finite rooted **K4**-frame and $\mathfrak{D}$ is a set of antichains in $\mathfrak{F}$, then whether a general frame $\mathfrak{G}$ refutes the canonical formula $\alpha(\mathfrak{F}, \mathfrak{D})$ depends on whether or not there is a cofinal subreduction f from $\mathfrak{G}$ to $\mathfrak{F}$ that satisfies the closed domain condition—a condition relating the subreduction f to the set of antichains $\mathfrak{D}$.

Our aim is to provide a purely algebraic account of canonical formulas. This requires an algebraic analysis of the two main ingredients of Zakharyaschev's proof. One of our main tools will be a generalization of the well-known duality between the category **MA** of modal algebras and modal algebra homomorphisms and the category **MS** of modal spaces and continuous p-morphisms. By the duality between **MA** and **MS**, continuous p-morphisms correspond to modal algebra homomorphisms. Subreductions are partial p-morphisms. We generalize the duality between **MA** and **MS** to a duality between the category $\mathbf{MA^R}$ of modal algebras and relativized modal algebra homomorphisms and the category $\mathbf{MS^P}$ of modal spaces and partial continuous p-morphisms. This yields that partial continuous p-morphisms correspond to relativized modal algebra homomorphisms. We also introduce cofinal relativized modal algebra homomorphisms and show that they correspond to cofinal partial continuous p-morphisms.

If A and B are **K4**-algebras and $\eta : A \to B$ is a relativized modal algebra homomorphism, then η does not preserve $\Diamond$ in general. We show that Zakharyaschev's closed domain condition exactly corresponds to preserving $\Diamond$ for all elements in some fixed subset D of A, where D is an algebraic analogue of Zakharyaschev's extra parameter of antichains.

For each finite subdirectly irreducible **K4**-algebra A and a subset D of A, generalizing the technique of Jankov and Rautenberg, we define the canonical formula $\alpha(A, D)$, and show that a **K4**-algebra B refutes $\alpha(A, D)$ iff there exist a homomorphic image C of B and a 1-1 cofinal relativized modal algebra homomorphism from A to C that preserves $\Diamond$ for all elements in D. This is an algebraic analogue of (ii).

For each formula α, we construct $(A_1, D_1), \ldots, (A_n, D_n)$, where each A_i is a finite subdirectly irreducible **K4**-algebra and $D_i \subseteq A_i$, and prove that a **K4**-algebra B refutes α iff there exist $i \leq n$, a homomorphic image C of B, and a 1-1 cofinal relativized modal algebra homomorphism from A_i to C that preserves $\Diamond$ for all elements in D_i. This is an algebraic analogue of (i). Our construction utilizes the results of [3], where an algebraic proof of the finite model property of cofinal subframe logics was given. (For frame-theoretic proofs see [9, 17].) Putting (i) and (ii) together provides a new algebraic proof of Zakharyaschev's theorem that each logic over **K4** is axiomatizable by canonical formulas.

We also give an algebraic account of negation-free canonical formulas, and show that the Jankov-Rautenberg, subframe, and cofinal subframe formulas are special cases of canonical formulas. This, in particular, gives the first axiomatization of subframe and cofinal subframe logics over **K4** by "algebra-based formulas," as opposed to their axiomatizations by frame-based formulas of Fine and Zakharyaschev [9, 17].

Our results complement the results of [1], where an algebraic account of canonical formulas for intuitionistic logic was given. As shown in [1], in the intuitionistic setting, partial p-morphisms correspond to $(\wedge, \to)$-preserving maps between Heyting algebras, cofinal partial p-morphisms correspond to $(\wedge, \to, 0)$-preserving maps, and partial p-morphisms satisfying the closed domain condition correspond to $(\wedge, \to)$-preserving maps that also preserve $\vee$ for a fixed set of pairs of elements. This provides an algebraic analogue of canonical formulas for intuitionistic logic that generalize Jankov's formulas, yielding an algebraic proof of Zakharyaschev's theorem that each intermediate logic is axiomatizable by canonical formulas.

The paper is organized as follows: In Section 2 we recall the duality between the category of modal algebras and modal algebra homomorphisms and the category of modal spaces and continuous p-morphisms. In Section 3 we extend this duality to a duality between the category of modal algebras and relativized modal algebra homomorphisms and the category of modal spaces and partial continuous p-morphisms. In Section 4 we restrict our attention to the transitive case. We give an algebraic characterization of Zakharyaschev's closed domain condition. We also define cofinal and dense relativized modal algebra homomorphisms and show that they correspond to Zakharyaschev's cofinal and dense subreductions. In Section 5 we provide an algebraic description of canonical formulas, and give a new algebraic proof of Zakharyaschev's theorem that each logic over **K4** is axiomatizable by canonical formulas. Finally, in Section 6 we give an algebraic description of negation-free canonical formulas and show that each logic over **K4** that is axiomatizable by negation-free formulas is also axiomatizable by negation-free canonical formulas. We also show that Jankov-Rautenberg, subframe, and cofinal subframe formulas are particular cases of canonical formulas. This in particular leads to a new axiomatization of subframe and cofinal subframe formulas via "algebra-based" formulas.

2. Preliminaries

In this section we briefly recall some basic facts about modal algebras and the duality between modal algebras and modal spaces that will be used in subsequent sections. The main references for this section are [5, 12, 14].

A *modal algebra* is a pair $(A, \Diamond)$, where A is a Boolean algebra and $\Diamond : A \to A$ is a unary function on A satisfying $\Diamond 0 = 0$ and $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$. As usual, we define $\Box : B \to B$ by $\Box a = \neg\Diamond\neg a$ for each $a \in A$. We denote modal algebras $(A, \Diamond)$ simply by A. Given two modal algebras A and B, we recall that $\eta : A \to B$ is a *modal algebra homomorphism* if η is a Boolean algebra homomorphism and $\eta(\Diamond a) = \Diamond\eta(a)$. When no confusion arises, we call modal algebra homomorphisms simply homomorphisms. Clearly modal algebras and modal algebra homomorphisms form a category which we denote by **MA**.

Let F be a filter in a modal algebra A. We recall that F is a $\Box$*-filter* if $a \in F$ implies $\Box a \in F$. It is well known that the lattice of congruences of a modal algebra A is isomorphic to the lattice of $\Box$-filters of A. In the lattice of $\Box$-filters of A, we have that A is the largest element and $\{1\}$ is the least element. Consequently, A is *subdirectly irreducible* if there exists a least $\Box$-filter properly containing $\{1\}$.

It is well known that **MA** has the *congruence extension property*; that is, for each $A, B \in$ **MA**, if A is subalgebra of B and F is a $\Box$-filter of A, then there exists a $\Box$-filter G of B such that $G \cap A = F$. The next lemma, which will be used in Section 5, is now an immediate consequence.

LEMMA 2.1. *If $A, B, C \in$ **MA**, $\eta : A \to B$ is a 1-1 homomorphism, and $\theta : A \to C$ is an onto homomorphism, then there exist $D \in$ **MA**, an onto homomorphism $\xi : B \to D$, and a 1-1 homomorphism $\zeta : C \to D$ such that $\xi \circ \eta = \zeta \circ \theta$.*

We briefly recall the duality between modal algebras and modal spaces that generalizes the celebrated Stone duality for Boolean algebras. We assume an elementary knowledge of category theory. In particular, we recall that two categories $\mathcal{C}$ and $\mathcal{D}$ are *dually equivalent* if there exist contravariant functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that each object c of $\mathcal{C}$ is isomorphic to $GF(c)$, each object d of $\mathcal{D}$ is isomorphic to $FG(d)$, and these isomorphisms are natural.

As usual, for a set X and a binary relation R on X, let $R(x) = \{y \in X : xRy\}$ and $R^{-1}(x) = \{y \in X : yRx\}$. Also, for $U \subseteq X$, let $R(U) = \bigcup\{R(x) : x \in U\}$ and $R^{-1}(U) = \bigcup\{R^{-1}(x) : x \in U\}$. If X is a topological space, then we recall that $U \subseteq X$ is *clopen* if U is both closed and open, that X is *zero-dimensional* if clopen subsets of X form a basis for X, and that X is a *Stone space* if X is compact, Hausdorff, and zero-dimensional.

A *modal space* is a pair (X, R), where X is a Stone space and R is a binary relation on X such that (i) $R(x)$ is closed for each $x \in X$, and (ii) $R^{-1}(U)$ is clopen for each clopen $U \subseteq X$. Given two modal spaces (X, R) and (Y, Q), we recall that a map $f : X \to Y$ is a *p-morphism* if for all $x, z \in X$ and $y \in Y$, (i) xRz implies $f(x)Qf(z)$, and (ii) $f(x)Qy$ implies there exists $z \in X$ such that xRz and $f(z) = y$. Clearly modal spaces and continuous p-morphisms form a category which we denote by **MS**. Then **MA** is dually equivalent to **MS**. This duality is a generalization of Stone duality and is obtained as follows.

First, the functor $(-)_* : \mathbf{MA} \to \mathbf{MS}$ is defined as follows. If A is a modal algebra, then A_* is the set of ultrafilters of A topologized by the basis $\{\varphi(a) : a \in A\}$ for open sets, where $\varphi(a) = \{x \in A_* : a \in x\}$. For $x, y \in A_*$, set xRy iff $(\forall a \in A)(a \in y \text{ implies } \Diamond a \in x)$. Then A_* is a modal space. For modal algebras A, B and a homomorphism $\eta : A \to B$, let $\eta_* = \eta^{-1} : B_* \to A_*$. Then η_* is a continuous p-morphism, and so $(-)_*$ is a contravariant functor. Next, the functor $(-)^* : \mathbf{MS} \to \mathbf{MA}$ is defined as follows. If (X, R) is a modal space, then $(X, R)^* = (\mathrm{Clopen}(X), \Diamond_R)$ is a modal algebra, where $\mathrm{Clopen}(X)$ is the Boolean algebra of clopen subsets of X and $\Diamond_R(U) = R^{-1}(U)$. For continuous p-morphism $f : X \to Y$, let $f^* = f^{-1} : \mathrm{Clopen}(Y) \to \mathrm{Clopen}(X)$. Then f^* is a modal algebra homomorphism, and so $(-)^*$ is a contravariant functor. Moreover, $\varphi : A \to {A_*}^*$ and $\varepsilon : X \to {X^*}_*$, given by $\varepsilon(x) = \{U \in X^* : x \in U\}$, are natural isomorphisms, and so $(-)_*$ and $(-)^*$ establish the desired dual equivalence between **MA** and **MS**.

3. Relativizations and generalized duality for modal algebras

In this section we generalize the concept of modal algebra homomorphism to that of relativized modal algebra homomorphism, the concept of continuous p-morphism to that of partial continuous p-morphism, and prove that the category of modal algebras and relativized modal algebra homomorphisms is dually equivalent to the category of modal spaces and partial continuous p-morphisms. This generalizes the well-known duality for modal algebras.

Let B be a Boolean algebra and $s \in B$. Then $[0, s] = \{x \in B : 0 \leq x \leq s\}$ also forms a Boolean algebra which we denote by B_s. The Boolean operations on B_s are defined as follows:

1. $x \wedge_s y = x \wedge y$;
2. $x \vee_s y = x \vee y$;
3. $0_s = 0$ and $1_s = s$;
4. $\neg_s x = \neg x \wedge s$.

We call B_s the *relativization* of B to s.

LEMMA 3.1. *Let A and B be Boolean algebras and $\eta : A \to B$ be a map. Then η is a Boolean algebra homomorphism from A to $B_{\eta(1)}$ iff η preserves $\wedge, \vee$, and 0 (that is, $\eta(a \wedge b) = \eta(a) \wedge \eta(b)$, $\eta(a \vee b) = \eta(a) \vee \eta(b)$, and $\eta(0) = 0$).*

PROOF. Let η be a Boolean algebra homomorphism from A to $B_{\eta(1)}$. Then:

$$\eta(a \wedge b) = \eta(a) \wedge_{\eta(1)} \eta(b) = \eta(a) \wedge \eta(b),$$
$$\eta(a \vee b) = \eta(a) \vee_{\eta(1)} \eta(b) = \eta(a) \vee \eta(b),$$
$$\eta(0) = 0_{\eta(1)} = 0.$$

Therefore, η preserves $\wedge, \vee$, and 0. Conversely, suppose that η preserves $\wedge, \vee$, and 0. Then:

$$\eta(a \wedge b) = \eta(a) \wedge \eta(b) = \eta(a) \wedge_{\eta(1)} \eta(b),$$
$$\eta(a \vee b) = \eta(a) \vee \eta(b) = \eta(a) \vee_{\eta(1)} \eta(b),$$
$$\eta(0) = 0 = 0_{\eta(1)} \text{ and } \eta(1) = 1_{\eta(1)}.$$

Moreover,

$$\eta(a) \wedge_{\eta(1)} \eta(\neg a) = \eta(a) \wedge \eta(\neg a) = \eta(a \wedge \neg a) = \eta(0) = 0_{\eta(1)},$$
$$\eta(a) \vee_{\eta(1)} \eta(\neg a) = \eta(a) \vee \eta(\neg a) = \eta(a \vee \neg a) = \eta(1) = 1_{\eta(1)}.$$

Therefore, $\neg_{\eta(1)}\eta(a) = \eta(\neg a)$, and so η is a Boolean algebra homomorphism from A to $B_{\eta(1)}$. ■

Let A be a modal algebra and $s \in A$. We define $\Diamond_s : A_s \to A_s$ by

$$\Diamond_s x = s \wedge \Diamond x$$

for each $x \in A_s$. Then A_s is a modal algebra because

$$\Diamond_s(x \vee y) = s \wedge \Diamond(x \vee y) = s \wedge (\Diamond x \vee \Diamond y) = (s \wedge \Diamond x) \vee (s \wedge \Diamond y) = \Diamond_s x \vee \Diamond_s y$$

and

$$\Diamond_s 0_s = \Diamond_s 0 = s \wedge \Diamond 0 = s \wedge 0 = 0 = 0_s.$$

Instead of modal algebra homomorphisms we will work with *relativized modal algebra homomorphisms*; that is, maps $\eta : A \to B$ such that η is a modal algebra homomorphism from A to the relativized modal algebra $B_{\eta(1)}$. Note that it may happen that $B_{\eta(1)} = \{0\}$. By Lemma 3.1, η is a relativized modal algebra homomorphism iff η preserves $\wedge, \vee, 0$ and $\eta(\Diamond a) = \Diamond_{\eta(1)}\eta(a)$ for each $a \in A$. When no confusion arises, we call relativized modal algebra homomorphisms simply relativized homomorphisms. Clearly each identity map $A \to A$ is a relativized homomorphism. If $\eta : A \to B$ and $\theta : B \to C$ are relativized homomorphisms, then $\theta \circ \eta : A \to C_{\theta(\eta(1))}$ is a homomorphism, and so $\theta \circ \eta : A \to C$ is a relativized homomorphism. It follows that modal algebras and relativized homomorphisms form a category which we denote by $\mathbf{MA^R}$. Clearly $\mathbf{MA}$ is a subcategory of $\mathbf{MA^R}$, $\mathbf{MA}$ and $\mathbf{MA^R}$ have the same objects, but not every morphism in $\mathbf{MA^R}$ is a morphism in $\mathbf{MA}$.

Next we introduce the concept dual to that of relativized homomorphism. Let X and Y be Stone spaces and $f : X \to Y$ be a partial map. We call f a *partial continuous map* if $\mathrm{dom}(f)$ is a clopen subset of X and f is a continuous map from $\mathrm{dom}(f)$ to Y. In particular, if $\mathrm{dom}(f) = X$, then $f : X \to Y$ is a continuous map, and so the concept of partial continuous map generalizes that of continuous map. Note that $f = \emptyset$ is a partial continuous map.

Definition 3.2. Let (X, R) and (Y, Q) be modal spaces and $f : X \to Y$ be a partial continuous map. We call f a *partial continuous p-morphism* if in addition f satisfies:

1. $x, z \in \mathrm{dom}(f)$ and xRz imply $f(x)Qf(z)$.
2. $x \in \mathrm{dom}(f)$ and $f(x)Qy$ imply there exists $z \in \mathrm{dom}(f)$ such that xRz and $f(z) = y$.

Clearly each identity map $X \to X$ is a partial continuous p-morphism. Let $f : X \to Y$ and $g : Y \to Z$ be partial continuous p-morphisms. Then $f^{-1}(\mathrm{dom}(g))$ is a clopen subset of $\mathrm{dom}(f)$, hence a clopen subset of X, and the restriction of $g \circ f$ to $f^{-1}(\mathrm{dom}(g))$ is a continuous p-morphism from $f^{-1}(\mathrm{dom}(g))$ to Z. Therefore, we define the composition of f and g as the partial map $g * f : X \to Z$ such that $\mathrm{dom}(g * f) = f^{-1}(\mathrm{dom}(g))$ and $(g * f)(x) = g(f(x))$ for $x \in \mathrm{dom}(g * f)$. Then $g * f : X \to Z$ is a partial continuous p-morphism. It follows that modal spaces and partial continuous p-morphisms form a category which we denote by $\mathbf{MS^P}$. Clearly $\mathbf{MS}$ is a subcategory of $\mathbf{MS^P}$, $\mathbf{MS}$ and $\mathbf{MS^P}$ have the same objects, but not every morphism in $\mathbf{MS^P}$ is a morphism in $\mathbf{MS}$. In order to prove that $\mathbf{MA^R}$ is dually equivalent to $\mathbf{MS^P}$, we need the following lemma.

Lemma 3.3.

1. *Let A, B be modal algebras, $\eta : A \to B$ be a relativized homomorphism, and $a \in A$. Then $\eta(\Diamond a) \leq \Diamond\eta(a)$.*
2. *Let $(X, R), (Y, Q)$ be modal spaces, $f : X \to Y$ be a partial p-morphism, and $x \in \mathrm{dom}(f)$. Then $fR(x) = Qf(x)$.*

Proof. (1) Let $a \in A$. Since η is a relativized homomorphism, we have:

$$\eta(\Diamond a) = \Diamond_{\eta(1)}\eta(a) = \eta(1) \wedge \Diamond\eta(a) \leq \Diamond\eta(a).$$

(2) First suppose that $y \in fR(x)$. Then there exists $z \in \mathrm{dom}(f)$ such that xRz and $f(z) = y$. Since $x \in \mathrm{dom}(f)$, by Definition 3.2.1, $f(x)Qf(z)$. Therefore, $f(x)Qy$, and so $y \in Qf(x)$. Next suppose that $y \in Qf(x)$. Then $f(x)Qy$. By Definition 3.2.2, there exists $z \in \mathrm{dom}(f)$ such that xRz and $f(z) = y$. Thus, $y \in fR(x)$, and so $fR(x) = Qf(x)$. ■

Theorem 3.4. *$\mathbf{MA^R}$ is dually equivalent to $\mathbf{MS^P}$.*

Proof. We define a contravariant functor $(-)_* : \mathbf{MA^R} \to \mathbf{MS^P}$ as follows. For a modal algebra A, let A_* be the modal space of A. If $\eta : A \to B$ is a relativized homomorphism and $x \in B_*$, then $\eta^{-1}(x) = \emptyset$ or $\eta^{-1}(x)$ is an ultrafilter of A_*. We set $\mathrm{dom}(\eta_*) = \{x \in B_* : \eta^{-1}(x) \neq \emptyset\}$ and for $x \in \mathrm{dom}(\eta_*)$ we set $\eta_*(x) = \eta^{-1}(x)$.

Claim 3.5. $\mathrm{dom}(\eta_*) = \varphi(\eta(1))$ *and* $\eta_*^{-1}(\varphi(a)) = \varphi(\eta(a))$ *for each* $a \in A$.

Proof. We have $x \in \mathrm{dom}(\eta_*)$ iff $\eta^{-1}(x) \neq \emptyset$ iff $1 \in \eta^{-1}(x)$ iff $\eta(1) \in x$ iff $x \in \varphi(\eta(1))$. Thus, $\mathrm{dom}(\eta_*) = \varphi(\eta(1))$. We also have $x \in \eta_*^{-1}(\varphi(a))$ iff $x \in \mathrm{dom}(\eta_*)$ and $\eta_*(x) \in \varphi(a)$ iff $x \in \mathrm{dom}(\eta_*)$ and $a \in \eta_*(x)$ iff $\eta(a) \in x$ iff $x \in \varphi(\eta(a))$. Thus, $\eta_*^{-1}(\varphi(a)) = \varphi(\eta(a))$. ■

Now since $\varphi(\eta(1))$ is a clopen subset of B_*, it follows that $\mathrm{dom}(\eta_*)$ is clopen. Moreover, as each clopen subset of A_* has the form $\varphi(a)$ for some $a \in A$, we obtain that η_* is a continuous map from $\mathrm{dom}(\eta_*)$ to A_*. Consequently, η_* is a partial continuous map. We show that η_* satisfies conditions (1) and (2) of Definition 3.2.

Let $x, z \in \mathrm{dom}(\eta_*)$, xRz, and $a \in \eta_*(z)$. Then $a \in \eta^{-1}(z)$, and so $\eta(a) \in z$. Since xRz, we have $\Diamond\eta(a) \in x$. As $x \in \mathrm{dom}(\eta_*)$, we also have $\eta^{-1}(x) \neq \emptyset$, so $1 \in \eta^{-1}(x)$, and so $\eta(1) \in x$. Therefore, $\eta(1) \wedge \Diamond\eta(a) \in x$, which means that $\Diamond_{\eta(1)}\eta(a) \in x$. Because η is a relativized homomorphism, $\Diamond_{\eta(1)}\eta(a) = \eta(\Diamond a)$. It follows that $\eta(\Diamond a) \in x$. Thus, $\Diamond a \in \eta^{-1}(x)$, so $\Diamond a \in \eta_*(x)$, and so $\eta_*(x)Q\eta_*(z)$. Consequently, η_* satisfies condition (1) of Definition 3.2.

Now let $x \in \mathrm{dom}(\eta_*)$ and $\eta_*(x)Qy$. Let F be the filter generated by $\eta[y] = \{\eta(a) : a \in y\}$ and I be the ideal generated by $\{a \in B : \Diamond a \notin x\} \cup \eta[A - y]$. If $F \cap I \neq \emptyset$, then there exist $a \in y$, $b \in B$ with $\Diamond b \notin x$, and $c \notin y$ such that $\eta(a) \leq b \vee \eta(c)$. Therefore, $\eta(a) \wedge \neg\eta(c) \leq b$. Since $\eta(\neg c) \leq \neg\eta(c)$, we have $\eta(a) \wedge \eta(\neg c) \leq b$. Thus, $\eta(a \wedge \neg c) \leq b$, and so $\Diamond\eta(a \wedge \neg c) \leq \Diamond b$. By Lemma 3.3.1, $\eta(\Diamond(a \wedge \neg c)) \leq \Diamond\eta(a \wedge \neg c)$. This yields $\eta(\Diamond(a \wedge \neg c)) \leq \Diamond b$. As $a \wedge \neg c \in y$, we have $\Diamond(a \wedge \neg c) \in \eta^{-1}(x)$, and so $\eta(\Diamond(a \wedge \neg c)) \in x$. Therefore, $\Diamond b \in x$, a contradiction. Thus, $F \cap I = \emptyset$, and so there exists an ultrafilter z of B such that $F \subseteq z$ and $I \cap z = \emptyset$. From $\{a \in B : \Diamond a \notin x\} \cap z = \emptyset$ it follows that xRz, and $F \subseteq z$ and $\eta[A - y] \cap z = \emptyset$ imply $\eta^{-1}(z) = y$. Therefore, xRz and $\eta^{-1}(z) = y$, which implies that $z \in \mathrm{dom}(\eta_*)$ and $\eta_*(z) = y$. Thus, there exists $z \in \mathrm{dom}(\eta_*)$ such that xRz and $\eta_*(z) = y$, and so η_* satisfies condition (2) of Definition 3.2. Consequently, η_* is a partial continuous p-morphism.

By the duality for modal algebras, if $\eta : A \to A$ is identity, then so is $\eta_* : A_* \to A_*$. Let $\eta : A \to B$ and $\theta : B \to C$ be relativized homomorphisms. We show that $\eta_* * \theta_* : C_* \to A_*$ is a partial continuous p-morphism and that $(\theta \circ \eta)_* = \eta_* * \theta_*$. We have that $\eta : A \to B_{\eta(1)}$ and $\theta : B_{\eta(1)} \to C_{\theta(\eta(1))}$ are homomorphisms. Therefore, by the duality for modal algebras and Claim 3.5, $\eta_* : \varphi(\eta(1)) \to A_*$ and $\theta_* : \varphi(\theta(\eta(1))) \to B_*$ are continuous p-morphisms. Thus, $\eta_* \circ \theta_* : \varphi(\theta(\eta(1))) \to A_*$ is a continuous p-morphism. Moreover, $\mathrm{dom}(\eta_* * \theta_*) = (\theta_*)^{-1}(\mathrm{dom}(\eta_*)) = (\theta_*)^{-1}(\varphi(\eta(1))) = \varphi(\theta(\eta(1)))$. Consequently, $\eta_* * \theta_* : C_* \to A_*$ is a partial continuous p-morphism. Furthermore, $\mathrm{dom}((\theta \circ \eta)_*) = \varphi(\theta(\eta(1))) = \mathrm{dom}(\eta_* * \theta_*)$, and for $x \in \mathrm{dom}((\theta \circ \eta)_*)$, we have $(\theta \circ \eta)_*(x) = (\theta \circ \eta)^{-1}(x) = \eta^{-1}(\theta^{-1}(x)) = \eta_*(\theta_*(x)) = (\eta_* * \theta_*)(x)$. Thus, $(-)_* : \mathbf{MA^R} \to \mathbf{MS^P}$ is a well-defined functor.

Next we define a contravariant functor $(-)_* : \mathbf{MS^P} \to \mathbf{MA^R}$ as follows. For a modal space (X, R), let $(X, R)^*$ be the modal algebra $(\mathrm{Clopen}(X), \Diamond_R)$. Also for a partial continuous p-morphism $f : X \to Y$, let $f^* : Y^* \to X^*$ be given by $f^*(U) = f^{-1}(U)$. It is easy to check that f^* is a relativized Boolean algebra homomorphism. Let $U \in Y^*$. We show that

$$f^*(\Diamond_Q U) = \Diamond_R f^*(U) \cap \mathrm{dom}(f).$$

We have $x \in f^*(\Diamond_Q U)$ iff $x \in \mathrm{dom}(f)$ and $f(x) \in \Diamond_Q U$ iff $x \in \mathrm{dom}(f)$ and $Qf(x) \cap U \neq \emptyset$. On the other hand, $x \in \Diamond_R f^*(U) \cap \mathrm{dom}(f)$ iff $R(x) \cap f^{-1}(U) \neq \emptyset$ and $x \in \mathrm{dom}(f)$ iff $fR(x) \cap U \neq \emptyset$ and $x \in \mathrm{dom}(f)$. Since $x \in \mathrm{dom}(f)$, by Lemma 3.3.2, $fR(x) = Qf(x)$. Therefore, $x \in f^*(\Diamond_Q U)$ iff $x \in \Diamond_R f^*(U) \cap \mathrm{dom}(f)$, and so $f^*(\Diamond_Q U) = \Diamond_R f^*(U) \cap \mathrm{dom}(f)$. Consequently, f^* is a relativized homomorphism.

By the duality for modal algebras, if $f : X \to X$ is identity, then so is $f^* : X^* \to X^*$. Let $f : X \to Y$ and $g : Y \to Z$ be partial continuous p-morphisms. Then $g * f : X \to Y$ is a partial continuous p-morphism with $\mathrm{dom}(g * f) = f^{-1}(\mathrm{dom}(g))$. Moreover, for $U \in Z^*$, we have $(g * f)^*(U) = (g * f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^*(g^*(U))$. Therefore, $(g * f)^* = f^* \circ g^*$, and so $(-)_* : \mathbf{MS^P} \to \mathbf{MA^R}$ is a well-defined functor.

Finally, it is obvious that the isomorphisms $\varphi : B \to {B_*}^*$ and $\varepsilon : X \to {X^*}_*$ given by the duality for modal algebras are still natural in this more general setting. Thus, $\mathbf{MA^R}$ is dually equivalent to $\mathbf{MS^P}$. ■

Remark 3.6. If $\eta : A \to B$ is a modal algebra homomorphism, then $\eta(1) = 1$, and so $\mathrm{dom}(\eta_*) = \varphi(\eta(1)) = \varphi(1) = B_*$. Thus, $\eta_* : B_* \to A_*$ is a total continuous p-morphism. Also, if $f : X \to Y$ is a total continuous p-morphism, then clearly $f^{-1} : Y^* \to X^*$ is a modal algebra homomorphism. Therefore, the dual equivalence of **MA** and **MS** is an easy consequence of Theorem 3.4.

Remark 3.7. Let **BA** denote the category of Boolean algebras and Boolean algebra homomorphisms, and **Stone** denote the category of Stone spaces and continuous maps. By Stone duality, **BA** is dually equivalent to **Stone**. Let also $\mathbf{BA^R}$ denote the category of Boolean algebras and relativized Boolean algebra homomorphisms, and $\mathbf{Stone^P}$ denote the category of Stone spaces and partial continuous maps. Then it is a consequence of Theorem 3.4 that $\mathbf{BA^R}$ is dually equivalent to $\mathbf{Stone^P}$. The proof of this is an obvious generalization of the proof that Stone duality is a consequence of the duality between **MA** and **MS**.

4. K4-algebras and the closed domain condition

In this section we restrict our attention to **K4**-algebras and their dual transitive spaces. For transitive spaces X and Y, we show that a partial continuous p-morphism $f : X \to Y$ satisfies the closed domain condition (CDC) iff the dual relativized homomorphism $f^* : Y^* \to X^*$ preserves $\Diamond_Q$ for some specified subset D of Y^*. This results in a purely algebraic characterization of (CDC). We also give an algebraic characterization of when $f : X \to Y$ is cofinal and when f is dense. We conclude the section by comparing our approach to that of Zakharyaschev.

We recall that a modal algebra A is a **K4**-*algebra* if $\Diamond\Diamond a \leq \Diamond a$ for each $a \in A$. Let **K4** denote the category of **K4**-algebras and modal algebra homomorphisms. Let A be a **K4**-algebra and $s \in A$. It is well known (see, e.g., [3, Lem. 4.8]) that the relativization A_s of A to s is also a **K4**-algebra. For each $a \in A$, we set $\Diamond^+ a = a \vee \Diamond a$. Then it is obvious that $a \leq \Diamond^+ a$ for each $a \in A$, and so $(A, \Diamond^+)$ is an **S4**-algebra. Moreover, $\Box^+ a = \neg\Diamond^+\neg a = a \wedge \Box a$, and $H = \Box^+(A) = \{\Box^+ a : a \in A\}$ is a Heyting algebra (see, e.g., [3, Sec. 3]). Next lemma will be used in Section 5. A Heyting algebra analogue of the lemma can be found in [15, Lem. 1].

LEMMA 4.1. *Let A be a* **K4**-*algebra, $a, b \in A$, and $\Box^+ a \not\leq b$. Then there exists a subdirectly irreducible* **K4**-*algebra B and an onto homomorphism $\eta : A \to B$ such that $\eta(\Box^+ a) = 1$ and $\eta(b) \neq 1$.*

PROOF. Let F be the filter of A generated by $\Box^+ a$. Since A is a **K4**-algebra, F is a $\Box$-filter of A. Moreover, $\Box^+ a \in F$ and $b \notin F$. Let Z be the set of $\Box$-filters of A containing $\Box^+ a$ and missing b. Then $F \in Z$, and so Z is nonempty. If we order Z by set inclusion, then it is easy to see that Z is an inductive set. Therefore, by Zorn's lemma, Z has a maximal element M. Let $B = A/{\sim}$, where $x{\sim}y$ iff $x \leftrightarrow y \in M$. For $x \in A$, let $[x] = \{y \in A : x{\sim}y\}$. Define $\eta : A \to B$ by $\eta(x) = [x]$ for each $x \in A$. Then it is well known that $\eta : A \to B$ is an onto modal algebra homomorphism. It is also clear that $\eta(\Box^+ a) = 1$ and $\eta(b) \neq 1$. Moreover, each $\Box$-filter of B corresponds to a $\Box$-filter of A containing M. Since M is a maximal $\Box$-filter of A containing $\Box^+ a$ and missing b, each $\Box$-filter of A properly containing M also contains b. Therefore, each $\Box$-filter of B which properly contains $\{1\}$, also contains $\eta(b)$. Thus, the filter of B generated by $\Box^+\eta(b) = \eta(\Box^+ b)$ is the smallest $\Box$-filter of B properly containing $\{1\}$. Consequently, B is subdirectly irreducible. ■

It is well known that the dual spaces of **K4**-algebras are *transitive spaces*; that is, modal spaces (X, R) in which R is transitive. Let **TS** denote the category of transitive spaces and continuous p-morphisms. Then the duality between **MA** and **MS** restricts to the duality between **K4** and **TS**. Moreover, if $A \in \mathbf{K4}$ and A_* is the dual transitive space of A, then $(A, \Diamond^+)_* = (A_*, R^+)$, where $R^+ = R \cup \{(x,x) : x \in A_*\}$ is the reflexive closure of R (see, e.g., [3, Sec. 3]).

Let X be a transitive space and $U \subseteq X$. We say that $x \in U$ is a *minimal point* of U if for each $y \in U$, from yRx it follows that xRy. We denote by $\min(U)$ the set of minimal points of U. It is well known (see, e.g., [7, Thm. III.2.1]) that for each closed subset F of X and $y \in F$ there exists $x \in \min(F)$ such that xR^+y. In fact, for each closed subset F of X, by selecting one point from each $C \cap F$, where C is a cluster with $C \cap \min(F) \neq \emptyset$, we can find an antichain $\mathfrak{d} \subseteq \min(F)$ such that $F \subseteq R^+(\mathfrak{d})$. In order to avoid such a selection, it is more convenient to work with quasi-antichains instead of antichains, where $\mathfrak{d} \subseteq X$ is a *quasi-antichain* if xRy implies yRx for each $x, y \in \mathfrak{d}$. Clearly each antichain is a quasi-antichain, but not the other way around. Nevertheless, they are closely related concepts; it is easy to see that if $\mathfrak{d}$ is a quasi-antichain, then by selecting one point from each cluster of $\mathfrak{d}$, we obtain an antichain $\mathfrak{d}_0$ such that $R^+(\mathfrak{d}) = R^+(\mathfrak{d}_0)$. One particular advantage of quasi-antichains over antichains is that if F is a closed subset of X, then $\min(F)$ is always a quasi-antichain, which in general may not be an antichain.

From now on we will mostly work with quasi-antichains, but we point out that it is only a convenient convention; all our results involving quasi-antichains can also be formulated by means of antichains.

DEFINITION 4.2. Let X and Y be transitive spaces and let $f : X \to Y$ be a partial continuous p-morphism. Let also $\mathfrak{D}$ be a (possibly empty) set of quasi-antichains in Y. We say that f satisfies the *closed domain condition (CDC)* for $\mathfrak{D}$ if:

$$f(R(x)) = R^+(\mathfrak{d}) \text{ for some } \mathfrak{d} \in \mathfrak{D} \text{ implies } x \in \mathrm{dom}(f).$$

Equivalently, f satisfies (CDC) for $\mathfrak{D}$ if

$$x \notin \mathrm{dom}(f) \text{ implies } f(R(x)) \neq R^+(\mathfrak{d}) \text{ for each } \mathfrak{d} \in \mathfrak{D}.$$

In particular, since $\min fR(x)$ is a quasi-antichain with
$fR(x) = R^+(\min fR(x))$, we have that $x \notin \mathrm{dom}(f)$ implies $\min fR(x) \notin \mathfrak{D}$.

LEMMA 4.3. *Let $(X,R), (Y,Q)$ be transitive spaces, $f : X \to Y$ be a partial continuous p-morphism, and U be a clopen subset of Y. We let*

$$\mathfrak{D}_U = \{\min fR(x) : fR(x) \cap U \neq \emptyset\}.$$

Then the following conditions are equivalent:

1. *f satisfies (CDC) for $\mathfrak{D}_U$.*
2. *$x \notin \mathrm{dom}(f)$ implies $\min fR(x) \notin \mathfrak{D}_U$.*
3. *$x \notin \mathrm{dom}(f)$ implies $fR(x) \cap U = \emptyset$.*
4. *$\Diamond_R f^{-1}(U) \subseteq f^{-1}\Diamond_Q(U)$.*

Proof. The implications (1)⇒(2)⇒(3) are obvious.

(3)⇒(4): Let $x \in \Diamond_R f^{-1}(U)$. Then $fR(x) \cap U \neq \emptyset$. By (3), $x \in \mathrm{dom}(f)$. By Lemma 3.3.2, $fR(x) = Qf(x)$. Therefore, $Qf(x) \cap U \neq \emptyset$, so $f(x) \in \Diamond_Q(U)$, and hence $x \in f^{-1}\Diamond_Q(U)$. Thus, $\Diamond_R f^{-1}(U) \subseteq f^{-1}\Diamond_Q(U)$.

(4)⇒(1): Let $x \notin \mathrm{dom}(f)$. If $\min(fR(x)) \in \mathfrak{D}_U$, then $fR(x) \cap U \neq \emptyset$. Therefore, $x \in \Diamond_R f^{-1}(U)$, and so, by (4), $x \in f^{-1}\Diamond_Q(U)$. Thus, $x \in \mathrm{dom}(f)$, a contradiction. Consequently, $\min(fR(x)) \notin \mathfrak{D}_U$, and it follows from the definition of $\mathfrak{D}_U$ that f satisfies (CDC) for $\mathfrak{D}_U$. ■

Remark 4.4. As follows from Lemma 4.3, a partial continuous p-morphism f satisfies (CDC) for $\mathfrak{D}_U$ iff $x \notin \mathrm{dom}(f)$ implies $fR(x) \cap U = \emptyset$. Therefore, we could take the latter condition as the definition of (CDC). We chose the former condition as the definition of (CDC) because we wanted to keep our approach close to that of Zakharyaschev.

Next we give an algebraic analogue of Lemma 4.3.

Theorem 4.5. *Let A and B be* **K4***-algebras, $\eta : A \to B$ be a relativized homomorphism, and $a \in A$. Then the following two conditions are equivalent:*

1. *$\eta(\Diamond a) = \Diamond\eta(a)$.*
2. *$\eta_* : B_* \to A_*$ satisfies (CDC) for $\mathfrak{D}_{\varphi(a)}$.*

Proof. The result follows from Lemmas 3.3.1, 4.3, and Theorem 3.4. ■

Corollary 4.6. *Let A and B be* **K4***-algebras, $\eta : A \to B$ be a relativized homomorphism, and $D \subseteq A$. Then the following two conditions are equivalent:*

1. *$\eta(\Diamond a) = \Diamond\eta(a)$ for each $a \in D$.*
2. *$\eta_* : B_* \to A_*$ satisfies (CDC) for $\mathfrak{D} = \bigcup\{\mathfrak{D}_{\varphi(a)} : a \in D\}$.*

Proof. Apply Theorem 4.5. ■

Next we recall the definitions of cofinal and dense partial continuous p-morphisms and give their dual algebraic descriptions.

Definition 4.7. Let (X, R) be a transitive space. We call $Y \subseteq X$ *cofinal* if $X = (R^+)^{-1}(Y)$.

Definition 4.8. Let X and Y be transitive spaces and let $f : X \to Y$ be a partial continuous p-morphism.

1. We say that f is *cofinal* if $\mathrm{dom}(f)$ is cofinal in X.
2. We say that f is *dense* if $\mathrm{dom}(f)$ is a *downset*; that is, $x \in \mathrm{dom}(f)$ and yRx imply $y \in \mathrm{dom}(f)$. In other words, f is dense if $x \notin \mathrm{dom}(f)$ implies $R(x) \cap \mathrm{dom}(f) = \emptyset$.

Lemma 4.9. *Let X and Y be transitive spaces and let $f : X \to Y$ be a partial continuous p-morphism. Then f is cofinal iff $\Diamond_{R^+}(\mathrm{dom} f) = X$.*

Proof. We have f is cofinal iff $X = (R^+)^{-1}\mathrm{dom}(f)$ iff $\Diamond_{R^+}(\mathrm{dom} f) = X$. ■

Definition 4.10. Let A and B be **K4**-algebras and let $\eta : A \to B$ be a relativized homomorphism.

1. We say that η is *cofinal* if $\Diamond^+\eta(1) = 1$.
2. We say that η is *dense* if $\Diamond\eta(1) \leq \eta(1)$.

Lemma 4.11. *Let A and B be* **K4**-*algebras and let $\eta : A \to B$ be a relativized homomorphism. Then:*

1. *η is cofinal iff $\eta_* : B_* \to A_*$ is cofinal.*
2. *η is dense iff $\eta_* : B_* \to A_*$ is dense.*

Proof. (1) We recall that $\varphi(\eta(1)) = \mathrm{dom}(\eta_*)$. Therefore, η is cofinal iff $\Diamond^+\eta(1) = 1$ iff $\varphi(\Diamond^+\eta(1)) = B_*$ iff $\Diamond_{R^+}\mathrm{dom}(\eta_*) = B_*$, which, by Lemma 4.9, holds iff η_* is cofinal.

(2) We have:

$$\begin{aligned}
\eta \text{ is dense } & \text{ iff } \Diamond\eta(1) \leq \eta(1) \\
& \text{ iff } \Diamond_R(\mathrm{dom}(\eta_*)) \subseteq \mathrm{dom}(\eta_*) \\
& \text{ iff } B_* - \mathrm{dom}(\eta_*) \subseteq B_* - \Diamond_R(\mathrm{dom}(\eta_*)) \\
& \text{ iff } x \notin \mathrm{dom}(\eta_*) \Rightarrow R(x) \cap \mathrm{dom}(\eta_*) = \emptyset \\
& \text{ iff } \eta_* \text{ is dense.}
\end{aligned}$$

■

Lemma 4.12. *If A and B are* **K4***-algebras and $\eta : A \to B$ is a cofinal and dense relativized homomorphism, then η is a modal algebra homomorphism.*

Proof. Since η is cofinal, $\Diamond^+\eta(1) = 1$. Therefore, $\eta(1) \vee \Diamond\eta(1) = 1$. But as η is dense, $\Diamond\eta(1) \leq \eta(1)$. Thus, $\eta(1) = 1$, and so η is a modal algebra homomorphism. ∎

Next lemma is an immediate consequence of Lemmas 4.11 and 4.12 and Theorem 3.4. Nevertheless, its direct proof is simple enough that we give it below.

Lemma 4.13. *If X and Y are transitive spaces and $f : X \to Y$ is a cofinal and dense partial continuous p-morphism, then f is a total continuous p-morphism.*

Proof. It is sufficient to show that $\mathrm{dom}(f) = X$. If not, then there exists $x \notin \mathrm{dom}(f)$. As f is cofinal, there exists $y \in \mathrm{dom}(f)$ such that xR^+y. Since $x \notin \mathrm{dom}(f)$, we have $x \neq y$. Therefore, xRy, and so $R(x) \cap \mathrm{dom}(f) \neq \emptyset$. As f is dense, x must be in $\mathrm{dom}(f)$. The obtained contradiction proves that $\mathrm{dom}(f) = X$, and so f is a total continuous p-morphism. ∎

Next lemma will play an important role in Section 5. Let A be a **K4**-algebra and (X, R) be the dual space of A. We recall that $Y \subseteq X$ is an *upset* of X if $x \in Y$ and xRy imply $y \in Y$, and that homomorphic images of A dually correspond to closed upsets of X (see, e.g., [5, Sec. 8.5]).

Lemma 4.14. *Let A and B be* **K4***-algebras, $s \in A$, and $\eta : A_s \to B$ be an onto homomorphism. Then there exists a* **K4***-algebra C and an onto homomorphism $\theta : A \to C$ such that B is isomorphic to the relativization of C to $\theta(s)$. Moreover, if s is cofinal in A, then $\theta(s)$ is cofinal in C.*

Proof. Let (X, R) be the dual space of A and let R_s be the restriction of R to $\varphi(s)$. Then $(\varphi(s), R_s)$ is homeomorphic to the dual space of A_s. Since B is a homomorphic image of A_s, the dual space of B is homeomorphic to a closed upset Y of $\varphi(s)$. Let $Z = Y \cup R(Y)$. Then Z is a closed upset of X. Therefore, (Z, R_Z) is a transitive space. Let C be the **K4**-algebra of clopen subsets of (Z, R_Z). Then C is a homomorphic image of A. Let $\theta : A \to C$ be the onto homomorphism. As Y is an upset of X, we have $Y = Z \cap \varphi(s) = \varphi(\theta(s))$. Therefore, Y is a clopen subset of Z, and so B is isomorphic to the relativization of C to $\theta(s)$. In addition, if s is cofinal in A, then $\varphi(s)$ is cofinal in X. Thus, Y is cofinal in Z, and so $\theta(s)$ is cofinal in C. ∎

We conclude this section by comparing our approach to that of Zakharyaschev. We will mostly follow [5, Sec. 9], which is a streamlined version of Zakharyaschev's earlier results. We point out that Zakharyaschev works with transitive general frames, whereas we work with transitive spaces. Although transitive spaces form a proper subcategory of the transitive general frames, they are sufficient for our purposes as they are duals of **K4**-algebras. Note that for transitive spaces, the notion of *subreduction* [5, p. 287] coincides with that of *onto* partial continuous p-morphism.

Definition 4.15. Let X and Y be transitive spaces and let $f : X \to Y$ be an onto partial continuous p-morphism.

1. [5, p. 295] We call f a *cofinal subreduction* if

$$R(\mathrm{dom}(f)) \subseteq (R^+)^{-1}(\mathrm{dom}(f)).$$

2. [5, p. 293] We call f a *dense subreduction*if

$$R^+(\mathrm{dom}(f)) \cap (R^+)^{-1}(\mathrm{dom}(f)) = \mathrm{dom}(f).$$

For an onto partial continuous p-morphism $f : X \to Y$, it is easy to see that if f is cofinal (resp. dense) in our sense (Definition 4.8), then it is cofinal (resp. dense) in Zakharyaschev's sense (Definition 4.15). However, the converse is clearly not true (see, e.g., [1, Ex. 4.3 and 4.6]). Nevertheless, each cofinal (resp. dense) subreduction $f : X \to Y$ gives rise to a cofinal (resp. dense) partial continuous p-morphism from the closed upset $R^+(\mathrm{dom}(f))$ of X onto Y. For a proof, we refer to [1, Lem. 4.5 and 4.7]. Note that [1] discusses only the intuitionistic case, but the proof for the modal case is unchanged.

Next we address Zakharyaschev's (CDC). We point out that Zakharyaschev only considers subreductions onto finite transitive frames. The main reason for this, of course, is that the canonical formulas he defines are associated with finite (rooted) transitive frames rather than any transitive space. On the other hand, our (CDC) applies to the infinite case as well (although the canonical formulas we will define will also be associated only with finite subdirectly irreducible **K4**-algebras). Therefore, we will not assume that the target space is finite. In addition, Zakharyaschev works with antichains, while we prefer to work with quasi-antichains. But as we mentioned earlier in this section, it is only a matter of convenience. Thus, we will modify Zakharyaschev's definition by replacing antichains by quasi-antichains.

Definition 4.16. [5, p. 298] Let Y be a transitive space and $\mathfrak{D}$ be a (possibly empty) set of quasi-antichains in Y. We say that a partial continuous p-morphism f from a transitive space X to Y satisfies *Zakharyaschev's closed domain condition* ($ZCDC$) *for* $\mathfrak{D}$ if:

$$x \in R(\mathrm{dom}(f)) \text{ and } f(R(x)) = R^+(\mathfrak{d}) \text{ for some } \mathfrak{d} \in \mathfrak{D} \text{ imply } x \in \mathrm{dom}(f).$$

Clearly (CDC) implies (ZCDC). However, the converse is not true in general. Nevertheless, (ZCDC) implies (CDC) for the restriction of f to $R^+(\mathrm{dom}(f))$. Again, the proof is the same as in the intuitionistic case [1, Cor. 4.9] and we skip it.

5. Canonical formulas for K4

In this section we give an algebraic description of canonical formulas. Our canonical formulas generalize the Jankov-Rautenberg formulas. The main result of the section is a new algebraic proof of Zakharyaschev's theorem that each logic over **K4** is axiomatizable by these formulas.

5.1. An algebraic description of canonical formulas

We assume that modal formulas are built from propositional variables and the constants $\top$ and $\bot$ by means of the connectives $\neg, \vee$ and the modal operator $\Diamond$. We also treat the connectives $\wedge, \to, \leftrightarrow$ and the modal operator $\Box$ as derived operations in the standard way; that is, $p \wedge q = \neg(\neg p \vee \neg q)$, $p \to q = \neg p \vee q$, $p \leftrightarrow q = (p \to q) \wedge (q \to p)$, and $\Box p = \neg\Diamond\neg p$. For modal formulas α and β, we use the following abbreviations: $\neg_\alpha \beta = \alpha \wedge \neg\beta$ and $\Diamond_\alpha \beta = \alpha \wedge \Diamond\beta$.

Let A be a finite subdirectly irreducible **K4**-algebra. Then it is well known that $H = \Box^+(A)$ is a subdirectly irreducible Heyting algebra, hence H has the second largest element which we denote by t. Let D be a subset of A. For each $a \in A$ we introduce a new variable p_a and define the *canonical formula* $\alpha(A, D)$ associated with A and D as follows:

$$\begin{aligned}\alpha(A, D) = \Box^+[\ &(\top \leftrightarrow \Diamond^+ p_1) \wedge (\bot \leftrightarrow p_0) \wedge \\ &\bigwedge\{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \wedge \\ &\bigwedge\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \wedge \\ &\bigwedge\{p_{\Diamond a} \leftrightarrow \Diamond_{p_1} p_a : a \in A\} \wedge \\ &\bigwedge\{p_{\Diamond a} \leftrightarrow \Diamond p_a : a \in D\}] \to (p_1 \to p_t).\end{aligned}$$

If we let

$$\begin{aligned}\Gamma = \quad & (\top \leftrightarrow \Diamond^+ p_1) \wedge (\bot \leftrightarrow p_0) \wedge \\ & \bigwedge \{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \wedge \\ & \bigwedge \{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \wedge \\ & \bigwedge \{p_{\Diamond a} \leftrightarrow \Diamond_{p_1} p_a : a \in A\} \wedge \\ & \bigwedge \{p_{\Diamond a} \leftrightarrow \Diamond p_a : a \in D\},\end{aligned}$$

then

$$\alpha(A, D) = \Box^+\Gamma \to (p_1 \to p_t).$$

LEMMA 5.1. *Let A be a finite subdirectly irreducible* **K4**-*algebra, $H = \Box^+(A)$, t be the second largest element of H, and D be a subset of A. Then $A \not\models \alpha(A, D)$.*

PROOF. Define a valuation ν on A by $\nu(p_a) = a$ for each $a \in A$. Then

$$\nu(\alpha(A, D)) = \Box^+ 1 \to (1 \to t) = 1 \to t = t.$$

Therefore, $A \not\models \alpha(A, D)$. ■

THEOREM 5.2. *Let A be a finite subdirectly irreducible* **K4**-*algebra, $D \subseteq A$, and B be a* **K4**-*algebra. Then $B \not\models \alpha(A, D)$ iff there exist a homomorphic image C of B and a 1-1 modal algebra homomorphism η from A into a cofinal relativization C_s of C such that $\eta(\Diamond a) = \Diamond\eta(a)$ for each $a \in D$.*

PROOF. First assume that there exist a homomorphic image C of B and a 1-1 modal algebra homomorphism η from A into a cofinal relativization C_s of C such that $\eta(\Diamond a) = \Diamond\eta(a)$ for each $a \in D$. By Lemma 5.1, there is a valuation ν on A refuting $\alpha(A, D)$. We define a valuation μ on C by $\mu(p_a) = \eta \circ \nu(p_a) = \eta(a)$ for each $a \in A$. We show that $\mu(\alpha(A, D)) = \eta(t)$. Since s is cofinal in C, we have $1_C = \Diamond^+\eta(1_A)$. Therefore,

$$\mu(\top \leftrightarrow \Diamond^+ p_1) = \mu(\top) \leftrightarrow \Diamond^+\mu(p_1) = 1_C \leftrightarrow \Diamond^+\eta(1_A) = \Diamond^+\eta(1_A) = 1_C.$$

As C_s is a relativization of C and $\eta : B \to C_s$ is a Boolean algebra homomorphism, $\eta(0_A) = 0_C$, $\eta(a \vee b) = \eta(a) \vee \eta(b)$, $\eta(a \wedge b) = \eta(a) \wedge \eta(b)$ for each $a, b \in A$. Thus,

$$\mu(\bot \leftrightarrow p_0) = 0_C \leftrightarrow \mu(p_0) = 0_C \leftrightarrow \eta(0_A) = 1_C,$$

$$\mu(p_{a\vee b} \leftrightarrow p_a \vee p_b) = \mu(p_{a\vee b}) \leftrightarrow \mu(p_a) \vee \mu(p_b) = \eta(a \vee b) \leftrightarrow \eta(a) \vee \eta(b) = 1_C,$$

$$\mu(p_{a\wedge b} \leftrightarrow p_a \wedge p_b) = \mu(p_{a\wedge b}) \leftrightarrow \mu(p_a)\wedge\mu(p_b) = \eta(a\wedge b) \leftrightarrow \eta(a)\wedge\eta(b) = 1_C.$$

Also, since $\eta : A \to C_s$ is a modal algebra homomorphism, for each $a \in A$ we have $\eta(\Diamond a) = \Diamond_{\eta(1_A)}\eta(a)$. Therefore,

$$\mu(p_{\Diamond a} \leftrightarrow \Diamond_{p_1} p_a) = \mu(p_{\Diamond a}) \leftrightarrow \Diamond_{\eta(1_A)}\mu(p_a) = \eta(\Diamond a) \leftrightarrow \Diamond_{\eta(1_A)}\eta(a) = 1_C.$$

Moreover, $\eta(\Diamond a) = \Diamond\eta(a)$ for each $a \in D$ implies

$$\mu(p_{\Diamond a} \leftrightarrow \Diamond p_a) = \mu(p_{\Diamond a}) \leftrightarrow \Diamond\mu(p_a) = \eta(\Diamond a) \leftrightarrow \Diamond\eta(a) = 1_C$$

for each $a \in D$. Thus, $\mu(\Gamma) = 1_C$, and so $\mu(\Box^+\Gamma) = \Box^+\mu(\Gamma) = \Box^+ 1_C = 1_C$. We also have that $\mu(p_t) = \eta(t)$. This yields

$$\mu(\alpha(A, D)) = \Box^+\mu(\Gamma) \to (\mu(p_1) \to \mu(p_t)) = 1_C \to (\eta(1_A) \to \eta(t)) = \eta(1_A) \to \eta(t).$$

It is obvious that $\eta(t) \leq \eta(1_A)$. If $\eta(1_A) \to \eta(t) = 1_C$, then $\eta(1_A) \leq \eta(t)$, so $\eta(1_A) = \eta(t)$, and so η is not 1-1, a contradiction. Therefore, $\eta(1_A) \not\leq \eta(t)$, yielding $\eta(1_A) \to \eta(t) \neq 1_C$. Consequently, $\alpha(A, D)$ is refuted on C. Now as C is a homomorphic image of B, we also have that $\alpha(A, D)$ is refuted on B.

Conversely, let $B \not\models \alpha(A, D)$. Then there exists a valuation μ on B such that $\mu(\alpha(A, D)) \neq 1_B$. Therefore, $\mu(\alpha(A, D)) = \Box^+\mu(\Gamma) \to (\mu(p_1) \to \mu(p_t)) \neq 1_B$. Thus, $\Box^+\mu(\Gamma) \not\leq \mu(p_1) \to \mu(p_t)$. By Lemma 4.1, there exist a subdirectly irreducible **K4**-algebra C and an onto homomorphism $\theta : B \to C$ such that $\theta(\Box^+\mu(\Gamma)) = 1_C$ and $\theta(\mu(p_1) \to \mu(p_t)) \neq 1_C$. Clearly $\nu = \theta \circ \mu$ is a valuation on C such that $\Box^+\nu(\Gamma) = 1_C$ and $\nu(p_1) \to \nu(p_t) \neq 1_C$. It follows that $\nu(\Gamma) = 1_C$.

Next define a map $\eta : A \to C$ by $\eta(a) = \nu(p_a)$ for each $a \in A$. Let $s = \eta(1_A)$. First we show that s is cofinal in C. Since $\nu(\Gamma) = 1_C$ and $\nu(\Gamma) \leq \nu(\top \leftrightarrow \Diamond^+ p_1)$, we obtain that $\nu(\top \leftrightarrow \Diamond^+ p_1) = 1_C$. But $\nu(\top \leftrightarrow \Diamond^+ p_1) = \nu(\top) \leftrightarrow \Diamond^+\nu(p_1) = 1_C \leftrightarrow \Diamond^+\nu(p_1)$, which implies that $\Diamond^+\nu(p_1) = 1_C$. But $\nu(p_1) = \eta(1_A) = s$. Therefore, $\Diamond^+ s = \Diamond^+\eta(1_A) = 1_C$, and so s is cofinal in C. Next we show that η is a 1-1 modal algebra homomorphism from A into C_s such that $\eta(\Diamond a) = \Diamond\eta(a)$ for each $a \in D$.

Let $a, b \in A$. Since $\nu(\Gamma) = 1_C$ and $\nu(\Gamma) \leq \nu(p_{a\wedge b}) \leftrightarrow (\nu(p_a) \wedge \nu(p_b))$, we obtain that $\nu(p_{a\wedge b}) \leftrightarrow (\nu(p_a) \wedge \nu(p_b)) = 1_C$. Therefore, $\nu(p_{a\wedge b}) = \nu(p_a) \wedge \nu(p_b)$. By a similar argument,

$$\nu(p_{a\vee b}) = \nu(p_a) \vee \nu(p_b),$$

$$\nu(p_0) = 0_C,$$

$\nu(p_{\Diamond a}) = \Diamond_{\nu(p_1)}\nu(p_a)$, and

$\nu(p_{\Diamond a}) = \Diamond\nu(p_a)$ for $a \in D$.

But $\nu(p_a) = \eta(a)$ for each $a \in A$. Therefore, for each $a, b \in A$, we have:

$\eta(a \wedge b) = \eta(a) \wedge \eta(b)$,
$\eta(a \vee b) = \eta(a) \vee \eta(b)$,
$\eta(0_A) = 0_C$,
$\eta(\Diamond a) = \Diamond_{\eta(1_A)}\eta(a) = \Diamond_s\eta(a)$, and
$\eta(\Diamond a) = \Diamond\eta(a)$ for $a \in D$.

By Lemma 3.1, η is a relativized Boolean algebra homomorphism, and hence η is a modal algebra homomorphism from A to C_s such that $\eta(\Diamond a) = \Diamond\eta(a)$ for each $a \in D$. It is left to be shown that η is 1-1. Let $a, b \in A$ with $a \not\leq b$. Then $a \to b \neq 1_A$, and so $\Box^+(a \to b) \leq t$. Therefore, $\eta(\Box^+(a \to b)) \leq \eta(t)$. Thus, $\Box_s^+(\eta(a) \to_s \eta(b)) \leq \eta(t) \leq \eta(1_A) = s$. If $\eta(1_A) \leq \eta(t)$, then $\nu(p_1) \leq \nu(p_t)$. So $\nu(p_1) \to \nu(p_t) = 1_C$, a contradiction. Consequently, $\eta(t) < \eta(1_A) = s$, and so $\Box_s^+(\eta(a) \to_s \eta(b)) < s$. If $\eta(a) \to_s \eta(b) = s$, then $\Box_s^+(\eta(a) \to_s \eta(b)) = \Box_s^+(s) = \Box_s^+(\eta(1_A)) = \eta(\Box^+ 1_A) = \eta(1_A) = s$, a contradiction. Thus, $\eta(a) \to_s \eta(b) < s$, so $\eta(a) \not\leq \eta(b)$, and hence η is 1-1. ∎

As an immediate consequence of Theorems 3.4, 5.2, and Corollary 4.6, we obtain:

Corollary 5.3. *Let A be a finite subdirectly irreducible* **K4***-algebra, $D \subseteq A$, and $\mathfrak{D} = \bigcup\{\mathfrak{D}_{\varphi(a)} : a \in D\}$ be the set of quasi-antichains in A_* associated with D. Then for each transitive space X, we have $X \not\models \alpha(A, D)$ iff there exist a closed upset Y of X and an onto cofinal partial continuous p-morphism $f : Y \to A_*$ such that f satisfies (CDC) for $\mathfrak{D}$.*

Remark 5.4. For an intuitionistic version of Theorem 5.2 see [1, Thm. 5.3]. Corollary 5.3 corresponds to [5, Thm. 9.39(i)]. Its intuitionistic analogues are [5, Thm. 9.40(i)] and [1, Cor. 5.5]. Also note that a transitive space X validates our canonical formulas iff X validates Zakharyachev's canonical formulas. Since the proof of this fact is the same as in the intuitionistic case, we refer the reader to [1, Rem. 5.6].

5.2. Axiomatization

We are ready to give a new algebraic proof of Zakharyaschev's theorem that every logic over **K4** is axiomatizable by canonical formulas. For this we first show that the refutability of a modal formula α in a **K4**-algebra B can be "coded" by means of finitely many pairs $(A_1, D_1), \dots, (A_m, D_m)$, where each A_i is a subdirectly irreducible **K4**-algebra and $D_i \subseteq A_i$.

THEOREM 5.5. *If* $\mathbf{K4} \not\vdash \alpha(p_1, \dots, p_n)$*, then there exist* $(A_1, D_1), \dots, (A_m, D_m)$ *such that each* A_i *is a finite subdirectly irreducible* **K4***-algebra,* $D_i \subseteq A_i$*, and for each* **K4***-algebra* B *we have* $B \not\models \alpha(p_1, \dots, p_n)$ *iff there exist* $i \leq m$*, a homomorphic image* C *of* B*, and a modal algebra homomorphism* η_i *from* A_i *into a cofinal relativization* C_u *of* C *such that* $\eta_i(\Diamond_i a) = \Diamond \eta_i(a)$ *for each* $a \in D_i$*.*

PROOF. Let F_n be the free n-generated **K4**-algebra and let $g_1, \dots, g_n$ be the generators of F_n. Since $\mathbf{K4} \not\vdash \alpha(p_1, \dots, p_n)$, we have $F_n \not\models \alpha(p_1, \dots, p_n)$. Therefore, $\alpha(g_1, \dots, g_n) \neq 1_{F_n}$. By [3, Main Lemma], there exist a cofinal $s \in F_n$ and a finite modal subalgebra B_s of $(F_n)_s$ such that $B_s \not\models \alpha(p_1, \dots, p_n)$. We briefly recall the construction of s. Let B_α be the Boolean subalgebra of F_n generated by the subpolynomials of $\alpha(g_1, \dots, g_n)$. Then B_α is finite. Let A_α denote the set of atoms of B_α. Let also $H_n = \Box^+(F_n)$. Then H_n is a Heyting algebra, where $\underset{H_n}{\longrightarrow}$ denotes the Heyting implication in H_n. Let H_α be the $(\wedge, \underset{H_n}{\longrightarrow})$-subalgebra of H_n generated by $\Box^+(B_\alpha)$. By Diego's Theorem, H_α is finite. Let

$$s = \bigvee_{a \in A_\alpha} \bigwedge_{h \in H_\alpha} \left(h_a \vee \Box_a^+ \neg_a h_a\right).$$

By [3, Lem. 5.3], s is cofinal. Finally, let B be the Boolean subalgebra of F_n generated by $B_\alpha \cup H_\alpha$, and let $B_s = \{b_s : b \in B\}$, where $b_s = s \wedge b$. Clearly B_s is finite. By [3, Rem. 5.8], B_s is a modal subalgebra of $(F_n)_s$ and $B_s \not\models \alpha(p_1, \dots, p_n)$.

Let $A_1, \dots, A_m$ be the subdirectly irreducible homomorphic images of B_s refuting $\alpha(p_1, \dots, p_n)$, and let $\theta_i : B_s \to B_i$ be the corresponding onto homomorphisms. Since each A_i refutes $\alpha(p_1, \dots, p_n)$, there exist $a_1, \dots, a_n \in A_i$ such that $\alpha(a_1, \dots, a_n) \neq 1_{A_i}$. Let A_i^α be the Boolean subalgebra of A_i generated by the subpolynomials of $\alpha(a_1, \dots, a_n)$. We set $D_i = \{\neg a \in A_i^\alpha : \Diamond_i a \in A_i^\alpha\}$.[1]

Given a **K4**-algebra B, we need to show that $B \not\models \alpha(p_1, \dots, p_n)$ iff there is $i \leq m$, a homomorphic image C of B, and a modal algebra homomorphism η_i from A_i into a cofinal relativization C_u of C such that $\eta_i(\Diamond_i d) = \Diamond \eta_i(d)$ for each $d \in D_i$.

[1] D_i could alternatively be defined as $\{a \in A_i^\alpha : \Box_i a \in A_i^\alpha\}$.

First suppose there exist $i \le m$, a homomorphic image C of B, and a modal algebra homomorphism η_i from A_i into a cofinal relativization C_u of C such that $\eta_i(\Diamond_i d) = \Diamond \eta_i(d)$ for each $d \in D_i$. Since $\eta_i : A_i \to C_u$ is a 1-1 modal algebra homomorphism, the formula $\alpha(p_1, \dots, p_n)$ is refuted on C_u. We show that $\alpha(p_1, \dots, p_n)$ is also refuted on C.

LEMMA 5.6. *Suppose that B is a* **K4***-algebra, $u \in B$, and $\Diamond^+ u = 1$. Let B_u be the relativization of B to u. Let also A be a* **K4***-algebra such that $\alpha(a_1, \dots, a_n) \neq 1_A$ for some $a_1, \dots, a_n \in A$. We let A_α be the Boolean subalgebra of A generated by the subpolynomials of $\alpha(a_1, \dots, a_n)$, and $D = \{\neg a \in A_\alpha : \Diamond a \in A_\alpha\}$. If there is a 1-1 modal algebra homomorphism η from A into B_u satisfying $\Diamond \eta(d) = \eta(\Diamond d)$ for each $d \in D$, then $\alpha(\eta(a_1), \dots, \eta(a_n)) \neq 1_B$.*

PROOF. Since $\alpha(a_1, \dots, a_n) \neq 1_A$ and there is a 1-1 modal algebra homomorphism $\eta : A \to B_u$, we have that $\alpha_{B_u}(\eta(a_1), \dots, \eta(a_n)) \neq u$. As $\Diamond \eta(d) = \eta(\Diamond d)$ for each $d \in D$, by Corollary 4.6, $\eta_* : B_* \to A_*$ satisfies (CDC) for $\mathfrak{D} = \bigcup\{\mathfrak{D}_{\varphi(d)} : d \in D\}$. Therefore, $x \notin \mathrm{dom}(\eta_*)$ implies $\eta_* R(x) \cap \varphi(d) = \emptyset$ for each $d \in D$. As $\mathrm{dom}(\eta_*) = \varphi(u)$ and $\eta_* R(x) \cap \varphi(d) = \emptyset$ iff $R(x) \cap \varphi(u) \cap \eta_*^{-1}\varphi(d) = \emptyset$ iff $R(x) \cap \varphi(u) \cap \varphi(\eta(d)) = \emptyset$, we obtain:

$$\begin{array}{ll}
\eta_* : B_* \to A_* \text{ satisfies (CDC) for } \mathfrak{D} = \bigcup\{\mathfrak{D}_{\varphi(d)} : d \in D\} & \text{iff} \\
R(x) \cap \varphi(u) \cap \varphi(\eta(d)) = \emptyset \text{ for each } d \in D \text{ and } x \notin \varphi(u) & \text{iff} \\
R(x) \cap \varphi(u) \subseteq \varphi(u) - \varphi(\eta(d)) \text{ for each } d \in D \text{ and } x \notin \varphi(u) & \text{iff} \\
R(x) \cap \varphi(u) \subseteq \varphi(\eta(1)) - \varphi(\eta(d)) \text{ for each } d \in D \text{ and } x \notin \varphi(u) & \text{iff} \\
R(x) \cap \varphi(u) \subseteq \varphi(\neg_{\eta(1)} \eta(d)) \text{ for each } d \in D \text{ and } x \notin \varphi(u) & \text{iff} \\
R(x) \cap \varphi(u) \subseteq \varphi(\eta(\neg d)) \text{ for each } d \in D \text{ and } x \notin \varphi(u). &
\end{array}$$

That $\alpha(\eta(a_1), \dots, \eta(a_n)) \neq 1_B$ now follows from the following claim.

CLAIM 5.7. *Let B_u^α be the Boolean subalgebra of B_u generated by the subpolynomials of $\alpha_{B_u}(\eta(a_1), \dots, \eta(a_n))$. If $R(x) \cap \varphi(u) \subseteq \varphi(\neg d)$ for each $d \in D$ and $x \notin \varphi(u)$, then*

$$u \wedge \alpha(b_1, \dots, b_n) = \alpha_{B_u}(b_1, \dots, b_n)$$

for each $b_1, \dots, b_n \in B_u^\alpha$. Consequently, if there exist $b_1, \dots, b_n \in B_u^\alpha$ such that $\alpha_{B_u}(b_1, \dots, b_n) \neq u$, then $\alpha(b_1, \dots, b_n) \neq 1_B$.

PROOF. Induction on the complexity of $\alpha(b_1, \dots, b_n)$.

If $\alpha(b_1, \dots, b_n) = 1$, then

$$u \wedge \alpha(b_1, \dots, b_n) = u \wedge 1 = 1_u = \alpha_{B_u}(b_1, \dots, b_n).$$

The case when $\alpha(b_1, \ldots, b_n) = 0$ is proved similarly.

If $\alpha(b_1, \ldots, b_n) = b_i$, then

$$u \wedge \alpha(b_1, \ldots, b_n) = u \wedge b_i = b_i = \alpha_{B_u}(b_1, \ldots, b_n).$$

If $\alpha(b_1, \ldots, b_n) = \beta \vee \gamma$, then

$$u \wedge \alpha(b_1, \ldots, b_n) = u \wedge (\beta \vee \gamma) = (u \wedge \beta) \vee (u \wedge \gamma) = \beta_u \vee \gamma_u = \alpha_{B_u}(b_1, \ldots, b_n).$$

If $\alpha(b_1, \ldots, b_n) = \neg\beta$, then

$u \wedge \alpha(b_1, \ldots, b_n) = u \wedge \neg\beta = u \wedge (\neg u \vee \neg\beta) = u \wedge \neg(u \wedge \beta) = \neg_u \beta_u = \alpha_{B_u}(b_1, \ldots, b_n)$.

Lastly, let $\alpha(b_1, \ldots, b_n) = \Diamond\beta$. Then

$$u \wedge \alpha(b_1, \ldots, b_n) = u \wedge \Diamond\beta$$

and

$$\alpha_{B_u}(b_1, \ldots, b_n) = \Diamond_u \beta_u = u \wedge \Diamond(u \wedge \beta).$$

We show that $u \wedge \Diamond\beta = u \wedge \Diamond(u \wedge \beta)$. It is obvious that $u \wedge \Diamond(u \wedge \beta) \leq u \wedge \Diamond\beta$. Conversely, let $x \in \varphi(u \wedge \Diamond\beta)$. Then $x \in \varphi(u)$ and $R(x) \cap \varphi(\beta) \neq \emptyset$. So there exists $y \in B_*$ such that xRy and $y \in \varphi(\beta)$. If $y \in \varphi(u)$, then $x \in \varphi(u \wedge \Diamond(u \wedge \beta))$. If $y \notin \varphi(u)$, then as $\Diamond^+ u = 1_B$, there exists $z \in \varphi(u)$ such that yRz. As R is transitive, xRz. Since $\Diamond\beta \in B_u^\alpha$, we have $\neg\beta \in D$. Therefore, $R(y) \cap \varphi(u) \subseteq \varphi(\beta)$. Thus, $z \in \varphi(\beta)$, and so $x \in \varphi(u \wedge \Diamond(u \wedge \beta))$. This implies that $u \wedge \Diamond\beta \leq u \wedge \Diamond(u \wedge \beta)$. Consequently, $u \wedge \Diamond\beta = u \wedge \Diamond(u \wedge \beta)$, and hence by induction we can conclude that $u \wedge \alpha(b_1, \ldots, b_n) = \alpha_{B_u}(b_1, \ldots, b_n)$.

Finally, if $\alpha_{B_u}(b_1, \ldots, b_n) \neq u$, then as $u \wedge \alpha(b_1, \ldots, b_n) = \alpha_{B_u}(b_1, \ldots, b_n) \neq u$, we obtain that $\alpha(b_1, \ldots, b_n) \neq 1_B$. ■

This concludes the proof of Lemma 5.6. ■

Lemma 5.6 yields that $\alpha(p_1, \ldots, p_n)$ is refuted on C. Since C is a homomorphic image of B, it follows that $\alpha(p_1, \ldots, p_n)$ is also refuted on B.

Conversely, suppose that $B \not\models \alpha(p_1, \ldots, p_n)$. Then there exist $a_1, \ldots, a_n \in B$ such that $\alpha(a_1, \ldots, a_n) \neq 1_B$. Let S_n be the subalgebra of B generated by $a_1, \ldots, a_n$. Then S_n is an n-generated **K4**-algebra, and so S_n is a homomorphic image of F_n. Let $\theta : F_n \to S_n$ be the onto homomorphism and let S_α be the Boolean subalgebra of S_n generated by the subpolynomials of $\alpha(a_1, \ldots, a_n)$. We construct a cofinal u and B_u in S_n exactly the same way we constructed s and B_s in F_n. We also let $D = \{\neg a \in S_\alpha : \Diamond a \in S_\alpha\}$. Clearly $\theta(s) = u$. Also, by [3, Lem. 5.7], $\Diamond_u b_u = u \wedge \Diamond b$ for each $b \in S_\alpha$.

Let $k : B_u \to (S_n)_u$, $l : (S_n)_u \to S_n$, and $m : S_n \to B$ be the corresponding embeddings. Then k and m are modal algebra homomorphisms, while l is a relativized modal algebra homomorphism. Moreover, the embedding $m \circ l \circ k : B_u \to B$ satisfies $\Diamond mlk(a) = mlk(\Diamond_u a)$ for each $a \in S_\alpha$.

Since $\theta : F_n \to S_n$ is an onto homomorphism and $\theta(s) = u$, the restriction of θ to B_s is a homomorphism from B_s onto B_u. As $B_u \not\models \alpha(p_1, \ldots, p_n)$, there is a subdirectly irreducible homomorphic image of B_u refuting $\alpha(p_1, \ldots, p_n)$. Since each homomorphic image of B_u is also a homomorphic image of B_s, we obtain that the subdirectly irreducible homomorphic image of B_u refuting $\alpha(p_1, \ldots, p_n)$ is A_i for some $i \leq m$. Let $\theta_i : B_u \to A_i$ be the onto homomorphism. Then, by Lemma 2.1, there exists a **K4**-algebra T, an onto homomorphism $\zeta : (S_n)_u \to T$, and a 1-1 homomorphism $n : A_i \to T$ such that $\zeta \circ k = n \circ \theta_i$. By Lemma 4.14, there exists a **K4**-algebra E and an onto homomorphism $\xi : S_n \to E$ such that T is isomorphic to the relativization of E to $\xi(u)$. Moreover, as u is cofinal in S_n, we also have that $\xi(u)$ is cofinal in E. Let $p : T \to E$ be the corresponding relativized modal algebra homomorphism from T into E. Then $\xi \circ l = p \circ \zeta$. Applying Lemma 2.1 again, we obtain a **K4**-algebra C, an onto homomorphism $\eta : A \to C$, and a 1-1 homomorphism $q : E \to C$ such that $\eta \circ m = q \circ \xi$. Therefore, we arrive at the following commutative diagram.

$$\begin{tikzcd}
B_u \arrow[r, tail, "k"] \arrow[d, two heads, "\theta_i"] & (S_n)_u \arrow[r, tail, "l"] \arrow[d, two heads, "\zeta"] & S_n \arrow[r, tail, "m"] \arrow[d, two heads, "\xi"] & B \arrow[d, two heads, "\eta"] \\
A_i \arrow[r, tail, "n"] & T \arrow[r, tail, "p"] & E \arrow[r, tail, "q"] & C
\end{tikzcd}$$

Let $\eta_i = q \circ p \circ n$ and let $(A_i)_\alpha$ be the Boolean subalgebra of A_i generated by the subpolynomials of $\alpha(\theta_i(a_1), \ldots, \theta_i(a_n))$. Then $(A_i)_\alpha = \theta_i[S_\alpha]$. Let $a \in (A_i)_\alpha$. Then there exists $b \in S_\alpha$ such that $a = \theta_i(b)$. As the diagram commutes and $\Diamond_B mlk(b) = mlk(\Diamond_u b)$ for each $b \in S_\alpha$, we have $\eta_i(\Diamond_i a) = \eta_i(\Diamond_i \theta_i(b)) = \eta_i\theta_i(\Diamond_u b) = \eta mlk(\Diamond_u b) = \eta \Diamond_B mlk(b) = \Diamond_C \eta mlk(b) = \Diamond_C \eta_i \theta_i(b) = \Diamond_C \eta_i(a)$. In particular, $\eta_i(\Diamond_i d) = \Diamond_C \eta_i(d)$ for each $d \in D$. Thus, we have found $i \leq m$, a homomorphic image C of B, and a relativized modal algebra homomorphism η_i from A_i into a cofinal relativization $C_{\eta(u)}$ of C such that $\eta_i(\Diamond_i d) = \Diamond \eta_i(d)$ for each $d \in D_i$. ■

As an immediate consequence of Theorems 3.4, 5.5, and Corollary 4.6, we obtain:

Corollary 5.8. *If* **K4** $\not\vdash \alpha(p_1, \ldots, p_n)$, *then there exist* $(A_1, D_1), \ldots, (A_m, D_m)$ *such that each* A_i *is a finite subdirectly irreducible* **K4**-*algebra,* $D_i \subseteq A_i$, *and for each transitive space X, we have $X \not\models \alpha(p_1, \dots, p_n)$ iff there exist $i \le m$, a closed upset Y of X, and a cofinal partial continuous p-morphism f_i from Y onto $(A_i)_*$ satisfying (CDC) for $\mathfrak{D}_i = \bigcup\{\mathfrak{D}_{\varphi(a)} : a \in D_i\}$.*

Corollary 5.9. *If* $\mathbf{K4} \not\vdash \alpha(p_1, \dots, p_n)$*, then there exist* $(A_1, D_1), \dots, (A_m, D_m)$ *such that each* A_i *is a finite subdirectly irreducible* $\mathbf{K4}$*-algebra,* $D_i \subseteq A_i$*, and for each* $\mathbf{K4}$*-algebra* B*, we have:*

$$B \models \alpha(p_1, \dots, p_n) \text{ iff } B \models \bigwedge_{i=1}^{m} \alpha(A_i, D_i).$$

Proof. Suppose that $\mathbf{K4} \not\vdash \alpha(p_1, \dots, p_n)$. Then, by Theorem 5.5, there exist $(A_1, D_1), \dots, (A_m, D_m)$ such that each A_i is a finite subdirectly irreducible $\mathbf{K4}$-algebra, $D_i \subseteq A_i$, and for each $\mathbf{K4}$-algebra B, we have $B \not\models \alpha(p_1, \dots, p_n)$ iff there exist $i \le m$, a homomorphic image C of B, and a modal algebra homomorphism η_i from A_i into a cofinal relativization C_u of C such that $\eta_i(\Diamond_i a) = \Diamond \eta_i(a)$ for each $a \in D_i$. The result now follows from Theorem 5.2. ■

Corollary 5.10. *If* $\mathbf{K4} \not\vdash \alpha(p_1, \dots, p_n)$*, then there exist* $(A_1, D_1), \dots, (A_m, D_m)$ *such that each* A_i *is a finite subdirectly irreducible* $\mathbf{K4}$*-algebra,* $D_i \subseteq A_i$*, and for each transitive space* X*, we have:*

$$X \models \alpha(p_1, \dots, p_n) \text{ iff } X \models \bigwedge_{i=1}^{m} \alpha(A_i, D_i).$$

Remark 5.11. For an intuitionistic version of Theorem 5.5 see [1, Thm. 5.7]; Lemma 5.6 corresponds to [5, Thm. 9.30]; Corollary 5.8 corresponds to [5, Thms. 9.34 and 9.36(i)]; for its intuitionistic version see [1, Cor. 5.5]; Corollary 5.9 corresponds to [5, Thm 9.43(i)]; for intuitionistic versions of Corollaries 5.9 and 5.10 see [1, Cor. 5.10 and 5.11].

As a consequence of Corollary 5.9, we obtain that every logic over $\mathbf{K4}$ is axiomatizable by canonical formulas.

Corollary 5.12 (Zakharyaschev's theorem). *Each logic* L *over* $\mathbf{K4}$ *is axiomatizable by canonical formulas. Moreover, if* L *is finitely axiomatizable, then* L *is axiomatizable by finitely many canonical formulas.*

Proof. Let L be a logic over $\mathbf{K4}$. Then L is obtained by adding $\{\alpha_i : i \in I\}$ to $\mathbf{K4}$ as new axioms. Therefore, $\mathbf{K4} \not\vdash \alpha_i$ for each $i \in I$. By Corollary 5.9, for each $i \in I$, there exist $(A_{i1}, D_{i1}), \dots, (A_{im_i}, D_{im_i})$ such that A_{ij} is a finite subdirectly irreducible **K4**-algebra, $D_{ij} \subseteq A_{ij}$, and for each **K4**-algebra B, we have $B \models \alpha_i$ iff $B \models \bigwedge_{j=1}^{m_i} \alpha(A_{ij}, D_{ij})$. Thus, $B \models L$ iff $B \models \{\alpha_i : i \in I\}$, which happens iff $B \models \{\bigwedge_{j=1}^{m_i} \alpha(A_{ij}, D_{ij}) : i \in I\}$. Consequently, $L = \mathbf{K4} + \{\bigwedge_{j=1}^{m_i} \alpha(A_{ij}, D_{ij}) : i \in I\}$, and so L is axiomatizable by canonical formulas. In particular, if L is finitely axiomatizable, then L is axiomatizable by finitely many canonical formulas. ∎

6. Negation-free canonical formulas, Jankov-Rautenberg, subframe, and cofinal subframe formulas for K4

In this section we consider negation-free canonical formulas. We show that all the results of the previous section hold for negation-free formulas if we remove the word "cofinal" in all the statements. We also show that Jankov-Rautenberg, subframe, and cofinal subframe formulas are particular cases of canonical formulas. This leads to a new axiomatization of subframe and cofinal subframe logics over **K4** with "algebra-based" formulas, as opposed to frame-based formulas.

6.1. Negation-free canonical formulas for K4

Suppose that A is a finite subdirectly irreducible **K4**-algebra, $H = \Box^+(A)$, t is the second largest element of H, and $D \subseteq A$. For each $a \in A$, we introduce a new variable p_a and define the *negation-free canonical formula* $\beta(A, D)$ associated with A and D as

$$\begin{aligned}\beta(A, D) = \Box^+[\ & (\bot \leftrightarrow p_0) \wedge \\ & \bigwedge \{p_{a \vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \wedge \\ & \bigwedge \{p_{a \wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \wedge \\ & \bigwedge \{p_{\Diamond a} \leftrightarrow \Diamond_{p_1} p_a : a \in A\} \wedge \\ & \bigwedge \{\Diamond p_a \leftrightarrow p_{\Diamond a} : a \in D\}] \to (p_1 \to p_t).\end{aligned}$$

Thus, $\beta(A, D)$ is obtained from $\alpha(A, D)$ by deleting the conjunct $\top \leftrightarrow \Diamond^+ p_1$.

THEOREM 6.1. *Let A be a finite subdirectly irreducible* **K4**-*algebra, $D \subseteq A$, and B be a* **K4**-*algebra. Then $B \not\models \beta(A, D)$ iff there exist a homomorphic image C of B and a relativized modal algebra homomorphism η from A into a relativization C_s of C satisfying $\eta(\Diamond a) = \Diamond \eta(a)$ for each $a \in D$.*

PROOF. The proof is a simplified version of the proof of Theorem 5.2. ∎

As an immediate consequence, we obtain:

COROLLARY 6.2. *Let A be a finite subdirectly irreducible* $\mathbf{K4}$*-algebra, $D \subseteq A$, and $\mathfrak{D} = \bigcup\{\mathfrak{D}_{\varphi(a)} : a \in D\}$ be the set of quasi-antichains in A_* associated with D. Then for each transitive space X, we have $X \not\models \beta(A, D)$ iff there exist a closed upset Y of X and an onto partial continuous p-morphism $f : Y \to A_*$ such that f satisfies (CDC) for $\mathfrak{D}$.*

We recall [5] that a modal formula α is *negation-free* if α is built from propositional variables and constants by means of $\wedge$, $\vee$, and $\Diamond$. The next theorem is an analogue of Theorem 5.5 for negation-free canonical formulas.

THEOREM 6.3. *If* $\mathbf{K4} \not\vdash \alpha(p_1, \dots, p_n)$*, where $\alpha(p_1, \dots, p_n)$ is negation-free, then there exist $(A_1, D_1), \dots, (A_m, D_m)$ such that each A_i is a finite subdirectly irreducible* $\mathbf{K4}$*-algebra, $D_i \subseteq A_i$, and for each* $\mathbf{K4}$*-algebra B, we have $B \not\models \alpha(p_1, \dots, p_n)$ iff there exist $i \leq m$, a homomorphic image C of B, and a modal algebra homomorphism η from A_i into a relativization C_s of C.*

PROOF. The proof is virtually the same as the proof of Theorem 5.5 with the only exception that we need to prove a version of Lemma 5.6 for negation-free canonical formulas. This we do in the next lemma.

LEMMA 6.4. *Let $\alpha(p_1, \dots, p_n)$ be a negation-free formula, A, B be* $\mathbf{K4}$*-algebras, and $s \in B$. Suppose there exist $a_1, \dots, a_n \in A$ such that $\alpha(a_1, \dots, a_n) \neq 1_A$. Let also A_α denote the Boolean subalgebra of A generated by the subpolynmials of $\alpha(a_1, \dots, a_n)$, and let $D = \{\neg a \in A_\alpha : \Diamond a \in A_\alpha\}$. If there exists a 1-1 relativized modal algebra homomorphism η from A into B_s satisfying $\eta(\Diamond d) = \Diamond \eta(d)$ for each $d \in D$, then $\alpha(\eta(a_1), \dots, \eta(a_n)) \neq 1_B$.*

PROOF. Since $\alpha(a_1, \dots, a_n) \neq 1_A$ and there is a 1-1 modal algebra homomorphism $\eta : A \to B_s$, we have that $\alpha_{B_s}(\eta(a_1), \dots, \eta(a_n) \neq s$. As $\eta(\Diamond d) = \Diamond \eta(d)$ for each $d \in D$, we have that $\eta_* : B_* \to A_*$ satisfies (CDC) for $\mathfrak{D} = \bigcup\{\mathfrak{D}_{\varphi(d)} : d \in D\}$. But $\eta_* : B_* \to A_*$ satisfies (CDC) for $\mathfrak{D}$ iff $R(y) \cap \varphi(u) \subseteq \varphi(\eta(a))$ for each $\neg a \in D$ (see the proof of Lemma 5.6). Therefore, the result follows from the following claim.

CLAIM 6.5. *Let B_s^α be the Boolean subalgebra of B_s generated by the subpolynomials of $\alpha_{B_s}(\eta(a_1), \dots, \eta(a_n))$. If $R(x) \cap \varphi(s) \subseteq \varphi(\neg d)$ for each $d \in D$ and $x \notin \varphi(s)$, then*

$$s \wedge \alpha(b_1, \dots, b_n) = \alpha_{B_s}(b_1, \dots, b_n)$$

for each $b_1, \dots, b_n \in B_s^\alpha$. Consequently, if there exist $b_1, \dots, b_n \in B_s^\alpha$ such that $\alpha_{B_s}(b_1, \dots, b_n) \neq s$, then $\alpha(b_1, \dots, b_n) \neq 1_B$.

PROOF. We prove the claim by induction on the complexity of $\alpha(b_1, \ldots, b_n)$. The cases $\alpha(b_1, \ldots, b_n) = 1$, $\alpha(b_1, \ldots, b_n) = 0$, $\alpha(b_1, \ldots, b_n) = b_i$, and $\alpha(b_1, \ldots, b_n) = \beta \vee \gamma$ are proved as in Claim 5.7. The case $\alpha(b_1, \ldots, b_n) = \beta \wedge \gamma$ is proved similarly.

Let $\alpha(b_1, \ldots, b_n) = \Diamond\beta$. It is sufficient to prove that $s \wedge \Diamond\beta \leq s \wedge \Diamond(s \wedge \beta)$. Let $x \in \varphi(s \wedge \Diamond\beta)$. Then $x \in \varphi(s)$ and there exists $y \in B_*$ such that xRy and $y \in \varphi(\beta)$. If $y \in \varphi(s)$, then we are done. Suppose that $y \notin \varphi(s)$. If $R(y) \cap \varphi(s) \neq \emptyset$, we proceed as in the proof of Claim 5.7. On the other hand, since $\alpha(p_1, \ldots, p_n)$ is negation-free, an easy induction on the complexity of subpolynomials γ of $\alpha(b_1, \ldots, b_n)$ shows that for each $z \notin \varphi(s)$ with $R(z) \cap \varphi(s) = \emptyset$, we have $z \notin \varphi(\gamma)$. Therefore, if $R(y) \cap \varphi(s) = \emptyset$, as β is a subpolynomial of $\alpha(b_1, \ldots, b_n)$, we obtain $y \notin \varphi(\beta)$, which is a contradiction. Thus, $s \wedge \Diamond\beta = s \wedge \Diamond(s \wedge \beta)$, and so by induction we can conclude that $s \wedge \alpha(b_1, \ldots, b_n) = \alpha_{B_s}(b_1, \ldots, b_n)$. Finally, if $\alpha_{B_s}(b_1, \ldots, b_n) \neq s$, then as $s \wedge \alpha(b_1, \ldots, b_n) = \alpha_{B_s}(b_1, \ldots, b_n) \neq s$, we obtain that $\alpha(b_1, \ldots, b_n) \neq 1_B$. ■

Thus, Lemma 6.4 is proved. ■

Consequently, Theorem 6.3 is also proved. ■

Theorem 6.3 has a number of useful corollaries. The proofs are similar to the ones given in the previous section, and we skip them.

COROLLARY 6.6. *If* $\mathbf{K4} \not\vdash \alpha(p_1, \ldots, p_n)$, *where* $\alpha(p_1, \ldots, p_n)$ *is negation-free, then there exist* $(A_1, D_1), \ldots, (A_m, D_m)$ *such that each* A_i *is a finite subdirectly irreducible* $\mathbf{K4}$*-algebra,* $D_i \subseteq A_i$*, and for each transitive space* X*, we have* $X \not\models \alpha(p_1, \ldots, p_n)$ *iff there exist* $i \leq m$*, a closed upset* Y *of* X*, and a partial continuous p-morphism* f_i *from* Y *onto* $(A_i)_*$ *satisfying* (CDC) *for* $\mathfrak{D}_i = \bigcup\{\mathfrak{D}_{\varphi(a)} : a \in D_i\}$.

COROLLARY 6.7. *If* $\mathbf{K4} \not\vdash \alpha(p_1, \ldots, p_n)$, *where* $\alpha(p_1, \ldots, p_n)$ *is negation-free, then there exist* $(A_1, D_1), \ldots, (A_m, D_m)$ *such that each* A_i *is a finite subdirectly irreducible* $\mathbf{K4}$*-algebra,* $D_i \subseteq A_i$*, and for each* $\mathbf{K4}$*-algebra* B*, we have:*

$$B \models \alpha(p_1, \ldots, p_n) \text{ iff } B \models \bigwedge_{i=1}^{m} \beta(A_i, D_i).$$

COROLLARY 6.8. *If* $\mathbf{K4} \not\vdash \alpha(p_1, \ldots, p_n)$, *where* $\alpha(p_1, \ldots, p_n)$ *is negation-free, then there exist* $(A_1, D_1), \ldots, (A_m, D_m)$ *such that each* A_i *is a finite subdirectly irreducible* $\mathbf{K4}$*-algebra,* $D_i \subseteq A_i$*, and for each transitive space*

X, we have:

$$X \models \alpha(p_1, \ldots, p_n) \text{ iff } X \models \bigwedge_{i=1}^{m} \alpha(A_i, D_i).$$

Corollary 6.9. *Each logic L over* **K4** *axiomatizable by negation-free formulas is axiomatizable by negation-free canonical formulas. Moreover, if L is axiomatizable by finitely many negation-free formulas, then L is axiomatizable by finitely many negation-free canonical formulas.*

Remark 6.10. Corollary 6.2 corresponds to [5, Thm. 9.39(ii)]; Lemma 6.4 corresponds to [5, Thm. 9.31]; Corollary 6.6 corresponds to [5, Thms. 9.34 and 9.36(ii)]; and Corollary 6.8 corresponds to [5, Thm. 9.43(ii)]. For an algebraic treatment of negation-free canonical formulas in the setting of intuitionistic logic see [1, Sec. 5.2].

6.2. Jankov-Rautenberg formulas for K4

Next we show that the Jankov-Rautenberg formulas are a particular case of our canonical formulas. Let A be a finite subdirectly irreducible **K4**-algebra, $H = \Box^+(A)$, and t be the second largest element of H. We recall that the Jankov-Rautenberg formula of A is

$$\begin{aligned}\chi(A) = \Box^+[&\bigwedge\{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\}\wedge \\ &\bigwedge\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\}\wedge \\ &\bigwedge\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\}\wedge \\ &\bigwedge\{p_{\Diamond a} \leftrightarrow \Diamond p_a : a \in A\}] \to p_t.\end{aligned}$$

It is well known that a **K4**-algebra B refutes $\chi(A)$ iff A is a subalgebra of a homomorphic image of B. We show that $\alpha(A, A)$ is equivalent to $\chi(A)$. Let

$$\begin{aligned}\chi'(A) = \Box^+[\ &(\top \leftrightarrow p_1) \wedge (\bot \leftrightarrow p_0)\wedge \\ &\bigwedge\{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\}\wedge \\ &\bigwedge\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\}\wedge \\ &\bigwedge\{p_{\Diamond a} \leftrightarrow \Diamond p_a : a \in A\}] \to p_t.\end{aligned}$$

Lemma 6.11. *Let A be a finite subdirectly irreducible* **K4***-algebra and let B be a* **K4***-algebra. The following three conditions are equivalent:*

1. $B \models \chi(A)$,

2. $B \models \chi'(A)$,

3. $B \models \alpha(A, A)$.

Proof. It is easy to see that (1) is equivalent to (2) as any lattice homomorphism between Boolean algebras is a Boolean algebra homomorphism iff it preserves 0 and 1.

(1)⇒(3): Suppose that $B \not\models \alpha(A, A)$. Then by Theorem 5.2, there exist a homomorphic image C of B and a 1-1 modal algebra homomorphism η from A into a cofinal relativization C_s of C such that $\eta(\Diamond a) = \Diamond\eta(a)$ for each $a \in A$. Then $1_C = \Diamond^+\eta(1_A) = \eta(1_A) \vee \Diamond\eta(1_A) = \eta(1_A) \vee \eta(\Diamond 1_A) = \eta(1_A \vee \Diamond 1_A) = \eta(1_A)$. Therefore, η is a modal algebra homomorphism, and so $B \not\models \chi(A)$.

(3)⇒(1): This is straightforward as every modal algebra homomorphism is also a cofinal relativized modal algebra homomorphism. ■

As a direct consequence of Lemma 6.11, we obtain:

Corollary 6.12. *Let A be a finite subdirectly irreducible* **K4***-algebra.*

1. For each **K4***-algebra B, we have $B \not\models \alpha(A, A)$ iff A is a subalgebra of a homomorphic image of B.*

2. For each transitive space X, we have $X \not\models \alpha(A, A)$ iff there exists a closed upset Y of X and a continuous p-morphism from Y onto A_*.

6.3. Subframe and cofinal subframe formulas for K4

We conclude the paper by showing that the subframe and cofinal subframe formulas for **K4** can be obtained from our canonical formulas by taking $D = \emptyset$. This yields a new axiomatization of subframe and cofinal subframe logics over **K4** using "algebra-based" formulas.

Let A be a finite subdirectly irreducible **K4**-algebra, $H = \Box^+(A)$, and t be the second largest element of H. Let

$$\begin{aligned}\alpha_{cs}(A) = \Box^+[\ &(\top \leftrightarrow \Diamond^+ p_1) \wedge (\bot \leftrightarrow p_0) \wedge \\ &\bigwedge\{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \wedge \\ &\bigwedge\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \wedge \\ &\bigwedge\{p_{\Diamond a} \leftrightarrow \Diamond_{p_1} p_a : a \in A\}] \to (p_1 \to p_t).\end{aligned}$$

Note that $\alpha_{cs}(A) = \alpha(A, \emptyset)$.

Corollary 6.13. *Let A be a finite subdirectly irreducible* **K4**-*algebra.*

1. *For each* **K4**-*algebra B, we have $B \not\models \alpha_{cs}(A)$ iff there exist a homomorphic image C of B and a 1-1 cofinal relativized homomorphism from A into C.*
2. *For each transitive space X, we have $X \not\models \alpha_{cs}(A)$ iff there exist a closed upset Y of X and a cofinal partial continuous p-morphism from Y onto A_*.*

Proof. Apply Theorem 5.2 and Corollary 5.3. ■

Subframe formulas are obtained from cofinal subframe formulas by removing the conjunct $\top \leftrightarrow \Diamond^+ p_1$. Thus, the subframe formula of a finite subdirectly irreducible **K4**-algebra A is

$$\begin{aligned}\alpha_s(A) = \Box^+[\ &(\bot \leftrightarrow p_0)\wedge \\ &\bigwedge\{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\}\wedge \\ &\bigwedge\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\}\wedge \\ &\bigwedge\{p_{\Diamond a} \leftrightarrow \Diamond_{p_1} p_a : a \in A\}] \to (p_1 \to p_t).\end{aligned}$$

Note that $\alpha_s(A) = \beta(A, \emptyset)$.

Corollary 6.14. *Let A be a finite subdirectly irreducible* **K4**-*algebra.*

1. *For each* **K4**-*algebra B, we have $B \not\models \alpha_s(A)$ iff there exist a homomorphic image C of B and a 1-1 relativized homomorphism from A into C.*
2. *For each transitive space X, we have $X \not\models \alpha_{cs}(B)$ iff there exist a closed upset Y of X and a partial continuous p-morphism from Y onto A_*.*

Proof. Apply Theorem 6.1 and Corollary 6.2. ■

Remark 6.15. Frame-based versions of subframe and cofinal subframe formulas are due to Fine [9] and Zakharyaschev [17]. An algebraic approach to subframe and cofinal subframe logics is developed in [3]. For an algebraic treatment of subframe and cofinal subframe formulas in the intuitionistic setting see [2], [1, Sec. 5.4], and [4, Sec. 3.3.3].

Let X be a transitive space. We recall that $Y \subseteq X$ is a *subframe* of X if Y is a clopen subset of X. If in addition $R(Y) \subseteq (R^+)^{-1}(Y)$, then Y is a *cofinal subframe* of X. (Note that the notion of a subframe in the intuitionistic setting is different; see [5, Sec 9.1] and [2].)

Let L be a logic over **K4**. We recall that L is a *subframe logic* if for each transitive space X and a subframe Y of X, from $X \models L$ it follows that $Y \models L$. We also recall that L is a *cofinal subframe logic* if for each transitive space X and a cofinal subframe Y of X, from $X \models L$ it follows that $Y \models L$.

It was proved by Fine [9] that each subframe logic over **K4** is axiomatizable by subframe formulas, and by Zakharyaschev [17] that each cofinal subframe logic over **K4** is axiomatizable by cofinal subframe formulas. It follows that each subframe logic over **K4** is axiomatizable by the formulas of the form $\alpha_s(A)$, and that each cofinal subframe logic over **K4** is axiomatizable by the formulas of the form $\alpha_{sc}(A)$. This yields a new "algebra-based" axiomatization of subframe and cofinal subframe logics.

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