Mathematical structures in logic
Exercise class 3

Heyting algebras, Boolean algebras

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1. Recall that in a topological space (X, τ), for a set $A \subseteq X$, the *interior of* A and the *closure of* A are defined respectively as the largest open set contained in A and the smallest closed set that contains A, i.e.:

$$\operatorname{Int} A = \bigcup\{U : U \in \tau \text{ and } U \subseteq A\}$$
$$\operatorname{Cl} A = \bigcap\{C : X \setminus C \in \tau \text{ and } A \subseteq C\}.$$

   Let $(P, \leq)$ be a poset. We know that $\operatorname{Up}(P)$, the upsets on P, form a topology on P.

   (a) Show that this is an Alexandroff topology, i.e. show that the open sets are closed under arbitrary intersections.

   (b) Show that for every $P' \subseteq P$, $\operatorname{Cl}(P') = {\downarrow}P'$.

   (c) Describe the interior of a set $P' \subseteq P$.

   (Here $\downarrow P'$ denotes the set $\{q \in P \mid \exists p \in P', q \leq p\}$.)

2. (Prime filters and maximal filters.) Let B be a Boolean algebra. Show that:

   (a) If F is a filter on B, then $I := \{\neg a \in B \mid a \in F\}$ is an ideal on B.

   (b) If F is a filter, then $B \setminus F$ may not be an ideal on B.

   (c) If F is a prime filter, then $B \setminus F$ is a prime ideal.

   (d) Which of these statements are true for Heyting algebras?

3. Let R be a pre-order, i.e. transitive and reflexive relation on a set X. Define $\Box_R \colon \wp(X) \to \wp(X)$ by

$$\Box_R(U) = \{x \in X : R[x] \subseteq U\},$$

   where $R[x] = \{x' \in X : xRx'\}$.

   (a) Show that $(\wp(X), \Box_R)$ is an **S4**-algebra[1] .

   (b) Determine the fixed points of the operator $\Box_R$.

   (c) Can you define a finite join preserving function $\Diamond_R \colon \wp(X) \to \wp(X)$ in a similar way? What are the fixed points for this operator.

4. (More on Alexandroff spaces and posets)

   (a) Let $f \colon P \to Q$ be a function between posets $(P, \leq_P)$ and $(Q, \leq_Q)$. Show that f is order-preserving iff f is continuous with respect to the topologies $\operatorname{Up}(P)$ and $\operatorname{Up}(Q)$.

[1] Also know as an *interior algebra*.

(b) Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be posets. Characterise the order-preserving maps $f \colon P \to Q$ with the property that f is an open map as a function between the induced Alexandroff spaces.

5. (a) Find an example of a Heyting algebra A and a subalgebra A' of A such that A' is not a homomorphic image of A.

   (b) Find an example of a Heyting algebra that has a homomorphic image B such that B is not isomorphic to a subalgebra of A.

   (*Hint:* finite such examples exist.)