MATHEMATICAL STRUCTURES IN LOGIC 2018
HOMEWORK 6

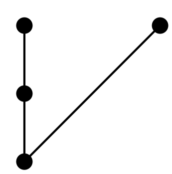
- Deadline: March 20 — at the **beginning** of the tutorial class.
- In exceptional cases homework can be submitted electronically (in a single pdf-file!) to Saul Fernandez (`saul.fdez.glez@gmail.com`).
- Grading is from 0 to 100 points.
- Discussion of problems is allowed, but each student should submit a homework they themselves have written.
- Good luck!

(1) (20pt) (From the final exam of 2017) Show that for every nontrivial variety V of Heyting algebras, if $\mathsf{V} \neq \mathsf{BA}$, then $\mathbf{Var}(\mathbf{3}) \subseteq \mathsf{V}$, where $\mathbf{3}$ is the 3-element Heyting algebra.

(Hint: show that $\mathbf{3}$ is a subalgebra of some subdirectly irreducible $A \in \mathsf{V}$.)

(2) (20pt) Prove that for Heyting algebras A and B we have $A \times B \simeq \mathsf{ClopUp}(X_A \sqcup X_B)$, where $X_A \sqcup X_B$ is the disjoint union of X_A and X_B. (For the definition of a disjoint union of posets see page 17 of "Lattices and Order" and for a disjoint union of topological spaces, see page 267.)

(3) (20pt) (From the final exam of 2017) Let A be the dual Heyting algebra of the poset shown in the figure below. Describe the lattice of subvarieties of the variety $\mathbf{Var}(A)$ generated by A. (Hint: use Jónsson's lemma and duality).



(4) (20pt) Let $\mathsf{KC} := \mathsf{IPC} + \neg\varphi \vee \neg\neg\varphi$.

(a) Let A be a non-trivial subdirectly irreducible HA. Show that A belongs to the variety of KC-algebras if and only if the bottom element of A is meet irreducible, i.e., $a \wedge b = 0$ implies $a \leq 0$ or $b \leq 0$.

(b) Recall that a variety V is finitely generated if $\mathsf{V} = \mathbf{Var}(A)$ for some finite algebra A. Is the variety V_{KC} of KC-algebras finitely generated? Justify your answer with a proof.

(5) (20pt) (Finite model property) Give full details of the algebraic proof[1] of the finite model property of HAs (**IPC**).

(6) **Bonus (+10pt)** Show that the intermediate logic KC enjoys the finite model property, i.e., show that the variety of KC-algebras is generated by finite KC-algebras.

[1] A sketch of this proof using $\vee$-free reducts can be found at `https://staff.fnwi.uva.nl/n.bezhanishvili//MSL/MSL2017/HA-FMP-Sketch.pdf`. You are free to choose whether to use $\vee$-free reducts or $\to$-free reducts.