AN ALGEBRAIC APPROACH TO FILTRATIONS FOR SUPERINTUITIONISTIC LOGICS

GURAM BEZHANISHVILI AND NICK BEZHANISHVILI

Dedicated to Albert Visser on the occasion of his 65th birthday.[1]

Abstract. There are two standard model-theoretic methods for proving the finite model property for modal and superintuionistic logics, the standard filtration and the selective filtration. While the corresponding algebraic descriptions are better understood in modal logic, it is our aim to give similar algebraic descriptions of filtrations for superintuitionistic logics via locally finite reducts of Heyting algebras. We show that the algebraic description of the standard filtration is based on the $\to$-free reduct of Heyting algebras, while that of selective filtration on the $\vee$-free reduct.

1. Introduction

The main tools for establishing the finite model property for modal and superintuitionistic logics are the methods of standard and selective filtrations. If a model $\mathfrak{M}$ refutes a formula φ, then we wish to filter it out so that the resulting model $\mathfrak{N}$ is finite and still refutes φ. The model $\mathfrak{N}$ can be constructed as a factor-model of $\mathfrak{M}$ (standard filtration) or as a submodel of $\mathfrak{M}$ (selective filtration).

The standard filtration (or simply filtration) was originally developed algebraically [24, 25], and later model-theoretically [23, 26]. The model-theoretic approach became a standard tool for proving the finite model property in modal logic. We refer to [14, 12] for a systematic exposition of the method, and its numerous applications. The algebraic and model-theoretic methods are closely related. For modal logics this was first discussed in [21, 22]. For a modern account of the connection see [20, 15, 5].

The method of selective filtration in modal logic was first discussed in [19], and further developed in [18, 29]. In [28] the method was applied to superintuitionistic logics. See [14] for an overview. An algebraic analogue of this technique for superintuionistic logics was developed in [7], and for modal logics in [8] (although [7, 8] do not discuss explicitly the connection of their algebraic method to selective filtration).

In this short note we would like to revisit the methods of standard and selective filtrations for superintuionistic logics from an algebraic point of view. We show that in both cases one has to work with appropriate locally finite reducts of Heyting algebras. In the case of filtrations, the $\to$-free reduct, and in the case of selective filtrations, the $\vee$-free reduct.

In order to define standard filtrations algebraically we need to work with free Boolean extensions of distributive lattices. This enables us to define the least and greatest filtrations algebraically. We show, via duality for Heyting algebras, that algebraically described standard and selective filtrations exactly correspond to their model-theoretic analogues. In [2, 4] the $\vee$-free and $\to$-free reducts of Heyting algebras were used for introducing $(\wedge, \to)$ and $(\wedge, \vee)$-canonical formulas which

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[1]Albert Visser has been our senior colleague and friend for many years now. The second author has especially pleasant memories of the time when he shared an office with Albert (and Rosalie Iemhoff) during his postdoc at Utrecht University in 2012–2013. Albert would always give his help and support when needed, and cheer you up by telling a nice joke or anecdote, often his personal encounters with famous logicians. But above all, he has always been a limitless source of inspirational ideas about many different aspects of logic. We are happy to dedicate this short paper to Albert's 65th birthday and wish him many more productive years to come.

axiomatize all superintuionistic logics. The proofs essentially use the algebraic versions of standard and selective filtrations.

We assume the reader's familiarity with the algebraic and relational semantics of intuitionistic logic. For basic definitions and facts we refer to [14, 9].

2. Standard filtration

A *Kripke frame* is a partially ordered set $\mathfrak{F} = (W, \leq)$. We call $U \subseteq W$ an *up-set* (upward closed set) if $w \in U$ and $w \leq v$ imply $v \in U$. A *valuation* ν on $\mathfrak{F}$ assigns to each propositional letter p an up-set $\nu(p)$ of $\mathfrak{F}$. A *Kripke model* is a pair $\mathfrak{M} = (\mathfrak{F}, \nu)$, where $\mathfrak{F}$ is a Kripke frame and ν is a valuation on $\mathfrak{F}$. We recall the definition of filtration (see, e.g., [14, Sec. 5.3]).

Definition 2.1 (Standard filtration model theoretically)**.** *Let Σ be a finite set of formulas closed under subformulas, and let $\mathfrak{M} = (\mathfrak{F}, \nu)$ be a model. Define an equivalence relation $\sim$ on W by*

$$w \sim v \text{ if } (\forall \varphi \in \Sigma)(w \in \nu(\varphi) \text{ iff } v \in \nu(\varphi)). \tag{1}$$

Let $W' = W/{\sim}$. Since Σ is finite, so is W'. In fact, $|W'| \leq 2^{|\Sigma|}$. Let $\leq'$ be a partial order on W' satisfying the following two conditions for all $w, v \in W$ and $\varphi \in \Sigma$:

$$w \leq v \quad \text{implies} \quad [w] \leq' [v], \tag{2}$$

$$[w] \leq' [v] \quad \text{and} \quad w \in \nu(\varphi) \quad \text{imply} \quad v \in \nu(\varphi). \tag{3}$$

Let ν' be a valuation on $\mathfrak{F}'$ such that

$$\nu'(p) = \{[w] : w \in \nu(p)\} \tag{4}$$

for each $p \in \Sigma$. Then $\mathfrak{M}' = (\mathfrak{F}', \nu')$ is a finite model called a standard filtration (*or simply a* filtration) *of the model $\mathfrak{M}$ through Σ.*

Remark 2.2. It is a consequence of (3) that $\{[w] : w \in \nu(\varphi)\}$ is an up-set of $\mathfrak{F}'$ for each $\varphi \in \Sigma$. Therefore, there always exists a valuation ν' on $\mathfrak{F}'$ satisfying (4). Thus, $\mathfrak{M}'$ is well defined.

The next lemma is well known (see, e.g., [14, Thm. 5.23]).

Lemma 2.3 (Filtration Lemma)**.** *Let $\mathfrak{M} = (\mathfrak{F}, \nu)$ be a model and let $\mathfrak{M}' = (\mathfrak{F}', \nu')$ be a filtration of $\mathfrak{M}$ through a finite set Σ of formulas closed under subformulas. Then for each $\varphi \in \Sigma$ and $w \in W$,*

$$w \in \nu(\varphi) \quad \text{iff} \quad [w] \in \nu'(\varphi). \tag{5}$$

Consequently, if $\varphi \in \Sigma$ is refuted on $\mathfrak{M}$, then it is refuted on $\mathfrak{M}'$ (see, e.g., [14, Cor. 5.25]).

Remark 2.4. As follows from Remark 2.2, Condition (3) is crucial for $\mathfrak{M}'$ to be well defined. On the other hand, Condition (2) is used in proving Condition (5). But Condition (2) itself is not necessary for refuting $\varphi \in \Sigma$ on $\mathfrak{M}'$ when it is refuted on $\mathfrak{M}$. For this purpose it is sufficient to have Condition (5). In some situations it is even disadvantageous to assume Condition (2); for example, when proving the finite model property for modal logics **GL** and **Grz** via standard filtration [13, 11]. However, it is common to require (2), so we will assume (2) throughout. This condition plays an important role in the study of stable logics [4].

As follows from [14, Sec. 5.3], among the filtrations of $\mathfrak{M}$ through Σ, there always exist the least and greatest filtrations. In other words, among the partial orders on W' that satisfy (2) and (3), there always exist the least and greatest ones. The least filtration is defined as follows. Let

$$[w] \preceq [v] \text{ iff there exist } w' \sim w \text{ and } v' \sim v \text{ such that } w' \leq v', \tag{6}$$

and let $\leq^l$ be the transitive closure of $\preceq$. On the other hand, the greatest filtration is given by

$$[w] \leq^g [v] \text{ iff } (\forall \varphi \in \Sigma)(w \in \nu(\varphi) \Rightarrow v \in \nu(\varphi)). \tag{7}$$

Next we give an algebraic description of filtrations. For a Heyting algebra A, let $\mathfrak{F}_A = (W_A, \subseteq)$ be the frame of prime filters of A ordered by inclusion. We call $\mathfrak{F}_A$ the *spectrum* of A. It is well known that A embeds into the Heyting algebra $\mathrm{Up}(\mathfrak{F}_A)$ of up-sets of $\mathfrak{F}_A$ by $\alpha(a) = \{w \in W_A : a \in w\}$. Each valuation ν on A gives rise to a valuation $\mu = \alpha \circ \nu$ on $\mathfrak{F}_A$.

Let Σ be a finite set of formulas closed under subformulas. Since Σ is finite, $\nu[\Sigma]$ is a finite subset of A. Let S be the bounded sublattice of A generated by $\nu[\Sigma]$. As bounded distributive lattices are locally finite, we see that S is finite. Therefore, S is a Heyting algebra, where

$$a \to_S b = \bigvee \{s \in S : a \wedge s \leq b\} \tag{8}$$

for each $a, b \in S$. Clearly $a \to_S b \leq a \to b$, and

$$a \to_S b = a \to b \ \text{ provided } \ a \to b \in S. \tag{9}$$

Lemma 2.5. *S gives rise to a filtration $\mathfrak{M}'_A = (\mathfrak{F}'_A, \mu')$ of $\mathfrak{M}_A = (\mathfrak{F}_A, \mu)$ through Σ.*

Proof. Define $\sim$ on W_A by $w \sim v$ iff $w \cap S = v \cap S$. We show that $w \sim v$ iff $w \cap \nu[\Sigma] = v \cap \nu[\Sigma]$. The left to right implication is clear. For the right to left implication, let $w \cap \nu[\Sigma] = v \cap \nu[\Sigma]$. If $a \in w \cap S$, then since $\nu[\Sigma]$ generates S, we have $a = \bigvee_{i=1}^{n} \bigwedge_{j=1}^{n_i} b_{ij}$ for some $b_{ij} \in \nu[\Sigma]$. As w is a prime filter of A, there is i with $\bigwedge_{j=1}^{n_i} b_{ij} \in w$, so $b_{ij} \in w$ for all j. Since $w \cap \nu[\Sigma] = v \cap \nu[\Sigma]$, we have $b_{ij} \in v$ for all j. Therefore, $\bigwedge_{j=1}^{n_i} b_{ij} \in v$, and so $\bigvee_{i=1}^{n} \bigwedge_{j=1}^{n_i} b_{ij} \in v$. Thus, $a \in v \cap S$, yielding $w \cap S \subseteq v \cap S$. The other inclusion is proved similarly, hence $w \sim v$.

Now $w \cap \nu[\Sigma] = v \cap \nu[\Sigma]$ is clearly equivalent to $(\forall \varphi \in \Sigma)(\nu(\varphi) \in w$ iff $\nu(\varphi) \in v)$, which in turn is equivalent to $(\forall \varphi \in \Sigma)(w \in \mu(\varphi)$ iff $v \in \mu(\varphi))$. Let $W'_A = W_A/{\sim}$, and set $[w] \leq' [v]$ if $w \cap S \subseteq v \cap S$. It is obvious that $\leq'$ is a partial order on W'_A. Let μ' be a valuation on $\mathfrak{F}'_A = (W'_A, \leq')$ such that $\mu'(p) = \{[w] : w \in \mu(p)\}$. To see that such a μ' exists, it is sufficient to show that $\{[w] : w \in \mu(\varphi)\}$ is an up-set for each $\varphi \in \Sigma$. Suppose $w \in \mu(\varphi)$ and $[w] \leq' [v]$. Then $\nu(\varphi) \in w$ and $w \cap S \subseteq v \cap S$. Therefore, $\nu(\varphi) \in v$, so $v \in \mu(\varphi)$, and hence $\{[w] : w \in \mu(\varphi)\}$ is indeed an up-set. Thus, $\mathfrak{M}'_A = (\mathfrak{F}'_A, \mu')$ is a model, and it is straightforward to see that it satisfies Conditions (2) and (3) of Definition 2.1. Thus, $\mathfrak{M}'_A$ is a filtration of $\mathfrak{M} = (\mathfrak{F}_A, \mu)$ through Σ. $\square$

Remark 2.6. More generally, the construction of Lemma 2.5 can be applied to any finite bounded sublattice L of A that contains $\nu[\Sigma]$, or equivalently has S as a bounded sublattice. In fact, one could define filtration algebraically as a pair (L, ν_L), where L is a sublattice of A that contains $\nu[\Sigma]$ and ν_L is the restriction of ν to L. The filtration lemma for (L, ν_L) holds (see Lemma 2.11 below), so if (A, ν) refutes a formula $\varphi \in \Sigma$, then so does (L, ν_L). But this definition of filtration does not match the model theoretic definition above because the equivalence relation defined from L can be more refined than the one defined from S. To ensure the match, we have to require that $w \cap L = v \cap L$ is equivalent to $w \cap S = v \cap S$. This is achieved by requiring that L and S have isomorphic free Boolean extensions.

Lemma 2.7. *Let L be a finite bounded sublattice of A that contains S as a bounded sublattice. Then the following two conditions are equivalent.*

(1) *$w \cap L = v \cap L$ is equivalent to $w \cap S = v \cap S$.*
(2) *$\alpha[L]$ and $\alpha[S]$ generate the same Boolean subalgebra of the powerset of the spectrum of A.*

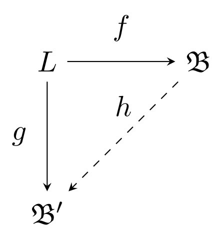
Proof. (1)⇒(2): Since $\nu[\Sigma]$ generates S, it is obvious that $\mu[\Sigma]$ and $\alpha[S]$ generate the same Boolean subalgebra $\mathfrak{B}$ of the powerset of W_A. Let $\sim$ be the equivalence relation given by $w \sim v$ iff $w \cap S = v \cap S$. Call $U \subseteq W_A$ *saturated* provided $w \in U$ and $w \sim v$ imply $v \in U$. Then $U \subseteq W_A$ is saturated iff U belongs to $\mathfrak{B}$. Therefore, (1) implies that $\alpha[L]$ and $\alpha[S]$ define the same equivalence relation $\sim$ on W_A, and hence both $\alpha[L]$ and $\alpha[S]$ generate the same Boolean algebra $\mathfrak{B}$.

(2)⇒(1): Suppose $\alpha[L]$ and $\alpha[S]$ generate the same Boolean algebra $\mathfrak{B}$. Since S is a bounded sublattice of L, it is clear that $w \cap L = v \cap L$ implies $w \cap S = v \cap S$. Conversely, suppose $w \cap S = v \cap S$. Let $a \in w \cap L$. Then $w \in \alpha(a) \in \alpha[L]$. Therefore, $\alpha(a) \in \mathfrak{B}$. Since $\alpha[S]$ generates $\mathfrak{B}$ and is closed under finite unions and intersections, we can write $\alpha(a) = \bigcup_{i=1}^{n} (\alpha(b_i) \cap \alpha(c_i)^c)$ with $b_i, c_i \in S$. Thus, $w \in \alpha(b_i)$ and $w \notin \alpha(c_i)$ for some i. This yields $b_i \in w$ and $c_i \notin w$ for some i. Since $w \cap S = v \cap S$, we conclude that $b_i \in v$ and $c_i \notin v$ for some i, so $v \in \alpha(a)$. This gives $w \cap L \subseteq v \cap L$, and the reverse inclusion is proved similarly. □

Lemma 2.8. *Each filtration $\mathfrak{M}'_A$ of $\mathfrak{M}_A$ through Σ gives rise to a finite bounded sublattice L of A that contains S as a bounded sublattice and such that $\alpha[L]$ and $\alpha[S]$ generate the same Boolean subalgebra of the powerset of the spectrum of A.*

Proof. Suppose $\mathfrak{M}'_A = (\mathfrak{F}'_A, \mu')$ is a filtration of $\mathfrak{M}_A = (\mathfrak{F}_A, \mu)$ through Σ. Then saturated subsets of W_A belong to the Boolean subalgebra $\mathfrak{B}$ of the powerset of W_A generated by $\mu(\Sigma)$. Let $\pi : W_A \to W'_A$ be the quotient map given by $\pi(w) = [w]$. Therefore, $\pi^{-1}(U) \in \mathfrak{B}$ for each $U \subseteq W'_A$. Let $\mathrm{Up}(\mathfrak{F}'_A)$ be the Heyting algebra of up-sets of $\mathfrak{F}'_A$. By Condition (2) of Definition 2.1, π is order-preserving. Therefore, $\pi^{-1}(U)$ is an up-set of W_A belonging to $\mathfrak{B}$. Thus, $\pi^{-1}(U) \in \alpha[A]$. This yields that $\pi^{-1}[\mathrm{Up}(\mathfrak{F}'_A)]$ is a bounded sublattice of $\alpha[A]$. It is finite since $\mathrm{Up}(\mathfrak{F}'_A)$ is finite, so $L := \alpha^{-1}\pi^{-1}[\mathrm{Up}(\mathfrak{F}'_A)]$ is a finite bounded sublattice of A. Clearly L contains $\nu[\Sigma]$, hence L contains S as a bounded sublattice. Moreover, for $w, v \in W_A$, we have $w \cap L = v \cap L$ iff $w \sim v$. Indeed, since $\nu[\Sigma] \subseteq L$, the left to right implication is obvious. For the right to left implication, let $w \sim v$. If $a \in w \cap L$, then $a \in w$ and $a = \alpha^{-1}\pi^{-1}(U)$ for some $U \in \mathrm{Up}(\mathfrak{F}'_A)$. This implies that $w \in \alpha(a)$ and $\alpha(a) = \pi^{-1}(U)$. Therefore, $w \in \pi^{-1}(U)$, and so $\pi(w) \in U$. As $w \sim v$, we have $\pi(w) = \pi(v)$. Thus, $\pi(v) \in U$, and hence $v \in \alpha(a)$. This yields that $a \in v \cap L$. Consequently, $w \cap L \subseteq v \cap L$, and the other inclusion is proved similarly. Therefore, we showed that $w \cap L = v \cap L$. Since $w \sim v$ iff $w \cap S = v \cap S$, by Lemma 2.7, $\alpha[L]$ and $\alpha[S]$ generate the same Boolean algebra. □

Remark 2.9. The Boolean algebra $\mathfrak{B}$ generated by $\alpha[S]$ is nothing more but the free Boolean extension of S. We recall (see, e.g., [1, p. 99, Def. 5]) that the *free Boolean extension* of a bounded distributive lattice L is a pair $(\mathfrak{B}, f)$ such that $\mathfrak{B}$ is a Boolean algebra, $f : L \to \mathfrak{B}$ is a bounded lattice embedding, and for any Boolean algebra $\mathfrak{B}'$ and a bounded lattice homomorphism $g : L \to \mathfrak{B}'$, there is a unique Boolean homomorphism $h : \mathfrak{B} \to \mathfrak{B}'$ such that $h \circ f = g$.



It is well known (see, e.g., [17, p. 39, Constr. 5.7]) that the free Boolean extension of a bounded distributive lattice L can be constructed as follows. Let $\mathfrak{F}_L$ be the spectrum of L and let $\mathfrak{B}$ be the Boolean subalgebra of the powerset of $\mathfrak{F}_L$ generated by $\alpha[L]$. Then $(\mathfrak{B}, \alpha)$ is isomorphic to the free Boolean extension of L. Therefore, up to isomorphism, the free Boolean extension of S is the Boolean subalgebra of the powerset of the spectrum $\mathfrak{F}_A$ of A generated by $\alpha[S]$.

It follows that there is a 1-1 correspondence between filtrations $\mathfrak{M}'_A$ of $\mathfrak{M}_A$ and finite bounded sublattices L of A that contain S as a bounded sublattice and have the same free Boolean extension as S. This motivates the following definition.

Definition 2.10 (Standard filtration algebraically)**.** *Let A be a Heyting algebra, ν be a valuation on A, and Σ be a finite set of formulas closed under subformulas. Suppose L is a finite bounded sublattice of A such that $\nu[\Sigma] \subseteq L$ and S, L have isomorphic free Boolean extensions. Let ν_L be a valuation on L such that $\nu_L(p) = \nu(p)$ for each $p \in \Sigma$. Then we call the pair (L, ν_L) a* filtration *of (A, ν) through Σ.*

Lemma 2.11 (Filtration Lemma)**.** *If (L, ν_L) is a filtration of (A, ν) through Σ, then $\nu_L(\varphi) = \nu(\varphi)$ for each $\varphi \in \Sigma$.*

Proof. Induction on the complexity of $\varphi \in \Sigma$. The case $\varphi = p$ follows from the definition of ν_L. The cases $\varphi = \bot$, $\varphi = \psi \wedge \chi$, and $\varphi = \psi \vee \chi$ follow from the fact that L is a bounded sublattice of A. Finally, let $\varphi = \psi \to \chi$. Then $\nu(\psi) \to \nu(\chi) = \nu(\psi \to \chi) \in \nu(\Sigma) \subseteq L$. Therefore, by (9), $\nu(\psi) \to \nu(\chi) = \nu(\psi) \to_L \nu(\chi)$. Thus, $\nu_L(\psi \to \chi) = \nu_L(\psi) \to_L \nu_L(\chi) = \nu(\psi) \to \nu(\chi) = \nu(\psi \to \chi)$. □

Remark 2.12. In order to prove the filtration lemma, it is not necessary to require that L is a sublattice of A. All that is needed is that $\nu_L(\varphi) \odot_L \nu_L(\psi) = \nu(\varphi) \odot \nu(\psi)$ for $\varphi \odot \psi \in \Sigma$, where $\odot \in \{\vee, \wedge, \to\}$. This is an algebraic import of the discussion in Remark 2.4.

Theorem 2.13. *Let Σ be a finite set of formula closed under subformulas, A be a Heyting algebra, and ν be a valuation on A. Define μ on the spectrum $\mathfrak{F}_A$ of A by $\mu = \alpha \circ \nu$. Then there is a 1-1 correspondence between the filtrations (L, ν_L) of (A, ν) through Σ and the filtrations $\mathfrak{M}'_A = (\mathfrak{F}'_A, \mu')$ of $\mathfrak{M}_A = (\mathfrak{F}_A, \mu)$ through Σ. Moreover, if (L, ν_L), (K, ν_K) are two filtrations of (A, ν) through Σ and $\mathfrak{M}'_A$, $\mathfrak{M}''_A$ are the corresponding filtrations of $\mathfrak{M}_A$ through Σ, then L is a bounded sublattice of K iff $[w] \le'' [v]$ implies $[w] \le' [v]$.*

Proof. If (L, ν_L) is a filtration of (A, ν) through Σ, then by Lemma 2.5 (see also Remark 2.6), $\mathfrak{M}'_A = (W'_A, \le', \mu')$ is a filtration of $\mathfrak{M}_A$ through Σ, where $[w] \le' [v]$ iff $w \cap L \subseteq v \cap L$. Conversely, if $\mathfrak{M}'_A$ is a filtration of $\mathfrak{M}_A$ through Σ, then by Lemma 2.8, (L, ν_L) is a filtration of (A, ν) through Σ, where $L = \alpha^{-1}\pi^{-1}[\mathrm{Up}(\mathfrak{F}'_A)]$. This correspondence between the filtrations (L, ν_L) of (A, ν) through Σ and the filtrations $\mathfrak{M}'_A = (\mathfrak{F}'_A, \mu')$ of $\mathfrak{M}_A = (\mathfrak{F}_A, \mu)$ through Σ is 1-1. To see this, it is sufficient to observe that if (L, ν_L) is a filtration of (A, ν) through Σ and $\mathfrak{M}'_A = (\mathfrak{F}'_A, \mu')$ is the corresponding filtration of $\mathfrak{M}_A$ through Σ, then $L = \alpha^{-1}\pi^{-1}[\mathrm{Up}(\mathfrak{F}'_A)]$; and conversely, if $\mathfrak{M}'_A = (\mathfrak{F}'_A, \mu')$ is a filtration of $\mathfrak{M}_A$ through Σ and (L, ν_L) is the corresponding filtration of (A, ν) through Σ, then $w \sim v$ iff $w \cap L = v \cap L$.

For the last statement of the theorem, if L is a bounded sublattice of K, then $w \cap K \subseteq v \cap K$ implies $w \cap L \subseteq v \cap L$. Therefore, $[w] \le'' [v]$ implies $[w] \le' [v]$. Conversely, if $[w] \le'' [v]$ implies $[w] \le' [v]$, then $\mathrm{Up}(\mathfrak{F}'_A)$ is a bounded sublattice of $\mathrm{Up}(\mathfrak{F}''_A)$. Thus, $L = \alpha^{-1}\pi^{-1}[\mathfrak{F}'_A]$ is a bounded sublattice of $K = \alpha^{-1}\pi^{-1}[\mathfrak{F}''_A]$. □

Among the filtrations (L, ν_L) of (A, ν), the filtration (S, ν_S) is clearly the least one. By Lemma 2.7, there is also the greatest filtration (T, ν_T) given by $T = \alpha^{-1}[\mathfrak{B}]$. By Theorem 2.13, (S, ν_S) corresponds to the greatest filtration $\mathfrak{M}^g = (W'_A, \le^g, \mu')$ of $\mathfrak{M}_A$ through Σ, while (T, ν_T) to the least filtration $\mathfrak{M}^l = (W'_A, \le^l, \mu')$. It is instructive to see a direct proof of this.

Lemma 2.14. *Let Σ be a finite set of formulas closed under subformulas, A be a Heyting algebra, and ν be a valuation on A. Define μ on the spectrum $\mathfrak{F}_A$ of A by $\mu = \alpha \circ \nu$. Then (S, ν_S) gives rise to the greatest filtration $\mathfrak{M}^g = (W'_A, \le^g, \mu')$, while (T, ν_T) to the least filtration $\mathfrak{M}^l = (W'_A, \le^l, \mu')$ of $\mathfrak{M}_A = (\mathfrak{F}_A, \mu)$ through Σ.*

Proof. Let $w, v \in W_A$. We have $w \cap S \subseteq v \cap S$ iff $w \cap \nu[\Sigma] \subseteq v \cap \nu[\Sigma]$, which is equivalent to $(\forall \varphi \in \Sigma)(w \in \mu(\varphi) \Rightarrow v \in \mu(\varphi))$. Therefore, $w \cap S \subseteq v \cap S$ iff $[w] \le^g [v]$. Thus, by Theorem 2.13, (S, ν_S) gives rise to the greatest filtration $\mathfrak{M}^g = (W'_A, \le^g, \mu')$ of $\mathfrak{M}_A$ through Σ.

First suppose that $[w] \preceq [v]$. Then there are $w' \sim w$ and $v' \sim v$ with $w' \le v'$. From $w \sim w'$ it follows that $w \cap \nu[\Sigma] = w' \cap \nu[\Sigma]$. Therefore, $w \cap S = w' \cap S$, and so $w \cap T = w' \cap T$ by Lemma 2.7. A similar argument gives $v \cap T = v' \cap T$, yielding $w \cap T \subseteq v \cap T$. Thus, if $\mathfrak{M}' = (W'_A, \le', \mu')$ is the filtration corresponding to T, then $[w] \preceq [v]$ implies $[w] \le' [v']$. Consequently, since $\le'$ is a partial order, we conclude that $[w] \le^l [v]$ implies $[w] \le' [v]$. Conversely, suppose $[w] \not\le^l [v]$. Then $[w] \cap {\downarrow}[v] = \varnothing$, where as usual, ${\downarrow}U = \{w : w \le u \text{ for some } u \in U\}$. If ${\downarrow}[v]$ is saturated, then ${\downarrow}[v] \in \mathfrak{B}$, so $W_A \setminus {\downarrow}[v]$ is an up-set belonging to $\mathfrak{B}$. Therefore, $W_A \setminus {\downarrow}[v] \in \alpha(A) \cap \mathfrak{B}$, and hence $W_A \setminus {\downarrow}[v] = \alpha(a)$ for some $a \in T$. Clearly $a \in w \cap T$, but $a \notin v \cap T$, so $w \cap T \not\subseteq v \cap T$.

If $\downarrow[v]$ is not saturated, then let $[\downarrow[v]]$ be the saturation of $\downarrow[v]$. Since $[w] \not\leq^l [v]$, we have $[w] \cap \downarrow[\downarrow[v]] = \varnothing$. If $\downarrow[\downarrow[v]]$ is saturated, then the same argument as above yields $w \cap T \not\subseteq v \cap T$. Otherwise we take the saturation of $\downarrow[\downarrow[v]]$. Since there are only finitely many saturated sets, it follows from the definition of $\leq^l$ that after finitely many steps we obtain a saturated down-set missing $[w]$. Thus, repeating the argument above, we conclude that $w \cap T \not\subseteq v \cap T$. Consequently, by Theorem 2.13, (T, ν_T) gives rise to the least filtration $\mathfrak{M}^l = (W'_A, \leq^l, \mu')$ of $\mathfrak{M}_A$ through Σ. □

3. Selective filtration

The idea of selective filtration is to work with submodels instead of quotient models. Given a model $\mathfrak{M} = (\mathfrak{F}, \nu)$ and a finite set Σ of formulas closed under subformulas, one wants to select a finite submodel $\mathfrak{M}' = (\mathfrak{F}', \nu')$ so that for each $w \in W'$ and $\varphi \in \Sigma$, we have $w \in \nu(\varphi)$ iff $w \in \nu'(\varphi)$. The method was introduced in [19]; see [14, Sec. 5.5] for details.

A more elaborate version of selective filtration was employed in [18] to prove that all transitive subframe logics have the finite model property. This was achieved by finding a subreduction (a p-morphism from a submodel) of $\mathfrak{M}$ onto a finite model $\mathfrak{M}'$. This method was further refined in [29], where it was shown that all transitive cofinal subframe logics have the finite model property. For an algebraic account of the method, as well as for its generalization to all weakly transitive subframe and cofinal subframe logics, consult [8]. Similar results for superintuitionistic logics are obtained in [28], and an algebraic account is given in [7].

We recall that a *p-morphism* between two frames $\mathfrak{F} = (W, \leq)$ and $\mathfrak{F}' = (W', \leq')$ is a map $f : W \to W'$ such that $w \leq v$ implies $f(w) \leq' f(v)$, and $f(w) \leq' u$ implies that there is $v \in W$ with $w \leq v$ and $f(v) = u$. A *subreduction* from a frame $\mathfrak{F}$ to a frame $\mathfrak{F}'$ is a p-morphism from a subframe $\mathfrak{G}$ of $\mathfrak{F}$ onto the frame $\mathfrak{F}'$. We denote the domain of a subreduction f by $\mathrm{dom}(f)$, and call the subreduction f *cofinal* if for each $w \in W$ there is $v \in \mathrm{dom}(f)$ with $w \leq v$.

It is well known (see, e.g., [14, p. 292, Thm. 9.7]) that if f is a subreduction from $\mathfrak{F}$ to $\mathfrak{F}'$, then $f^* : \mathrm{Up}(\mathfrak{F}') \to \mathrm{Up}(\mathfrak{F})$, given by $f^*(U) = W \setminus f^{-1}(W' \setminus U)$, is a $(\wedge, \to)$-homomorphism (meaning that $f^*(U \cap V) = f^*(U) \cap f^*(V)$ and $f^*(U \to V) = f^*(U) \to f^*(V)$). Moreover, f is cofinal iff f^* is a $(\wedge, \to, 0)$-homomorphism (meaning that in addition $f^*(\varnothing) = \varnothing$); see, e.g., [2, Lem. 3.22]. Note that $w \in f^*(U)$ iff $f[\uparrow w] \subseteq U$, where as usual, $\uparrow w = \{v : w \leq v\}$.

A *subreduction* from a model $\mathfrak{M} = (\mathfrak{F}, \nu)$ to a model $\mathfrak{M}' = (\mathfrak{F}', \nu')$ is a subreduction f from the frame $\mathfrak{F}$ to the frame $\mathfrak{F}'$ satisfying $f^*(\nu'(\varphi)) = \nu(\varphi)$.

Definition 3.1 (Selective filtration model theoretically)**.** *Let Σ be a finite set of formulas closed under subformulas and let $\mathfrak{M} = (\mathfrak{F}, \nu)$ be a model. We call a finite model $\mathfrak{M}' = (\mathfrak{F}', \nu')$ a* selective filtration *of $\mathfrak{M}$ through Σ if there is a subreduction f of $\mathfrak{M}$ to $\mathfrak{M}'$ such that*

$$(\forall w \in W)(\forall \varphi \in \Sigma)(\exists v \in \mathrm{dom}(f) : w \leq v \;\&\; w \in \nu(\varphi) \Leftrightarrow v \in \nu(\varphi)). \tag{10}$$

Remark 3.2. It follows from Condition (10) that if $\mathfrak{M}'$ is a selective filtration of $\mathfrak{M}$ through Σ, then the subreduction f is cofinal.

Lemma 3.3 (Filtration Lemma)**.** *Let $\mathfrak{M} = (\mathfrak{F}, \nu)$ be a model and let $\mathfrak{M}' = (\mathfrak{F}', \nu')$ be a selective filtration of $\mathfrak{M}$ through a finite set Σ of formulas closed under subformulas. Then for each $\varphi \in \Sigma$ and $w \in \mathrm{dom}(f)$, we have*

$$w \in \nu(\varphi) \text{ iff } f(w) \in \nu'(\varphi). \tag{11}$$

Proof. We have $w \in \nu(\varphi)$ iff $w \in f^*(\nu'(\varphi))$, which happens iff $f[\uparrow w] \subseteq \nu'(\varphi)$. Since $w \in \mathrm{dom}(f)$, we have $f[\uparrow w] = \uparrow f(w)$, so the last condition is equivalent to $\uparrow f(w) \subseteq \nu'(\varphi)$. Because $\nu'(\varphi)$ is an up-set, this is equivalent to $f(w) \in \nu'(\varphi)$. □

As an immediate consequence of Lemma 3.3, we obtain:

Lemma 3.4. *Let $\mathfrak{M} = (\mathfrak{F}, \nu)$ be a model and let $\mathfrak{M}' = (\mathfrak{F}', \nu')$ be a selective filtration of $\mathfrak{M}$ through a finite set Σ of formulas closed under subformulas. If $\mathfrak{M}$ refutes $\varphi \in \Sigma$, then so does $\mathfrak{M}'$.*

Proof. If $\mathfrak{M}$ refutes $\varphi \in \Sigma$, then there is $w \in W$ such that $w \notin \nu(\varphi)$. Let f be the subreduction of $\mathfrak{M}$ to $\mathfrak{M}'$. By (10), there is $v \in \mathrm{dom}(f)$ such that $w \le v$ and $v \notin \nu(\varphi)$. Therefore, by (11), $f(v) \notin \nu'(\varphi)$. Thus, $\mathfrak{M}'$ refutes φ. □

We next give an algebraic description of selective filtrations. Let A be a Heyting algebra, ν be a valuation on A, and Σ be a finite set of formulas closed under subformulas. Then $\nu[\Sigma]$ is a finite subset of A. Let S be the bounded implicative subsemilattice of A generated by $\nu[\Sigma]$ (so S is closed under $\wedge, \to, 0$, but not necessarily under $\vee$). By Diego's Theorem [16], S is finite. Therefore, S is a Heyting algebra, where

$$a \vee_S b = \bigwedge \{s \in S : a, b \le s\} \tag{12}$$

for each $a, b \in S$. It follows from the definition that $a \vee b \le a \vee_S b$, and that

$$a \vee b = a \vee_S b \ \text{ provided } \ a \vee b \in S. \tag{13}$$

Definition 3.5 (Selective filtration algebraically)**.** *Let A be a Heyting algebra, ν be a valuation on A, and Σ be a finite set of formulas closed under subformulas. Suppose L is a finite bounded implicative subsemilattice of A containing $\nu[\Sigma]$. Let ν_L be a valuation on L such that $\nu_L(p) = \nu(p)$ for each $p \in \Sigma$. Then we call (L, ν_L) a* selective filtration *of (A, ν) through Σ.*

Lemma 3.6 (Filtration Lemma)**.** *If (L, ν_L) is a selective filtration of (A, ν) through Σ, then $\nu_L(\varphi) = \nu(\varphi)$ for each $\varphi \in \Sigma$.*

Proof. Induction on the complexity of $\varphi \in \Sigma$. The case $\varphi = p$ follows from the definition of ν_L. The cases $\varphi = \bot$, $\varphi = \psi \wedge \chi$, and $\varphi = \psi \to \chi$ follow from the fact that L is a bounded implicative subsemilattice of A. Finally, let $\varphi = \psi \vee \chi$. Then $\nu(\psi) \vee \nu(\chi) = \nu(\psi \vee \chi) \in \nu(\Sigma) \subseteq L$. Therefore, by (13), $\nu(\psi) \vee \nu(\chi) = \nu(\psi) \vee_L \nu(\chi)$. Thus, $\nu_L(\psi \vee \chi) = \nu_L(\psi) \vee_L \nu_L(\chi) = \nu(\psi) \vee \nu(\chi) = \nu(\psi \vee \chi)$. □

Remark 3.7. Like with standard filtrations (see Remark 2.12), in order to prove the filtration lemma for selective filtrations, it is not necessary to require that L is an implicative subsemilattice of A. All that is needed is that $\nu_L(\varphi) \odot_L \nu_L(\psi) = \nu(\varphi) \odot \nu(\psi)$ for $\varphi \odot \psi \in \Sigma$, where $\odot \in \{\vee, \wedge, \to\}$.

Remark 3.8. While the construction of the selective filtration (S, ν_S) of (A, ν) through Σ is relatively straightforward (see also [7, Sec. 7]), it is rather involved in modal logic, both model-theoretically (see [18] and [14, Thm. 9.34]) and algebraically (see [8]).

Theorem 3.9. *Let Σ be a finite set of formula closed under subformulas. If $\mathfrak{M}' = (\mathfrak{F}', \nu')$ is a selective filtration of $\mathfrak{M} = (\mathfrak{F}, \nu)$ through Σ, then $(\mathrm{Up}(\mathfrak{F}'), \nu')$ is a selective filtration of $(\mathrm{Up}(\mathfrak{F}), \nu)$ through Σ. Conversely, suppose (L, ν_L) is a selective filtration of (A, ν) through Σ. Let $\mathfrak{F}_A$ be the spectrum of A and $\mathfrak{F}_L$ be the spectrum of L. Define μ on $\mathfrak{F}_A$ by $\mu = \alpha \circ \nu$ and μ' on $\mathfrak{F}_L$ by $\mu' = \alpha \circ \nu_L$. Then $(\mathfrak{F}_L, \mu')$ is a selective filtration of $(\mathfrak{F}_A, \mu)$ through Σ.*

Proof. First suppose that $\mathfrak{M}'$ is a selective filtration of $\mathfrak{M}$ through Σ. Let f be the subreduction. Then $f^*[\mathrm{Up}(\mathfrak{F}')]$ is a finite bounded implicative subsemilattice of $\mathrm{Up}(\mathfrak{F})$. Since $\nu(\varphi) = f^*(\nu'(\varphi))$ for each $\varphi \in \Sigma$, we see that $\nu[\Sigma]$ is contained in $f^*[\mathrm{Up}(\mathfrak{F}')]$. Therefore, $(\mathrm{Up}(\mathfrak{F}'), \nu')$ is a selective filtration of $(\mathrm{Up}(\mathfrak{F}), \nu)$ through Σ.

Next suppose that (L, ν_L) is a selective filtration of (A, ν) through Σ. Then L is a finite bounded implicative subsemilattice of A. Therefore, there is a cofinal subreduction f from $\mathfrak{F}_A$ to $\mathfrak{F}_L$ such that $f^*(\mu'(\varphi)) = \mu(\varphi)$ for each $\varphi \in \Sigma$ (see [2, Thm. 3.14, Lem. 3.29, and Lem. 3.31]). To see that $(\mathfrak{F}_L, \mu')$ satisfies Condition (10), let $w \in W_A$ and $\varphi \in \Sigma$. Since f is cofinal, there is $v \in \mathrm{dom}(f)$ with $w \le v$. Clearly $w \in \mu(\varphi)$ implies $v \in \mu(\varphi)$. On the other hand, if $w \notin \mu(\varphi)$, then $w \notin f^*(\mu'(\varphi))$. Thus, $f[{\uparrow} w] \not\subseteq \mu'(\varphi)$. So there is $u \in \mathrm{dom}(f)$ with $w \le u$ and $f(u) \notin \mu'(\varphi)$. Since $u \in \mathrm{dom}(f)$, by

Lemma 3.3, $u \notin \mu(\varphi)$. Therefore, replacing v with u, we see that Condition (10) is satisfied. Thus, $(\mathfrak{F}_L, \mu')$ is a selective filtration of $(\mathfrak{F}_A, \mu)$ through Σ. □

Remark 3.10. The 1-1 correspondence of Theorem 2.13 between filtrations (L, ν_L) of (A, ν) and filtrations $\mathfrak{M}'_A = (\mathfrak{F}'_A, \mu')$ of $\mathfrak{M}_A = (\mathfrak{F}_A, \mu)$ through Σ does not have an immediate analogue for selective filtrations. The reason is that different subreductions of $\mathfrak{M}_A$ may give rise to the same bounded implicative subsemilattice of A (see [2, Sec. 4]). To remedy this, we need to strengthen the notion of a subreduction to that of an onto partial Esakia morphism of [2]. In order to keep the definition of selective filtrations relatively simple, we decided to work with a more familiar concept of subreductions. This discrepancy (roughly speaking, there are more model-theoretic selective filtrations than algebraic ones) goes away if in Definition 3.1, the subreduction f is strengthened to be an onto partial Esakia morphism.

Remark 3.11. Clearly among selective filtrations of (A, ν) through Σ there is a least one, namely (S, ν_S). However, there is no greatest selective filtration of (A, ν). This means that among selective filtrations of $\mathfrak{M}$ through Σ there is a greatest one, but there is no least selective filtration of $\mathfrak{M}$. This is in contrast with the theory of standard filtrations, where we always have the least and greatest filtrations. However, we would get the same situation for standard filtrations if we defined them as suggested in Remark 2.6.

Remark 3.12. Let L be a superintuionistic logic. Following [14, p. 142, Sec. 5.3], we say that L *admits filtration* if for each non-theorem φ of L and some countermodel $\mathfrak{M} = (\mathfrak{F}, \nu)$ of φ, there is a filtration $\mathfrak{M}' = (\mathfrak{F}', \nu')$ of $\mathfrak{M}$ through some finite set Σ closed under subformulas and containing φ such that $\mathfrak{F}' \models L$. Clearly every superintuitionistic logic admitting filtration has the finite model property. This notion depends on at least three different parameters: formulas, models, frames. In [4] *stable superinutionistic logics* are introduced as the logics that are sound and complete with respect to a class of frames closed under order-preserving images. Every stable logic admits filtration, and hence has the finite model property. Thus, stable logics in some sense formalize the notion of admitting filtration by avoiding mentioning models and formulas. However, since a notion of filtration is not unique (there is a whole spectrum of them between the least and greatest ones), not every logic that admits filtration is stable. In [4] it is shown how to axiomatize these logics by special types of formulas, called *stable formulas*. We refer to [4] and [6] for more details.

Similarly, we can define when a superintuitionistic logic admits selective filtration. Fine [18] formalizes this notion by defining subframe transitive modal logics, and Zakharyaschev [28, 29] defines subframe and cofinal subframe modal and superintuitionistic logics. These are logics that are sound and complete with respect to a class of frames closed under subframes or cofinal subframes. An algebraic characterization of these logics can be found in [7, 2, 8, 3]. These logics are axiomatizable by the so-called subframe and cofinal subframe formulas [18, 29, 14]. They are also axiomatizable by the NNIL-formulas of Visser et al. [27], while stable logics are axiomatizable by ONNILLI-formulas (only NNIL to the left of implications) [10]. Thus, we see yet another influence of Albert Visser's ideas!

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Guram Bezhanishvili: Department of Mathematical Sciences, New Mexico State University, Las Cruces NM 88003, gbezhani@nmsu.edu

Nick Bezhanishvili: Institute for Logic, Language and Computation, University of Amsterdam, P.O. Box 94242, 1090 GE Amsterdam, The Netherlands, N.Bezhanishvili@uva.nl