Mathematical structures in logic
Exercise class 2

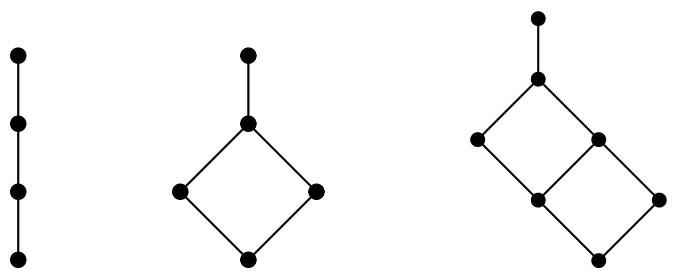
Heyting algebras, Boolean algebras

February 13, 2018

1. Let A be a Boolean algebra. Show that $a \wedge b = \neg(\neg a \vee \neg b)$ and $a \vee b = \neg(\neg a \wedge \neg b)$.

2. We know that the lattice $(\mathrm{Fin}(\mathbb{N}) \cup \{\mathbb{N}\}, \subseteq)$ of finite subsets of $\mathbb{N}$ (together with $\mathbb{N}$) is a complete bounded distributive lattice. Is it a Heyting algebra?

3. Let A_1, A_2 and A_3 be the following posets



   (a) Convince yourself that A_1, A_2 and A_3 are all Heyting algebras.

   (b) Identify the joins and the pseudo-complements in A_1, A_2 and A_3.

   (c) Is A_1 isomorphic to a bounded sublattice of A_2 or A_3? Is it isomorphic to a Heyting subalgebra of A_2 and A_3?

4. (Atoms and co-atoms) Recall that, if $(L, \leq)$ is a bounded lattice, $a \in L$ is called an *atom* if $b < a$ implies $b = 0$ and a *coatom* if $a < b$ implies $b = 1$.

   (a) Describe atoms and co-atoms on a Boolean algebra of the form $\mathcal{P}(X)$.

   (b) Show that in every Boolean algebra, if a is an atom, then $\neg a$ is a co-atom.

   (c) Find a Heyting algebra A with an atom a such that $\neg a$ is not a co-atom

5. Show that not every bounded distributive lattice is isomorphic to the lattice of upsets of some poset.

6. We abbreviate $a \to 0$ with $\neg a$. Show that in every Heyting algebra

   (a) $a \wedge \neg a = 0$ but not necessarily $a \vee \neg a = 1$;

   (b) $a \leq b$ iff $a \to b = 1$;

   (c) $a \leq \neg\neg a$;

   (d) $\neg a \wedge \neg b = \neg(a \vee b)$;

   (e) $\neg a \vee \neg b \leq \neg(a \wedge b)$ but not necessarily $\neg(a \wedge b) \leq \neg a \vee \neg b$;

   (f) $a \to (b \to c) = (a \wedge b) \to c$;

   (g) $b \leq c$ implies $a \to b \leq a \to c$;

(h) $b \leq c$ implies $c \to a \leq b \to a$.

7. A *topological space* is a pair (X, τ) where X is a set and $\tau \subseteq \mathcal{P}(X)$ is a collection of subsets of X such that

   i. $\varnothing \in \tau$ and $X \in \tau$;
   ii. If $U, V \in \tau$, then $U \cap V \in \tau$;
   iii. If $\sigma \subseteq \tau$, then $\bigcup \sigma \in \tau$.

   Given $P \subseteq X$, we can define the *interior of* P as $\operatorname{Int} P = \bigcup \{U \in \tau : U \subseteq P\}$.

   (a) Prove that $(\tau, \subseteq)$ is a Heyting algebra.
   (b) Characterise $\bigvee \sigma$ and $\bigwedge \sigma$ for $\sigma \subseteq \tau$.

Additional exercises

8. Let A_2 and A_3 be as in exercise 1.

   (a) Is A_2 isomorphic to a bounded sublattice of A_3? Is it isomorphic to a Heyting subalgebra of A_3?
   (b) Is there a surjective bounded lattice homomorphism from A_3 to A_2? Is there a surjective Heyting algebra homomorphism from A_3 to A_2?

9. Let L be a bounded distributive lattice. Show that there is a 1-to-1 correspondence between pairs of complemented elements of L (i.e. pairs $\langle a, b \rangle \in L^2$ such that $a \wedge b = 0$ and $a \vee b = 1$) and decompositions of the form $L \simeq L_1 \times L_2$ where L_1 and L_2 are bounded distributive lattices. (Hint: Try to understand first what this means for powerset lattices.)

10. *For people who know some category theory:* Given a poset $(P, \leq)$ we can see it as a category having P as objects and there is a morphism from p to q iff $p \leq q$. Try to connect the notions of lattice theory that we encountered so far (suprema, infima, bounds, Heyting implications, complements, ...) to categorical structure (such as products, coproducts, ...).