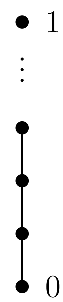
MATHEMATICAL STRUCTURES IN LOGIC 2018
HOMEWORK 5

- Deadline: March 13 — at the **beginning** of the tutorial class.
- In exceptional cases homework can be submitted electronically (in a single pdf-file!) to Saul Fernandez (`saul.fdez.glez@gmail.com`).
- Grading is from 0 to 100 points.
- Discussion of problems is allowed, but each student should submit a homework they themselves have written.
- Good luck!

(1) (20pt) Draw the (underlying poset of the) Esakia space dual to the Heyting algebra $\omega + 1$ shown below. Determine the clopen up-sets of this Esakia space.



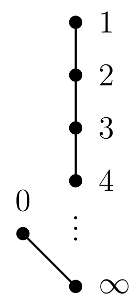
(2) (30pt) Let X be a poset. X is called *rooted* if there is $r \in X$ such that $X = \uparrow r$. Then r is called the *root* of X.

(a) Let X be a non-trivial Esakia space (i.e., $X \neq \emptyset$). Show that X is rooted if and only if in its dual Heyting algebra A for each $a, b \in A$ we have $a \vee b = 1$ implies $a = 1$ or $b = 1$.

(b) Call a rooted Esakia space X with a root r *strongly rooted* if $\{r\}$ is open. Show that an Esakia space X is strongly rooted if and only if the Heyting algebra dual to X has the second largest element. (Recall that in Hausdorff spaces every point is closed.)

(c) Give an example of a rooted Esakia space, which is not strongly rooted.

(3) (30pt) Consider the ordered topological space $(\mathfrak{X}, \leq)$ drawn below where the space $\mathfrak{X}$ is the Alexandroff compactification $\alpha\mathbb{N}$ of $\mathbb{N}$. In other words, the clopen sets are finite subsets of $\mathbb{N}$ and cofinite subsets of $\mathbb{N}$ together with the point ∞.



(a) Show that this space is a Priestley space.

(b) Show that it is not an Esakia space.

(c) Draw the distributive lattice dual to this space.

(4) (20pt) [From the final exam of 2017]

(a) Let $(X, \leq)$ be an Esakia space. Show that for any up-set $U \subseteq X$ we have that its closure $\mathrm{cl}(U)$ is also an up-set.

(b) Give an example of a Priestley space $(X, \leq)$ and an up-set $U \subseteq X$ such that its closure $\mathrm{cl}(U)$ is not an up-set.

Recall that if X is a Stone space and $x \in X$, then $x \in \mathrm{cl}(U)$ iff for every clopen set V with $x \in V$ we have $V \cap U \neq \emptyset$.