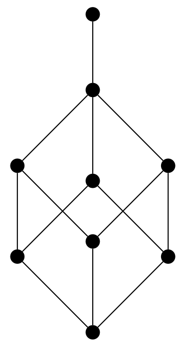
Mathematical structures in logic
Exercise class 6

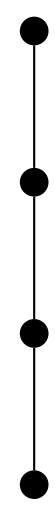
Varieties

March 13, 2018

1. Let A and B be the following Heyting algebras:



A



B

   (a) Compute the lattice of subvarieties of $\mathsf{V}_1 = Var(\{A\})$.
   (b) Compute the lattice of subvarieties of $\mathsf{V}_2 = Var(\{B\})$.
   (c) Compute the lattice of subvarieties of $\mathsf{V}_3 = Var(\{A, B\})$.

2. Show that the set of superintuionistic logics $\Lambda(\mathbf{IPC})$ forms a bounded lattice with respect to $\subseteq$. Describe the bounds, meets and joins in this lattice.

3. Show that the subvarieties of the variety of Heyting algebras $\Lambda(\mathsf{HA})$ form a lattice with respect to $\subseteq$. Describe the bounds, meets and joins in this lattice.

Additional exercises

4. Let L be a superintuitionistic logic. Recall that V_L is the variety axiomatized by $\{\varphi \approx 1 : L \vdash \varphi\}$. Show that for superintuionistic logics L, L' the following holds

$$L \subseteq L' \text{ implies } \mathsf{V}_L \supseteq \mathsf{V}_{L'}.$$

and

$$L \nsubseteq L' \text{ implies } \mathsf{V}_L \nsupseteq \mathsf{V}_{L'}.$$

(Hint: use that L is complete wrt V_L. That is, for each formula φ we have

$$L \vdash \varphi \text{ iff } \mathsf{V}_L \models \varphi \approx 1.)$$

5. Let V be a variety of Heyting algebras. Recall that L_{V} is the least superintuionistic logic containing the set $\{\varphi \leftrightarrow \psi : \mathsf{V} \models \varphi \approx \psi\}$. Show that for each variety of Heyting algebras V and each superintuionistic logic L we have

$$\mathsf{V}_{L_{\mathsf{V}}} = \mathsf{V} \text{ and } L_{\mathsf{V}_L} = L.$$

6. Deduce that the lattice of superintuionistic logics $(\Lambda(\mathbf{IPC}), \subseteq)$ is dually isomorphic to the lattice $(\Lambda(\mathsf{HA}), \subseteq)$ of varieties of Heyting algebras.